Multiscale analysis in image processing

Preliminaries on wavelets

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bpascal-fr.github.io/talks

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Multiple scales in real data

Albert Marquet, *Paysage, baie méditerranéenne, vue d'Agay*, 1905

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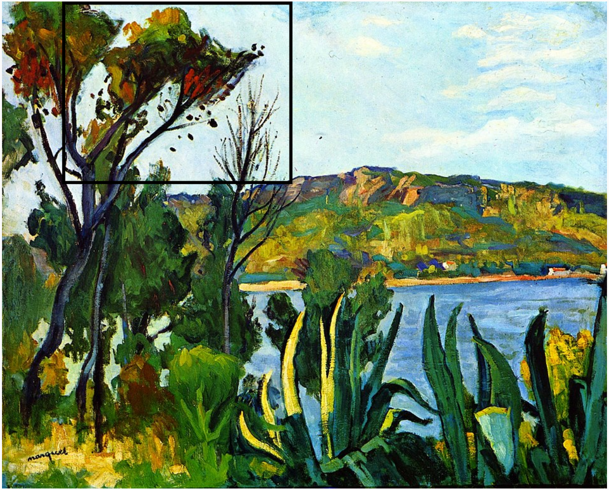

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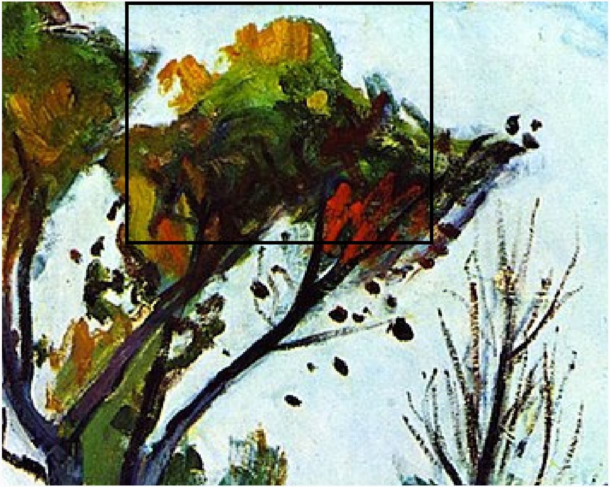

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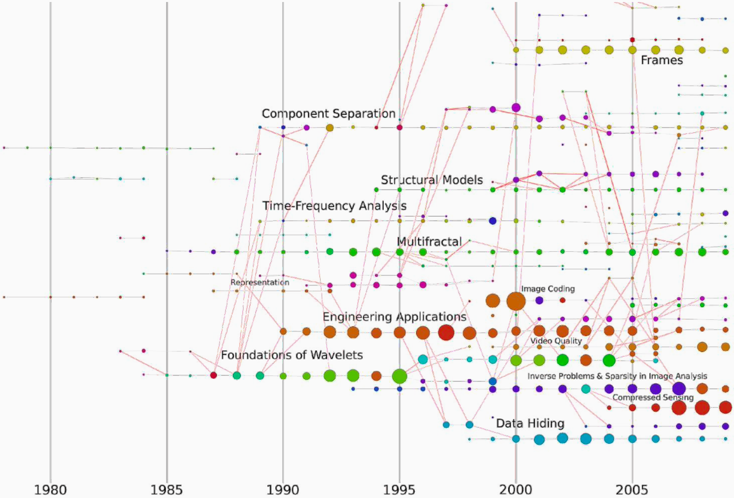
Overview of the history of wavelets

Revealing evolutions in dynamical networks

Matteo Morini, Patrick Flandrin, Eric Fleury, Tommaso Venturini, Pablo Jensen[1]
IXXI, ENS de Lyon, INRIA, CNRS, LIP UMR 5668, LP UMR 5672



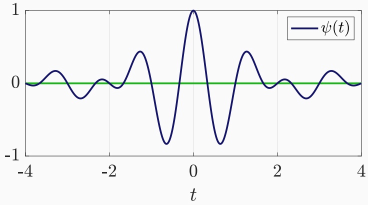
Wavelet Transforms

Continuous Wavelet Transform

Wavelet: $\psi \in L^2(\mathbb{R})$ locally oscillating, integrable with $\int_{\mathbb{R}} \psi(s)\,\mathrm{d}s = 0$

Example: real Shannon wavelet

$$\psi(t) = \frac{\sin(2\pi t) - \sin(\pi t)}{\pi t}$$



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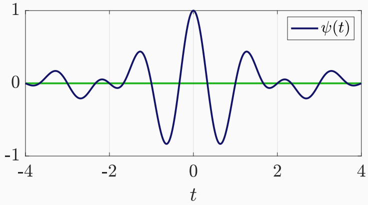

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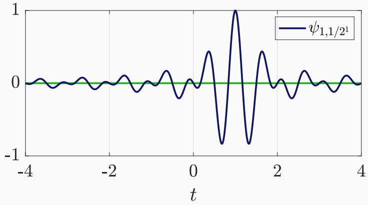

$\Longrightarrow \{\psi_{t,a},\, (t,a) \in \mathbb{R} \times \mathbb{R}_+\}$ atoms with **different time supports**

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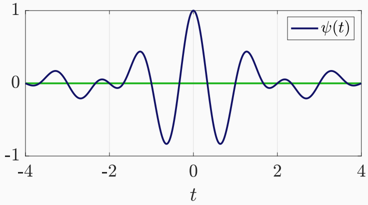

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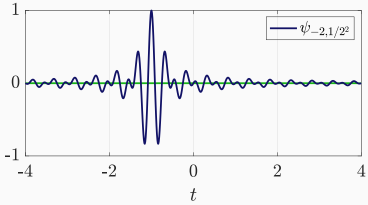

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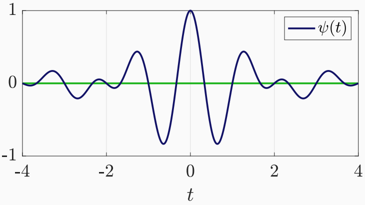

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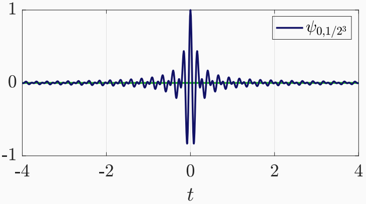

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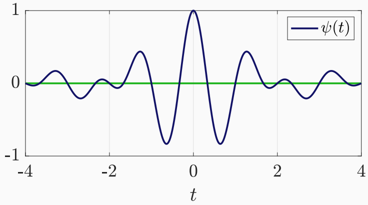

Continuous Wavelet Transform of a finite-energy signal $f \in L^2(\mathbb{R})$

$$\mathcal{W}_f(t,a) = \langle f, \psi_{t,a} \rangle = \int_{\mathbb{R}} f(s) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{s-t}{a}\right)}\,\mathrm{d}s$$

$\langle \cdot, \cdot \rangle$ scalar product in $L^2(\mathbb{R})$, $\overline{\cdot}$ complex conjugate

$\Longrightarrow \{\psi_{t,a},\, (t,a) \in \mathbb{R} \times \mathbb{R}_+\}$ atoms with **different time supports**

Example of a gravitational wave

Physics: two objects $M = m_1 + m_2$ at distance R, $\mu^{-1} = m_1^{-1} + m_2^{-1}$

$$f(t) = A(t_0 - t)^{-1/4} \cos(d(t_0 - t)^{5/8} + \varphi)\mathbf{1}_{(-\infty;t_0[}(t)$$

chirp: amplitude $a(t) = A(t_0 - t)^{-\frac{1}{4}}$, frequency $\omega(t) = \dfrac{10\pi d}{8}(t_0 - t)^{-\frac{3}{8}}$

- t_0: time of coalescence,
- d: instantaneous frequency parameter $d \simeq 241\mathcal{M}_{\odot}^{-5/8}$,
- A: amplitude reference $A \simeq 3.37 \times 10^{-21}\mathcal{M}_{\odot}^{5/4}/R$.

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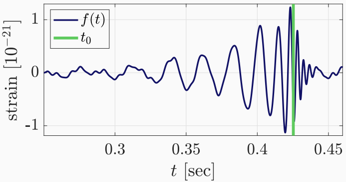

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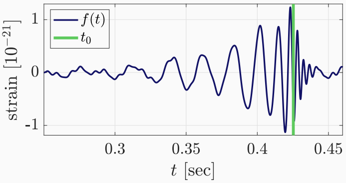

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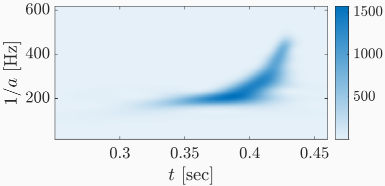

Morlet scalogram $|\mathcal{W}_f(t,a)|^2$

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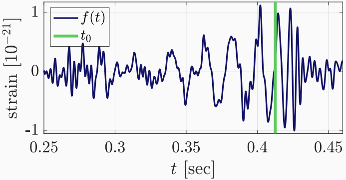

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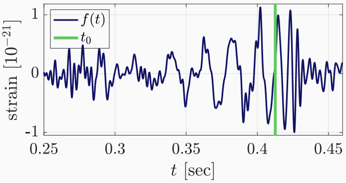

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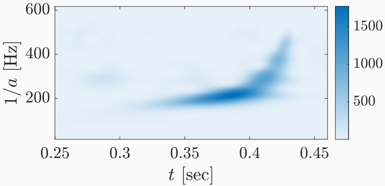

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Continuous Wavelet Transform

Reconstruction formula For $\widetilde{\psi}$ the Fourier transform of ψ,

if $C_\psi = \int_{\mathbb{R}} \frac{|\widetilde{\psi}(\omega)|^2}{|\omega|} \mathrm{d}\omega < \infty$ then ψ is **admissible** and for $f \in L^2(\mathbb{R})$

$$f(t) = \frac{1}{C_\psi} \int_{\mathbb{R}\times\mathbb{R}_+} \mathcal{W}_f(s,a) \frac{1}{\sqrt{a}} \psi\left(\frac{s-t}{a}\right) \mathrm{d}s \frac{\mathrm{da}}{a^2}$$

with $\mathcal{W}_f(s,a) = \langle f, \psi_{s,a} \rangle$ [A. P. Calderón, 1964, *Stud. Math.*; A. Grossmann & J. Morlet, 1984, *SIAM J. Math. Anal.*]

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Reproducing kernel $\mathcal{W}_f(t,a)$ **redundant** representation of f

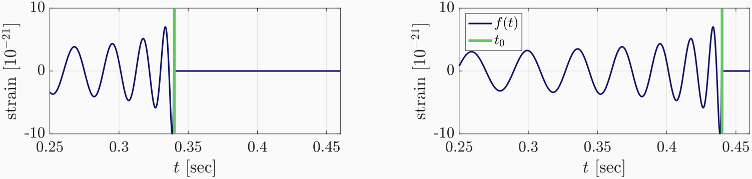
$$\mathcal{W}_f(t',a') = \frac{1}{C_\psi} \int_{\mathbb{R}\times\mathbb{R}_+} \mathcal{K}(t',t;a',a)\mathcal{W}_f(t,a)\,\mathrm{d}t \frac{\mathrm{da}}{a^2}$$

with $\mathcal{K}(t',t;a',a) = \langle \psi_{t,a}, \psi_{t',a'} \rangle$ correlations between wavelets

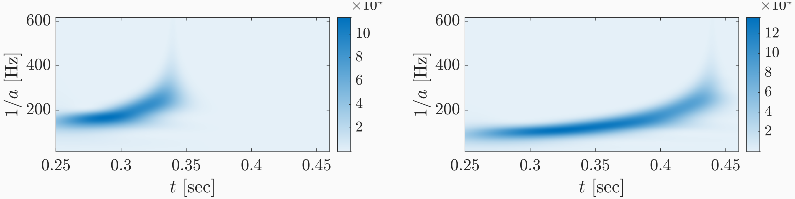
Continuous Wavelet Transform

Translation invariance

Let $f^{\Delta t}(t) = f(t - \Delta t)$, then $\mathcal{W}_{f^{\Delta t}}(t, a) = \mathcal{W}_f(t - \Delta t, a)$.



$t_0 = 0.44$ [sec] $t_0 = 0.34$ [sec]



Discrete Signals and Wavelets

From continuous signals to discrete vectors

- f continuous on $[0,1]$, discretized in $\mathrm{z}_n = f\left(\frac{n}{N}\right)$, $n = 0, 1, \ldots, N$

  discrete wavelet transform can be computed at scales $N^{-1} < a^j < 1$
- **discrete scales**: $a = 2^{1/v} \implies v$ intermediate scales in octave $[2^j, 2^{j+1})$

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Practical implementation

Integral representation

$$\mathcal{W}_f(t,a) = \int_{\mathbb{R}} f(s) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{s-t}{a}\right)} \,\mathrm{d}s$$

Discrete convolution $\mathrm{z} * \overline{\psi_{t,a}(-\cdot)}$

$$\mathcal{W}_z[n,j] = \sum_{m=0}^{N-1} z_m \frac{1}{\sqrt{a^j}} \overline{\psi\left(\frac{m-n}{a^j}\right)}$$

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complexity $\mathcal{O}\left(vN\left(\log_2 N\right)^2\right)$

[S. Mallat, 2009, *Academic Press, Elsevier*]

Multiresolution Analysis

Aims and principles of multiresolution analysis

Motivation: process only details at relevant **discrete** resolutions

[P. J. Burt & E. H.Adelson, 1983, *Proc. IEEE Int. Conf. Commun.*]

Multiresolution analysis

Definition: A multiresolution analysis of $L^2(\mathbb{R})$ is a subpaces sequence

$\{0\} \subset \ldots \subset V_1 \subset V_0 \subset V_{-1} \subset \ldots V_{-j} \ldots \subset V_{-(j+1)} \subset \ldots \subset L^2(\mathbb{R})$ satisfying

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- self-similarity in **scale**: $(\forall j \in \mathbb{Z}) \quad f \in V_j \Longleftrightarrow f\left(\frac{\cdot}{2}\right) \in V_{j+1}$
- **regularity**: φ **father wavelet** or **scaling function** such that

  $\{\varphi(t-k), \quad k \in \mathbb{Z}\}$ is an orthonormal basis of V_0
- **completeness**: $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\cap_{j\in\mathbb{Z}} V_j = \{0\}$

$\Longrightarrow V_j$ **approximation space** at *scale* 2^j, i.e., *resolution* 2^{-j}.

[P. J. Burt & E. H.Adelson, 1983, *Proc. IEEE Int. Conf. Commun.*;
S. Mallat, 1989, *Trans. Amer. Math. Soc.*;
S. Mallat, 1989, *IEEE Trans. Pattern Anal. Mach. Intell.*;
Y. Meyer, 1992, *Cambridge University Press*]

Aims and principles of multiresolution analysis

Motivation: process only details at relevant **discrete** resolutions

scale $4 \Longleftrightarrow$ resolution $1/4$ | scale $2 \Longleftrightarrow$ resolution $1/2$ | scale $1 \Longleftrightarrow$ resolution 1

$$\text{scale } 2^j \Longleftrightarrow \text{resolution } 2^{-j}$$

[P. J. Burt & E. H.Adelson, 1983, *Proc. IEEE Int. Conf. Commun.*]

Multiresolution analysis – Daubechies 2 wavelets

From **time** and **scale** invariance and **regularity** condition:

$$\{\sqrt{2^{-j}}\varphi(t/2^j - k), \quad k \in \mathbb{Z}\} \text{ is an orthonormal basis of } V_j$$

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Orthogonal projection of f onto V_j

$$f|_{V_j} = \sum_{k \in \mathbb{Z}} \phi_{j,k} \sqrt{2^{-j}} \varphi(t/2^j - k)$$

with *approximation coefficients*

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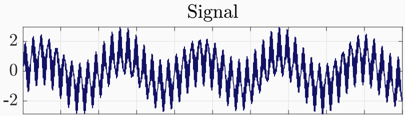

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with *approximation coefficients*

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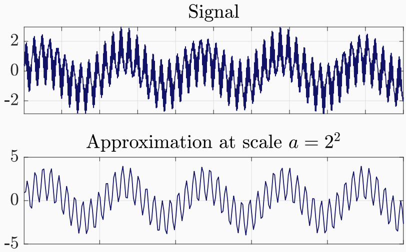

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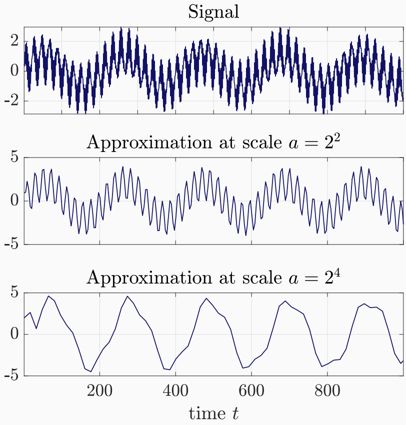

Multiresolution analysis – Daubechies 2 wavelets

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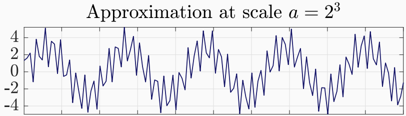

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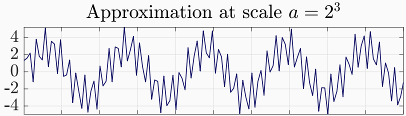

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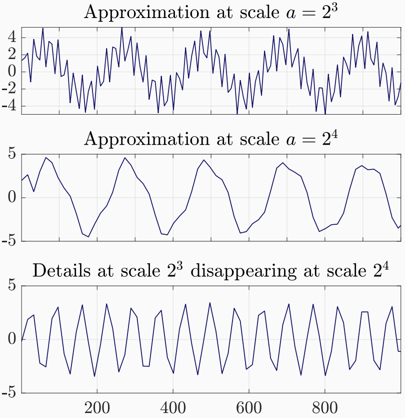

Orthonormal wavelet basis

By the **completeness** property of the multiresolution analysis

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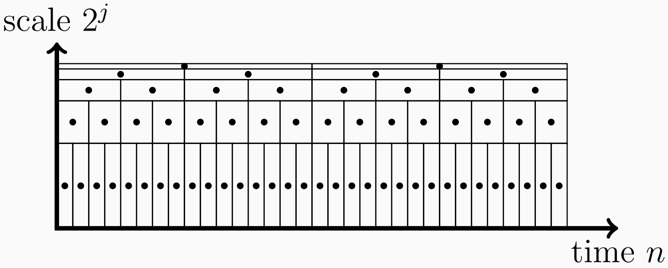
Tiling of the time–scale half-plane



Vanishing Moments of Wavelets

Theorem: Let φ scaling function, ψ mother wavelet and $\widetilde{\psi}$ its Fourier transform s.t.

$$|\varphi(t)| \underset{|t|\to\infty}{=} \mathcal{O}\left((1+t^2)^{-n_\psi/2-1}\right), \quad |\psi(t)| \underset{|t|\to\infty}{=} \mathcal{O}\left((1+t^2)^{-n_\psi/2-1}\right)$$

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iii) $(\forall 0 \leq k < p) \quad t \mapsto \displaystyle\sum_{n=-\infty}^{\infty} n^k \varphi(t-n)$ is a polynomial of degree k (Fix-Strang)

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Interpretation:

- mother wavelet **orthogonal** to polynomials of degree at most $n_\psi - 1$
- if signal f is $\mathcal{C}^k$, $k < n_\psi$

  wavelet coefficients $\zeta_{j,n} = \langle f, \psi_{j,n} \rangle$ **small at fine scales**

Support size and number vanishing moments trade-off

If f has an **isolated singularity** at t_0 contained in the support of $\psi_{j,n}$ then the wavelet coefficient $\langle f, \psi_{j,n} \rangle$ is **large**.

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If ψ has a compact support of size $\Delta \in \mathbb{N}^*$, at scale 2^j

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$\Longrightarrow$ to reduce the number of significant coefficients, **reduce support size** of ψ.

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A **Daubechies** wavelet has a minimum-size support $[-n_\psi + 1, n_\psi]$ and the support of the corresponding scaling function is $[0, 2n_\psi - 1]$.

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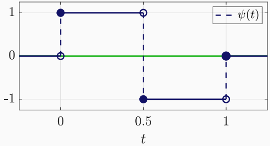

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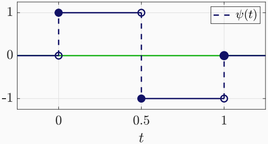

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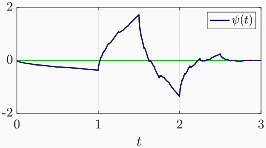
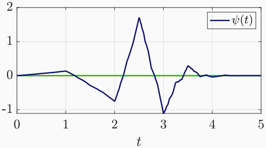
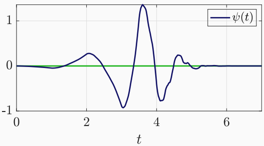


$n_\psi = 2$



$n_\psi = 3$



$n_\psi = 4$

Decompositions on Wavelet Frames

Theory of Frames

$\mathcal{H}$: Hilbert space, e.g., $L^2(\mathbb{R})$ or subspace of $L^2(\mathbb{R})$; $\mathbb{I} \subset \mathbb{N}$: set of indices

Definition A family of elements of $\mathcal{H}$, $\{\mathrm{e}_n, n \in \mathbb{I}\}$, s.t.

$$(\forall f \in \mathcal{H}) \quad \underline{\mu}\|f\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, \mathrm{e}_n\rangle|^2 \leq \overline{\mu}\|f\|^2$$

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Stable analysis and synthesis

$f \in \mathcal{H} \mapsto (\langle f, \mathrm{e}_n\rangle)_{n\in\mathbb{N}}\, \ell^2(\mathbb{I})$ bounded linear operator

$(f_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{I}) \mapsto \sum_{n\in\mathbb{Z}} f_n \mathrm{e}_n \in \mathcal{H}$ bounded linear operator

Theory of Frames

Initially: reconstruction of irregularly sampled band-limited signals

[Duffin & Schaeffer, 1952, *Trans. Amer. Math. Soc.*]

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Motivations in multiresolution analysis:

Continuous Wavelet Transform: highly redundant, computationally costly

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Orthonomal wavelet basis decomposition: not invariant under translations

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Wavelet frames: $\psi_{j,n}^{(\gamma,b)} = \left\{ \frac{1}{\sqrt{\gamma^j}} \widetilde{\psi}\left(\frac{t - bn\gamma^j}{\gamma^j}\right), (j,n) \in \mathbb{Z}^2 \right\}, \ \gamma > 1, \ b > 0$

$\Longrightarrow$ more freedom in the design of the wavelet $\psi^{(\gamma,b)}$

Wavelet Decomposition of Images

Mathematical representation of images

Real-valued square-integrable **field**

$$\mathsf{X} : \mathbb{R}^2 \to \mathbb{R}$$

restricted to a **rectangular** domain $\Omega = [0, n_1 - 1] \times [0, n_2 - 1]$

Two-dimensional wavelet decomposition

Separable wavelet bases:

φ and ψ the scaling function and mother wavelet of a **1D multiresolution analysis**

Define
$$\begin{cases} \psi^{(0)}(\underline{x}) = \varphi(x_1)\varphi(x_2), \quad \psi^{(1)}(\underline{x}) = \psi(x_1)\varphi(x_2) \\ \psi^{(2)}(\underline{x}) = \varphi(x_1)\psi(x_2), \quad \psi^{(3)}(\underline{x}) = \psi(x_1)\psi(x_2). \end{cases}$$

Then, the family

$$\left\{2^{-j}\psi^{(m)}\left(\underline{x}2^{-j} - \underline{n}\right), m \in \{1,2,3\}, \underline{x} = (x_1, x_2) \in \mathbb{R}^2, \underline{n} = (n_1, n_2) \in \mathbb{Z}^2\right\}$$

defines an **orthonormal wavelet basis** of $L^2(\mathbb{R})$.

The **wavelet coefficients** of a 2D field $\mathsf{X} \in L^2(\mathbb{R}^2)$ are defined as

$$\zeta_{j,\underline{n}}^{(m)} = \langle \mathsf{X}, \psi_{j,\underline{n}}^{(m)} \rangle, \quad \psi_{j,\underline{n}}^{(m)}(\underline{x}) = 2^{-j}\psi^{(m)}\left(\underline{x}2^{-j} - \underline{n}\right)$$

Wavelet transform of images

Albert Marquet, *Paysage, baie méditerranéenne, vue d'Agay*, 1905

Daubechies wavelet transform with $n_\psi = 2$ vanishing moments at scale 2^1

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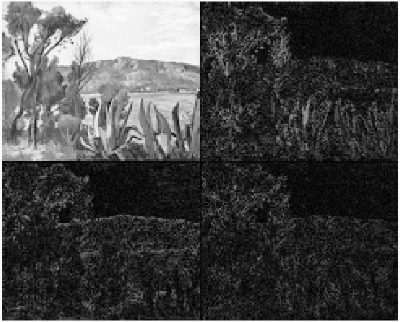

Daubechies wavelet transform with $n_\psi = 2$ vanishing moments at scale 2^3

Wavelet transform of images

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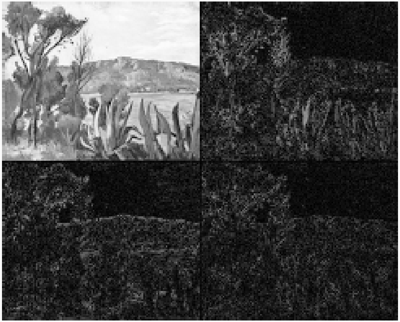
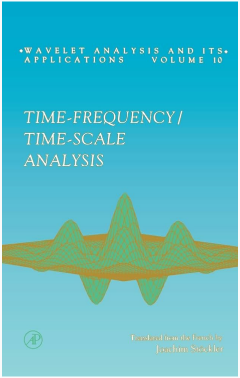
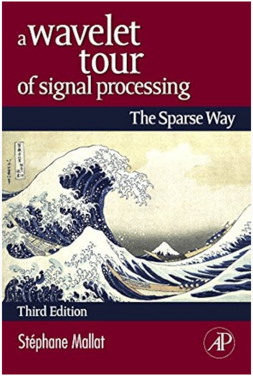


Daubechies wavelet transform with $n_\psi = 2$ vanishing moments at scale 2^3

Application: compression of images and videos: JPEG2000, MPEG-4

References and further readings





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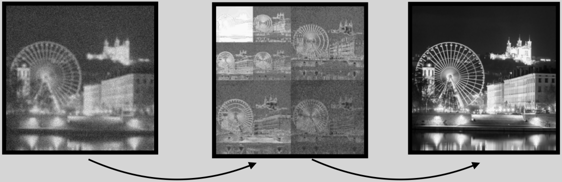
Multiresolution/multilevel

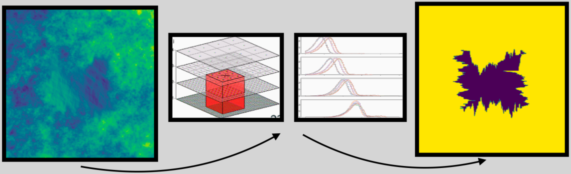
Multiresolution
to perform
image restoration

(~2000–2015)



Multiresolution
to perform
texture
segmentation

(~2014- now)



Multiresolution
to accelerate
algorithms

(~2016- now)