Bilevel optimization for automated data-driven inverse problem resolution

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(bpascal-fr.github.io)

Joint work with Patrice Abry, Nelly Pustelnik, Valérie Vidal, and Samuel Vaiter

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Learning and Optimization in Luminy

Centre International de Rencontres Mathématiques (CIRM), Marseille

Observation model

$$\boldsymbol{y} \sim \mathcal{B}\left(\boldsymbol{\Phi}\overline{\boldsymbol{x}}\right)$$

- $\boldsymbol{y} \in \mathbb{R}^P$: degraded observations;
- $\overline{\boldsymbol{x}} \in \mathbb{R}^N$: unknown quantity of interest;
- $\boldsymbol{\Phi} : \mathbb{R}^N \to \mathbb{R}^P$: known deformation;
- $\mathcal{B}$: random measurement noise.

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Goal: Estimate $\overline{\boldsymbol{x}}$

- ill-conditioned, rank deficient $\boldsymbol{\Phi}$

Inpainting

(Guillemot et al., 2013, *IEEE Sig. Process. Mag.*)

Super-resolution

(Marquina et al., 2008, *J. Sci. Comput.*)

Deblurring

(Pan, 2016, *IEEE Trans. Pattern Anal. Mach. Intell.*)

Inverse problems in signal and image processing

Observation model

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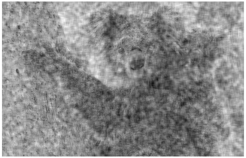
Goal: Estimate $\overline{\boldsymbol{x}}$

- ill-conditioned, rank deficient $\boldsymbol{\Phi}$
- correlated, data-dependent $\mathcal{B}$

Correlated



(Pascal et al., 2021, *J. Math. Imaging Vis.*)

Data-dependent

(Luisier et al., 2010, *IEEE Trans. Image Process.*)

Multiplicative

(Shama, 2016, *Appl. Math. Comput.*)

Variational estimator

$$\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) \in \underset{\boldsymbol{x}\in\mathbb{R}^N}{\operatorname{Argmin}}\, \mathcal{D}(\boldsymbol{y}, \boldsymbol{\Phi}\boldsymbol{x})$$

- $\mathcal{D}(\boldsymbol{y};\cdot) = -\log \mathbb{P}(\boldsymbol{y}|\cdot)$: negative log-likelihood

Ex: $\mathcal{D}(\boldsymbol{y};\boldsymbol{x}) = \|\boldsymbol{y}-\boldsymbol{x}\|_2^2$

No regularization

$\mathcal{R} = 0$

The variational framework: penalized log-likelihood

Variational estimator

$$\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) \in \operatorname*{Argmin}_{\boldsymbol{x}\in\mathbb{R}^N} \mathcal{D}(\boldsymbol{y},\boldsymbol{\Phi}\boldsymbol{x}) + \lambda\mathcal{R}(\boldsymbol{x})$$

- $\mathcal{D}(\boldsymbol{y};\cdot) = -\log \mathbb{P}(\boldsymbol{y}|\cdot)$: negative log-likelihood — Ex: $\mathcal{D}(\boldsymbol{y};\boldsymbol{x}) = \|\boldsymbol{y}-\boldsymbol{x}\|_2^2$
- $\mathcal{R}$: regularization term encoding a priori knowledge — Ex: $\mathcal{R}(\boldsymbol{x}) = \|\mathbf{D}\boldsymbol{x}\|_q^q$

(Giovannelli & Idier, 2015, *Wiley*)

No regularization

$\mathcal{R} = 0$

Smooth

$\mathcal{R}(\boldsymbol{x}) = \|\mathbf{D}_1\boldsymbol{x}\|_2^2$

(Tikhonov et al., 1977, *Wiley*)

Piecewise constant

$\mathcal{R}(\boldsymbol{x}) = \|\mathbf{D}_1\boldsymbol{x}\|_1$

(Rudin et al., 1992, *Physica D*)

Fine-tuning of the regularization parameter

Example: $$\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) \in \underset{\boldsymbol{x}\in\mathbb{R}^N}{\operatorname{Argmin}} \|\boldsymbol{y}-\boldsymbol{x}\|_2^2 + \lambda\|\mathbf{D}_1\boldsymbol{x}\|_2^2 \quad \text{(Tikhonov)}$$

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$\lambda = 1$

not enough

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not enough

$\lambda = 25$

too regularized

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Oracle-based hyperparameter selection

$$\lambda^\dagger \in \operatorname*{Argmin}_{\lambda\in\Lambda} \mathcal{O}(\boldsymbol{y};\lambda)$$

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- golden case: $\mathcal{O}(\boldsymbol{y};\lambda) = \|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \overline{\boldsymbol{x}}\|^2 \Longrightarrow$ efficient bi-level $(\boldsymbol{x},\lambda)$ minimization

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$\lambda = 1$	$\boldsymbol{\lambda^\dagger = 5}$	$\lambda = 25$
		
not enough	**optimal**	too regularized

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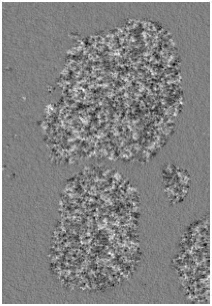
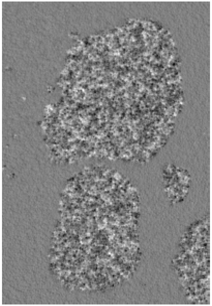
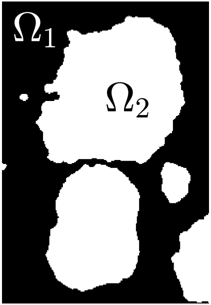
- golden case: $\mathcal{O}(\boldsymbol{y};\lambda) = \|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \overline{\boldsymbol{x}}\|^2 \Longrightarrow$ efficient bi-level $(\boldsymbol{x},\lambda)$ minimization
- practical case: ground truth $\overline{\boldsymbol{x}}$ **not available!** $\Longrightarrow$ data-driven $\mathcal{O}(\boldsymbol{y};\lambda)$

Image processing:

Texture segmentation

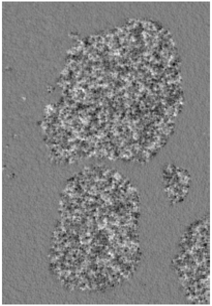


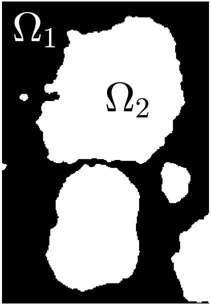




Goal: obtain a partition of the image into K homogeneous textures

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_K$$





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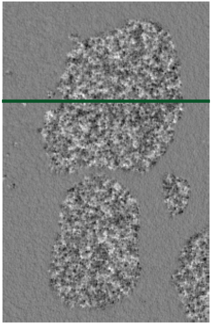
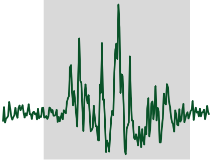
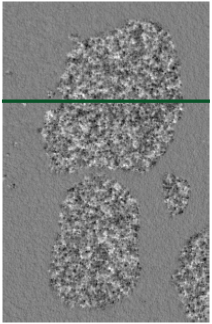
Piecewise monofractal model

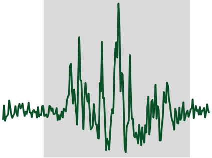




Fractal attributes

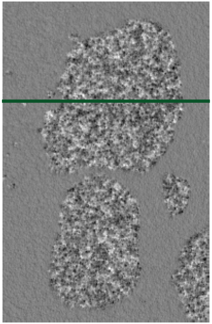
- variance σ^2 *amplitude of variations*

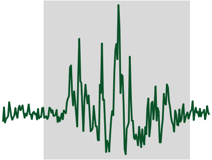




Fractal attributes

- variance σ^2 *amplitude of variations*
- local regularity h *scale invariance*





Fractal attributes

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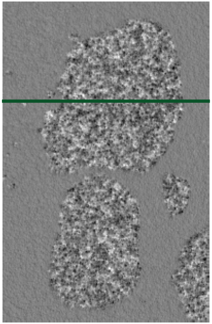
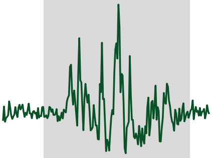
$$|f(x) - f(y)| \leq \sigma(x)|x - y|^{h(x)}$$





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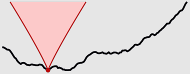
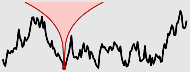
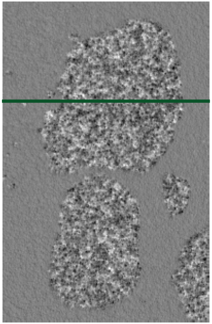
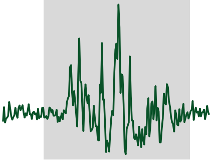


$h(x) \equiv h_1 = 0.9$



$h(x) \equiv h_2 = 0.3$





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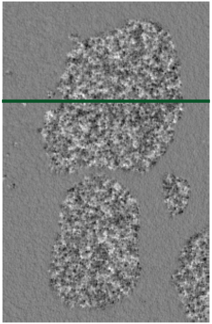
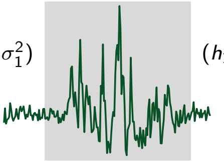


(h_2, σ_2^2)

(h_1, σ_1^2)



(h_1, σ_1^2)

Segmentation

- σ^2 and h piecewise constant

Fractal attributes

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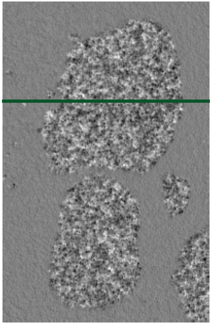
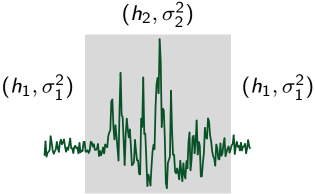
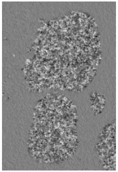




Segmentation

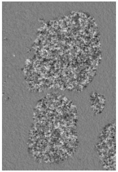
- σ^2 and h piecewise constant
- region Ω_k characterized by (σ_k^2, h_k)

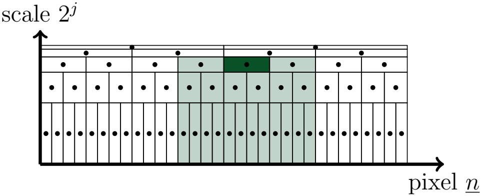
Textured image



Textured image

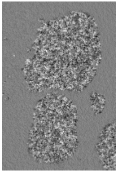
Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$





Multiscale analysis

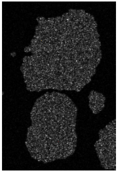
Textured image



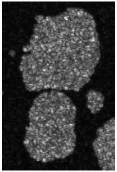
Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$

Scale

$a = 2^1$

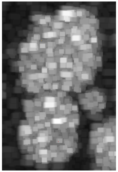


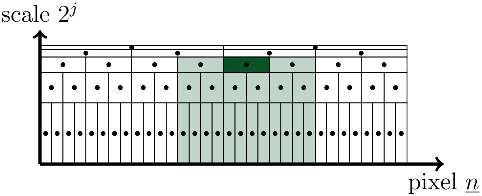
$a = 2^2$



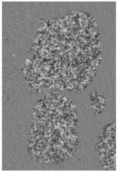
...

$a = 2^5$





Textured image



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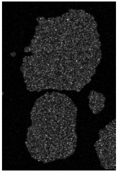

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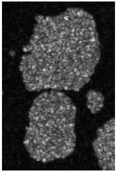

$\dots$

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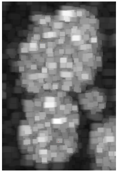

Proposition (Jaffard, 2004, *Proc. Symp. Pure Math.*; Wendt et al., 2009, *Signal Process.*)

$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underset{\text{regularity}}{\boldsymbol{h}} + \underset{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}{\boldsymbol{v}}$$

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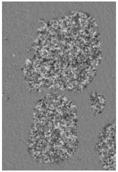

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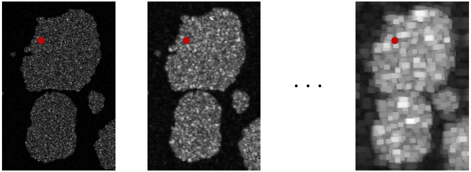

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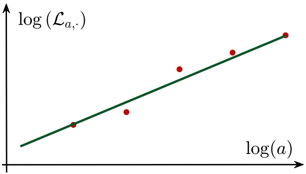

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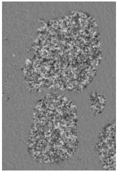

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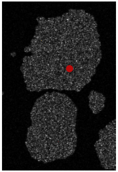

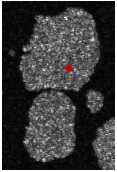

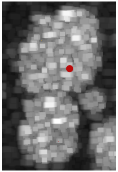

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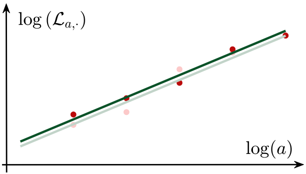

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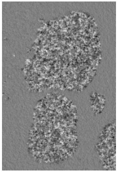

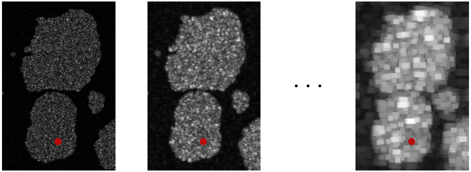

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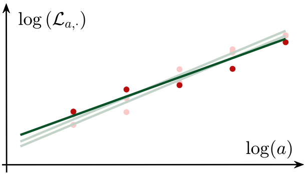
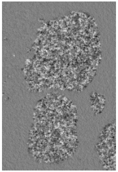


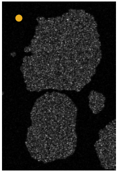
Textured image



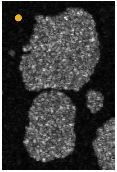
Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$

Scale

$a = 2^1$

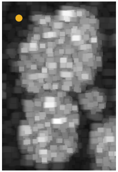


$a = 2^2$



...

$a = 2^5$



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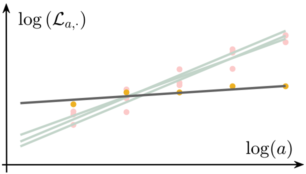
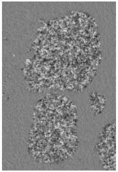
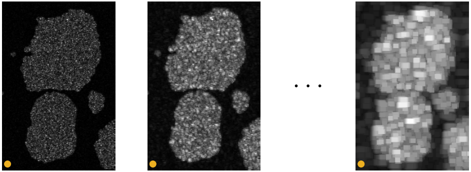


Textured image



Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$

Scale $a = 2^1$ $a = 2^2$ ... $a = 2^5$



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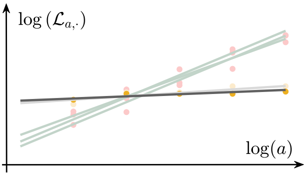
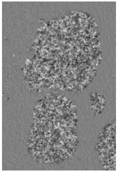


Textured image



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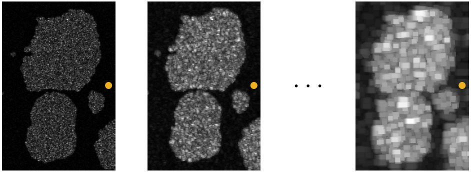

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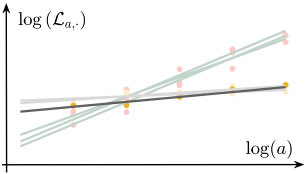
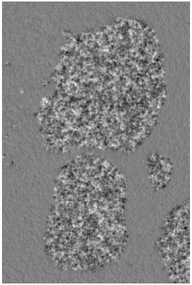


Linear regression $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

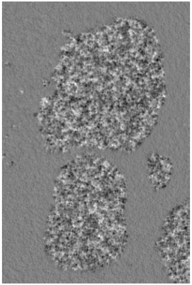
Textured image



Linear regression $\log(\mathcal{L}_{a,\cdot}) \simeq \log(a) \underset{\text{regularity}}{\boldsymbol{h}} + \underset{\propto \log(\boldsymbol{\sigma}^2)}{\boldsymbol{v}}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\| \log(\mathcal{L}_{a,\cdot}) - \log(a)\boldsymbol{h} - \boldsymbol{v} \right\|^2$$

Textured image



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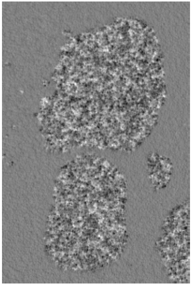
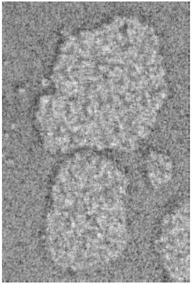
Textured image



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$

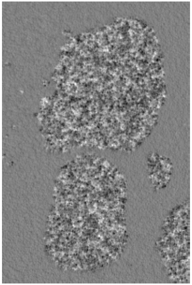


Direct punctual estimation

Linear regression $\underbrace{\mathbb{E} \log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{expected value}} = \log(a) \underbrace{\bar{\boldsymbol{h}}}_{\text{regularity}} + \underbrace{\bar{\boldsymbol{v}}}_{\propto \log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\| \log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v} \right\|^2$$

Textured image



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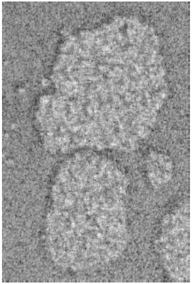
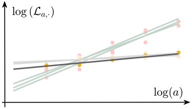


$\longrightarrow$ large estimation variance

Functionals with either free or co-localized contours

$$\sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}}$$

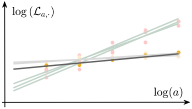
$\rightarrow$ fidelity to the log-linear model

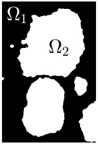


$$\sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to the log-linear model

$\rightarrow$ favors piecewise constancy





$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \;\; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;\; + \;\; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

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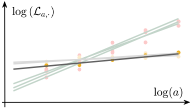

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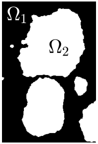
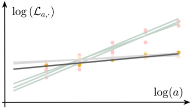


$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model



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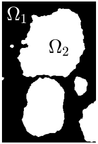
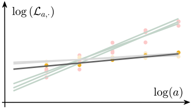


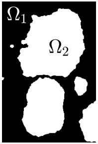
Finite differences $\mathbf{D}_1\boldsymbol{x}$ (horizontal), $\mathbf{D}_2\boldsymbol{x}$ (vertical) at each pixel

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to the log-linear model

$\rightarrow$ favors piecewise constancy





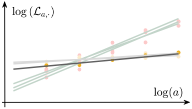
Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

Free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

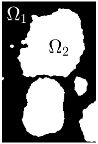
$$\mathcal{Q}_{\mathsf{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\text{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\text{Total Variation}}$$

→ fidelity to the log-linear model



→ favors piecewise constancy



Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

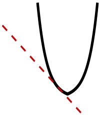
Free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

$$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

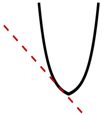
Co-localized: $\boldsymbol{h}$, $\boldsymbol{v}$ are **concomitantly** piecewise constant

$$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \|[\alpha\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}]\|_{2,1}$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



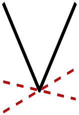
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$ $\qquad \boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth

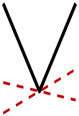


- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$ $\qquad \boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})$
- implicit subgradient descent: proximal point algorithm

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \boldsymbol{u}^n, \; \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \;\Leftrightarrow\; \boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



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$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \boldsymbol{u}^n, \ \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \ \Leftrightarrow \ \boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

- splitting proximal algorithm (Chambolle et al., 2011, *J. Math. Imaging Vis.*)

$$\boldsymbol{y}^{n+1} = \mathrm{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma\mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

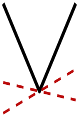
$$\boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}}-\boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau\mathbf{D}^\top\boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

Accelerated algorithm based on strong-convexity

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_{a} \underset{\textbf{Least-Squares}}{\underline{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}} \quad + \quad \lambda \; \underset{\textbf{Total Variation}}{\underline{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}}$$

nonsmooth



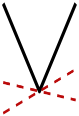
Primal-dual algorithm (Chambolle et al., 2011, *J. Math. Imaging Vis.*)

δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$

Accelerated algorithm based on strong-convexity

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



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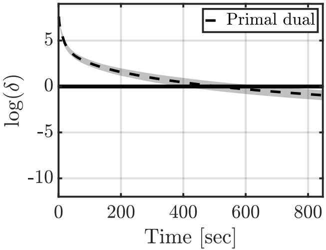
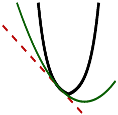
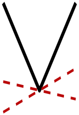


$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underset{\textbf{Least-Squares}}{\underline{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}} \; + \; \lambda \, \underset{\textbf{Total Variation}}{\underline{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}}$$

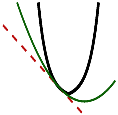


μ-strongly convex

nonsmooth

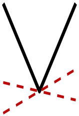


$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



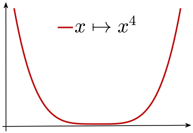
$\boldsymbol{\mu}$-strongly convex

nonsmooth

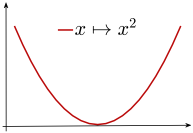


Strong-convexity

- φ $\boldsymbol{\mu}$-strongly convex iff $\varphi - \dfrac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convex

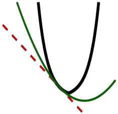


✓ strictly convex
✗ non strongly convex



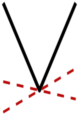
✓ strictly convex
✓ 1-strongly convex

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



$\boldsymbol{\mu}$-strongly convex

nonsmooth



Strong-convexity

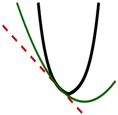
- φ $\boldsymbol{\mu}$-strongly convex iff $\varphi - \frac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convex

Proposition (Pascal et al. , 2018, *Proc. Int. Conf. Image Process.*; Pascal et al., 2021, *Appl. Comput. Harmon. Anal.*)

$$\sum_{a=a_{\min}}^{a_{\max}} \|\log \boldsymbol{\mathcal{L}} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 \text{ is } \boldsymbol{\mu}\text{-strongly convex.}$$

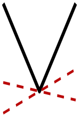
$a_{\min} = 2^1$,	$a_{\max}$	2^2	2^3	2^4	2^5	2^6
$\boldsymbol{\mu}$		0.29	**0.72**	1.20	1.69	2.20

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda\, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



μ-strongly convex

nonsmooth



Accelerated Primal-dual algorithm *(Chambolle et al., 2011, J. Math. Imaging Vis.)*

for $n = 0, 1, \ldots$ $\qquad \boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})$

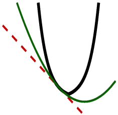
$$\boldsymbol{y}^{n+1} = \mathrm{prox}_{\sigma_n(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma_n \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

$$\boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau_n\|\boldsymbol{\mathcal{L}}-\boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau_n \mathbf{D}^\top \boldsymbol{y}^{n+1}\right)$$

$$\theta_n = \sqrt{1 + 2\mu\tau_n}, \quad \tau_{n+1} = \tau_n/\theta_n, \quad \sigma_{n+1} = \theta_n\sigma_n$$

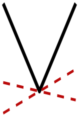
$$\bar{\boldsymbol{x}}^{n+1} = \boldsymbol{x}^{n+1} + \theta_n^{-1}\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



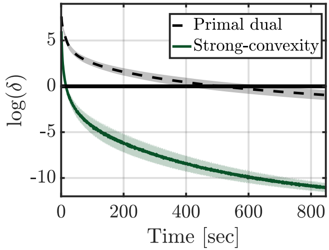
μ-strongly convex

nonsmooth



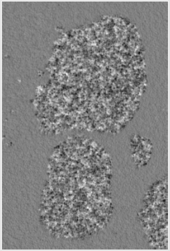
Accelerated Primal-dual algorithm *(Chambolle et al., 2011, J. Math. Imaging Vis.)*

δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \xrightarrow[n\to\infty]{} 0$



$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

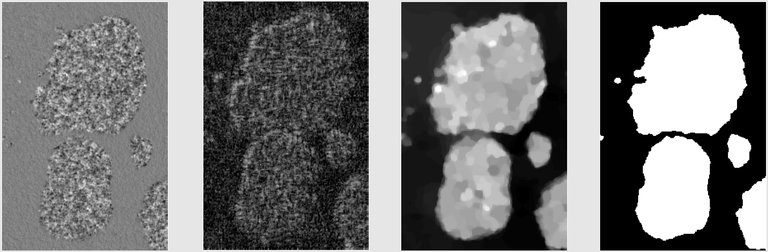
Textured image | Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$



Segmentation *via* iterated thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \ + \ \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image	Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$	Co-localized contours $\widehat{\boldsymbol{h}}^{\mathrm{C}}$	Threshold estimate† $T\widehat{\boldsymbol{h}}^{\mathrm{C}}$



†(Cai et al., 2013, *J. Sci. Comput.*)

Threshold-ROF on $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

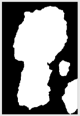
(Nelson et al., 2016, *IEEE Trans. Image Process.*; Pustelnik et al., 2016, *IEEE Trans. Comput. Imaging*)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Lin. reg.

ROF

Threshold

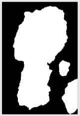


Only based on regularity $\boldsymbol{h}$.

Threshold-ROF on $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

(Nelson et al., 2016, *IEEE Trans. Image Process.*; Pustelnik et al., 2016, *IEEE Trans. Comput. Imaging*)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \; \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

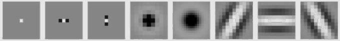
Lin. reg.	ROF	Threshold
		

Only based on regularity $\boldsymbol{h}$.

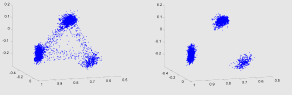
Factorization based segmentation†

(Yuan et al., 2015, *IEEE Trans. Image Process.*)

(i) local histograms



(ii) matrix factorization



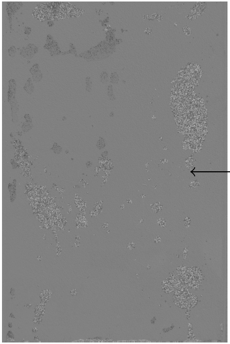
(a) (b)

Fig. 2. Scatterplot of features in subspace. (a) Scatterplot of features projected onto the 3-d subspace. (b) Scatterplot after removing features with high edgeness.

† `https://sites.google.com/site/factorizationsegmentation/`

Multiphase flow through porous media

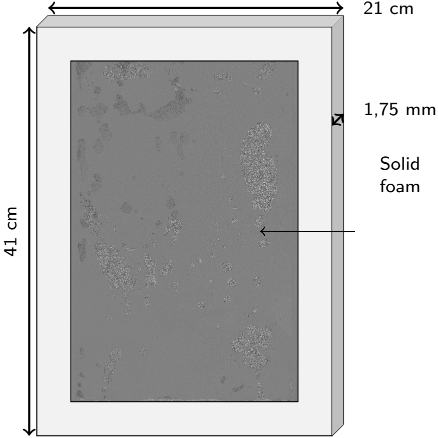
Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



Solid foam

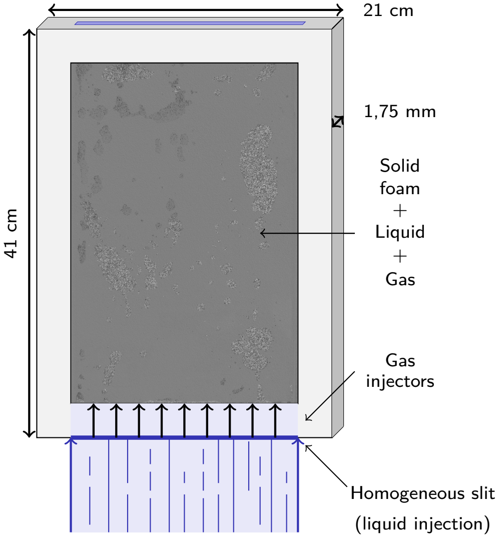
Multiphase flow through porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)

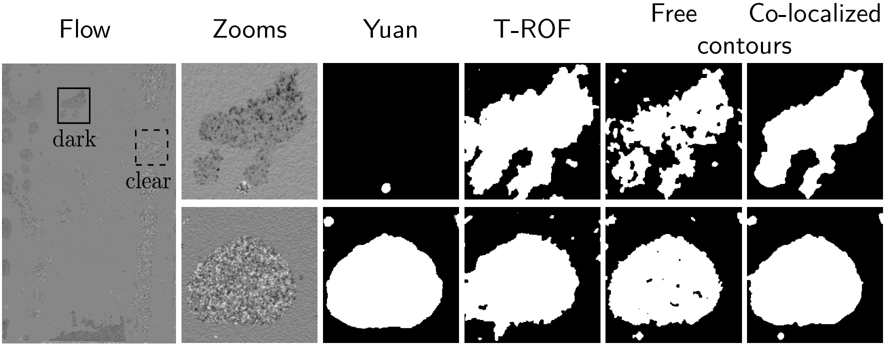


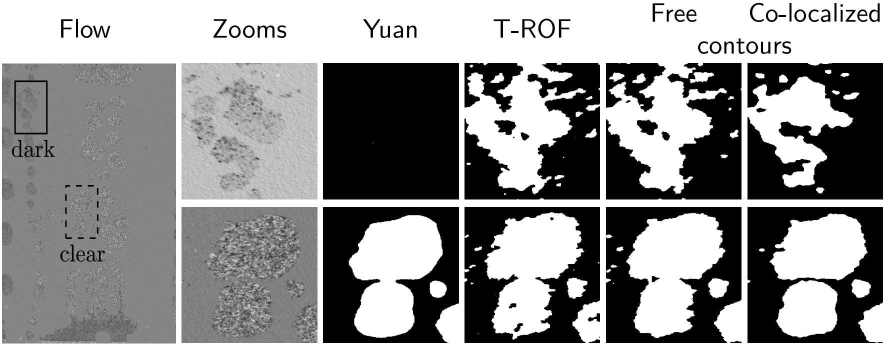
Multiphase flow through porous media

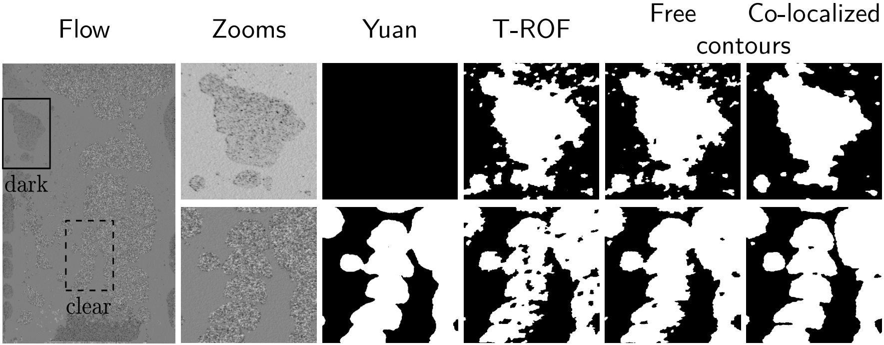
Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



- → 1600 × 1100 pixels
- → video: ∼ 1000 images
- → phase diagram: ∼ 10 flow rates

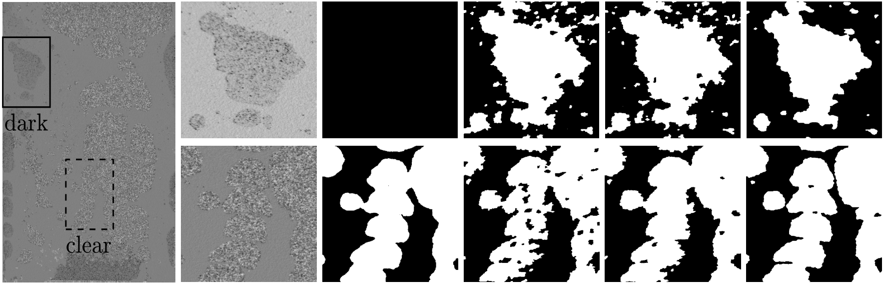

Flow
Zooms
Yuan
T-ROF
Free
Co-localized
contours
dark
clear


Flow
Zooms
Yuan
T-ROF
Free
Co-localized
contours
dark
clear


Flow
Zooms
Yuan
T-ROF
Free
Co-localized
contours
dark
clear

High activity: $Q_{\mathrm{G}} = 1200\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$

Flow	Zooms	Yuan	T-ROF	Free contours	Co-localized contours



Computational load		1s	12s	700s	**2100s**

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \operatorname*{argmin}_{\boldsymbol{h},\boldsymbol{v}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

$(\lambda, \alpha) = (0, 0)$

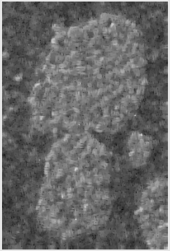
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

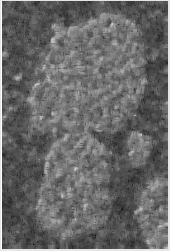
$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha)=\underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}}\sum_{a}\|\log\boldsymbol{\mathcal{L}}_{a,\cdot}-\log(a)\boldsymbol{h}-\boldsymbol{v}\|^2+\lambda\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

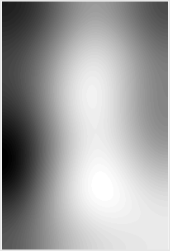
$(\lambda,\alpha)=(0,0)$

$(\lambda,\alpha)=(0.5,0.5)$



too small

$(\lambda,\alpha)=(500,500)$

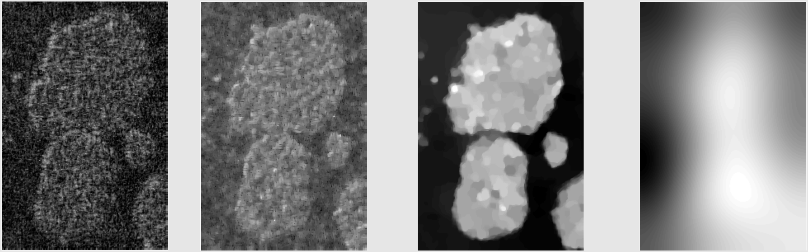


too large

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$ | $(\lambda, \alpha) = (0.5, 0.5)$ | $(\lambda^\dagger, \alpha^\dagger) = (11.5, 0.8)$ | $(\lambda, \alpha) = (500, 500)$



too small | optimal | too large

What *optimal* means? How to determine $\lambda^\dagger$ and $\alpha^\dagger$?

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\mathcal{L}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$

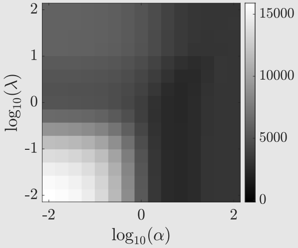
Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



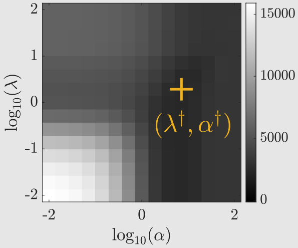
Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$

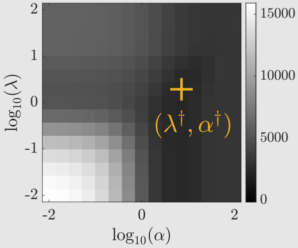


$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

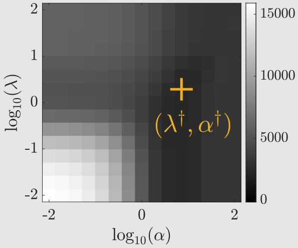
?

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

?

Stein Unbiased Risk Estimate
(SURE)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \, \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

$$\textbf{Ex.} \quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Quadratic error $R(\lambda) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \stackrel{?}{=} \mathbb{E}_{\boldsymbol{\zeta}} \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ unknown

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

$$\textbf{Ex.} \quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \operatorname*{argmin}_{\boldsymbol{x}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Quadratic error $R(\lambda) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \overset{?}{=} \mathbb{E}_{\boldsymbol{\zeta}} \widehat{R}(\boldsymbol{y}; \lambda)$ $\quad$ $\bar{\boldsymbol{x}}$ unknown

Theorem (Stein, 1981, *Ann. Stat.*)

Let $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\boldsymbol{\zeta} \mapsto \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \boldsymbol{\zeta} \rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$.

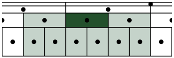
$$\widehat{R}(\boldsymbol{y}; \lambda) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\rho^2 \operatorname{tr}\left(\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)\right) - \rho^2 P$$
$$\Longrightarrow R(\lambda) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{y}; \lambda)].$$

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\quad$ $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$ $\quad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

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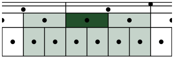

$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

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Theorem (Pascal et al., 2021, *J. Math. Imaging Vis.*)

Let $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ be an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\boldsymbol{\zeta} \mapsto \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \mathbf{A}\boldsymbol{\zeta}\rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$.

$$\widehat{R}(\boldsymbol{\Lambda}) \triangleq \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right) - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

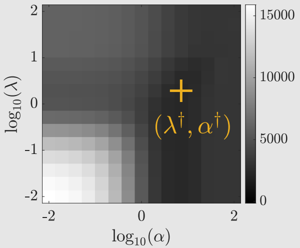
$$\Longrightarrow R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{\Lambda})].$$

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \operatorname*{argmin}_{\boldsymbol{h},\boldsymbol{v}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \left\| \log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v} \right\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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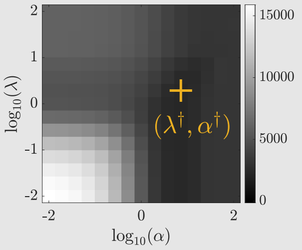

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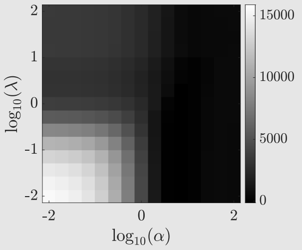

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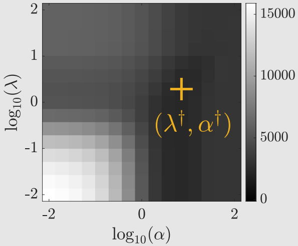

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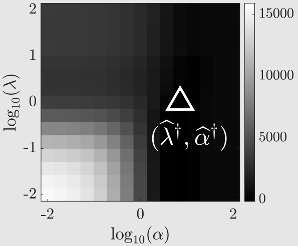
$$\widehat{R}_{\nu, \boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$



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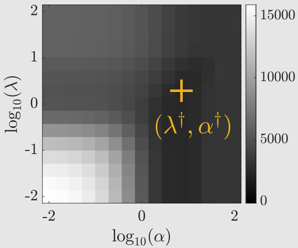

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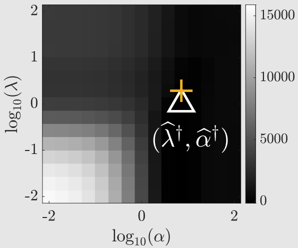

Generalized Finite Difference Monte Carlo SURE

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) =$$
$$\|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

Generalized Finite Difference Monte Carlo SUGAR

$$\begin{aligned}\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) &= 2\left(\mathbf{A}\boldsymbol{\Phi}\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)^\top \mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right) \\ &\quad + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\left(\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle\end{aligned}$$

Theorem *(Pascal et al., 2021, J. Math. Imaging Vis.)*

Let $(\boldsymbol{y};\boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})$ be an estimator of $\bar{\boldsymbol{x}}$

- uniformly-Lipschitz continuous w.r.t. $\boldsymbol{y}$
- such that $\forall \boldsymbol{\Lambda} \in \mathbb{R}^L$, $\widehat{\boldsymbol{x}}(\mathbf{0}_P;\boldsymbol{\Lambda}) = \mathbf{0}_N$,
- uniformly L-Lipschitz continuous w.r.t. $\boldsymbol{\Lambda}$, L independently of $\boldsymbol{y}$. Then

$$\partial_{\boldsymbol{\Lambda}} R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \lim_{\nu\to 0}\mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}}\left[\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}})\right]$$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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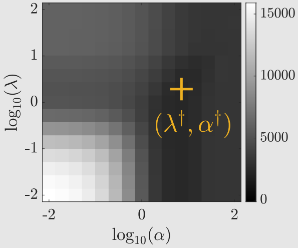
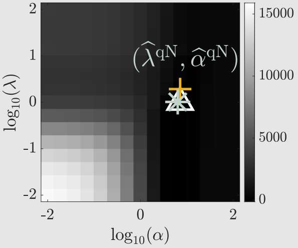


$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

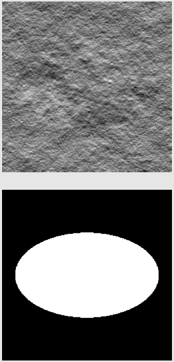


L-BFGS-B quasi-Newton algorithm: $\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$ and $\partial_{\boldsymbol{\Lambda}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}})$

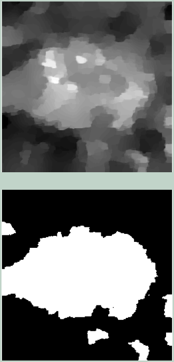
Automated selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

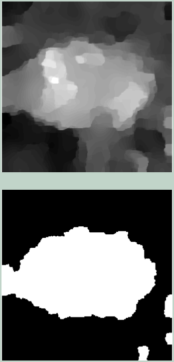
Example



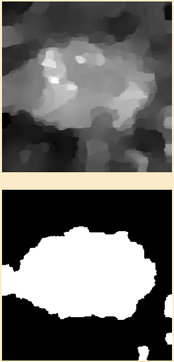
$\widehat{\boldsymbol{h}}^{\mathrm{F}}(\boldsymbol{\mathcal{L}}; \lambda^\dagger, \alpha^\dagger)$ (grid)



$\widehat{\boldsymbol{h}}^{\mathrm{F}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^\dagger, \widehat{\alpha}^\dagger)$ (grid)



$\widehat{\boldsymbol{h}}^{\mathrm{F}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\mathrm{qN}}, \widehat{\alpha}^{\mathrm{qN}})$ (quasi-Newton)

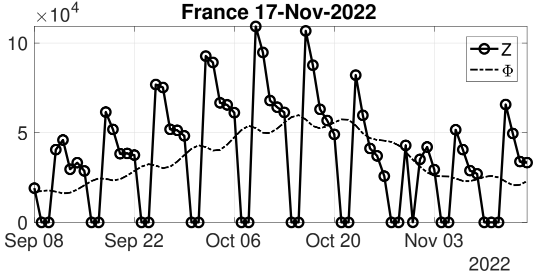


225 calls of the estimator over the grid v.s. **40** for quasi-Newton

Time series analysis:

Epidemiological indicator estimation

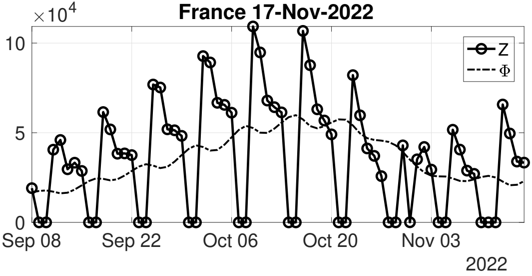
Counts of daily new infections



data from National Health Agencies collected by Johns Hopkins University

$\Longrightarrow$ number of cases not informative enough: need to capture the **dynamics**

Counts of daily new infections



data from National Health Agencies collected by Johns Hopkins University

$\Longrightarrow$ number of cases not informative enough: need to capture the **dynamics**

Design adapted counter measures and evaluate their effectiveness

$\longrightarrow$ efficient monitoring tools *epidemiological model,*

$\longrightarrow$ robust to low quality of the data *managing erroneous counts.*

Reproduction number in Cori model

"averaged number of secondary cases generated by a typical infectious individual"

(Cori et al., 2013, *Am. Journal of Epidemiology*; Liu et al., 2018, *PNAS*)

Pandemic study: modeling at the service of monitoring

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Interpretation: at day t

$\mathrm{R}_t > 1$ the virus propagates at exponential speed,

$\mathrm{R}_t < 1$ the epidemic shrinks with an exponential decay,

$\mathrm{R}_t = 1$ the epidemic is stable.

$\Longrightarrow$ one single indicator accounting for the overall pandemic mechanism

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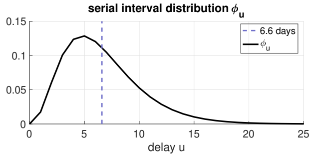
Principle: Z_t new infections at day t

$$\mathbb{E}\left[\mathrm{Z}_t\right] = \mathrm{R}_t \Phi_t, \quad \Phi_t = \sum_{u=1}^{\tau_\Phi} \phi_u \mathrm{Z}_{t-u}$$



with Φ_t global "infectiousness" in the population

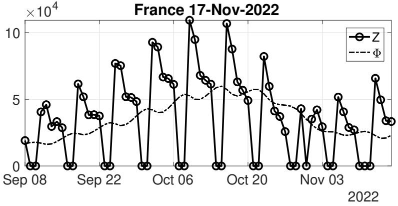
$\{\phi_u\}_{u=1}^{\tau_\Phi}$ distribution of delay between onset of symptoms in primary and secondary cases

Gamma distribution truncated at 25 days, of mean 6.6 days and standard deviation 3.5 days

Data: daily counts $\mathbf{Z} = (Z_1, \ldots, Z_T)$

Model: Poisson distribution

$$\mathbb{P}(\mathsf{Z}_t | \mathbf{Z}_{t-\tau_\Phi:t-1}, \mathsf{R}_t) = \frac{(\mathsf{R}_t \Phi_t)^{\mathsf{Z}_t} \mathrm{e}^{-\mathsf{R}_t \Phi_t}}{\mathsf{Z}_t!}$$



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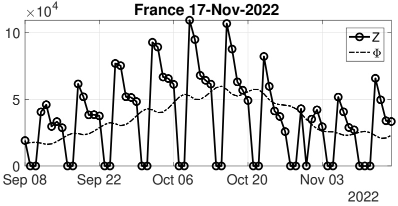
$\Longrightarrow$ At day t: $\quad \mathsf{Z}_t \sim \mathcal{P}(\mathsf{R}_t \Phi_t)$



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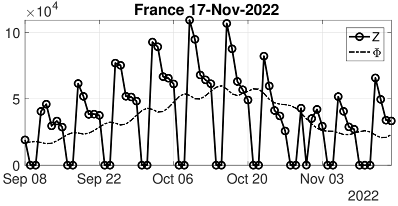

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$\Longrightarrow$ At day t: $\quad \mathrm{Z}_t \sim \mathcal{P}(\mathrm{R}_t \Phi_t)$

Inverse problem formalism:

$$\mathbf{Z} \sim \mathcal{P}\left(\mathbf{\Phi} \mathbf{R}\right)$$

- $\mathbf{Z} \in \mathbb{N}^T$: reported infection counts,
- $\mathbf{R} = (\mathrm{R}_1, \ldots, \mathrm{R}_T) \in \mathbb{R}_+^T$: daily unknown reproduction number,
- $\mathbf{\Phi} = \mathrm{diag}\left(\Phi_1, \ldots, \Phi_T\right)$: linear operator,
- $\mathcal{P}$: data-dependent Poisson noise.

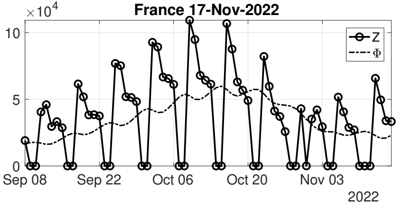
$$\Longrightarrow \mathcal{D}(\mathbf{Z}, \mathbf{\Phi R}) = -\log \mathbb{P}(\mathbf{Z} | \mathbf{R})$$

Pandemic study: modeling at the service of monitoring

Data: daily counts $\mathbf{Z} = (Z_1, \ldots, Z_T)$

Model: Poisson distribution

$$\mathbb{P}(Z_t | \mathbf{Z}_{t-\tau_\Phi:t-1}, R_t) = \frac{(R_t \Phi_t)^{Z_t} e^{-R_t \Phi_t}}{Z_t!}$$



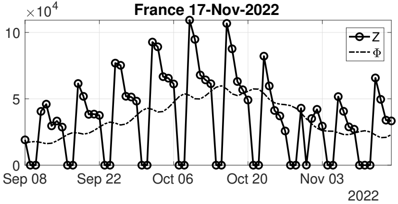
Maximum Likelihood Estimate (MLE)

$$\begin{aligned}
&\ln\left(\mathbb{P}(Z_t | \mathbf{Z}_{t-\tau_\Phi:t-1}, R_t)\right) \\
&= Z_t \ln(R_t \Phi_t) - R_t \Phi_t - \ln(Z_t!) \\
&\underset{Z_t \gg 1}{\simeq} Z_t \ln(R_t \Phi_t) - R_t \Phi_t - Z_t \ln(Z_t) + Z_t \\
&\underset{\text{(def.)}}{=} -d_{KL}(Z_t | R_t \Phi_t) \quad \text{(Kullback-Leibler)}
\end{aligned}$$

Data: daily counts $\mathbf{Z} = (Z_1, \ldots, Z_T)$

Model: Poisson distribution

$$\mathbb{P}(\mathsf{Z}_t | \mathbf{Z}_{t-\tau_\Phi:t-1}, \mathsf{R}_t) = \frac{(\mathsf{R}_t \Phi_t)^{\mathsf{Z}_t} \mathrm{e}^{-\mathsf{R}_t \Phi_t}}{\mathsf{Z}_t!}$$



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&\underset{\text{(def.)}}{=} -\mathrm{d}_{\mathrm{KL}}(\mathsf{Z}_t | \mathsf{R}_t \Phi_t) \quad \text{(Kullback-Leibler)}
\end{aligned}$$

$$\Longrightarrow \widehat{\mathsf{R}}_t^{\mathrm{MLE}} = \mathsf{Z}_t / \Phi_t = \mathsf{Z}_t / \sum_{u=1}^{\tau_\Phi} \phi_u \mathsf{Z}_{t-u}$$

ratio of moving averages

Pandemic study: modeling at the service of monitoring

Data: daily counts $\mathbf{Z} = (Z_1, \ldots, Z_T)$

Model: Poisson distribution

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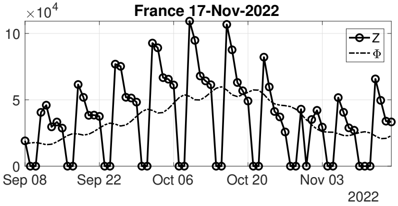

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$$\begin{aligned} &\ln\left(\mathbb{P}(\mathsf{Z}_t | \mathbf{Z}_{t-\tau_\Phi:t-1}, \mathsf{R}_t)\right) \\ &= \mathsf{Z}_t \ln(\mathsf{R}_t \Phi_t) - \mathsf{R}_t \Phi_t - \ln(\mathsf{Z}_t!) \\ &\underset{\mathsf{Z}_t \gg 1}{\simeq} \mathsf{Z}_t \ln(\mathsf{R}_t \Phi_t) - \mathsf{R}_t \Phi_t - \mathsf{Z}_t \ln(\mathsf{Z}_t) + \mathsf{Z}_t \\ &\underset{\text{(def.)}}{=} -d_{\mathrm{KL}}(\mathsf{Z}_t | \mathsf{R}_t \Phi_t) \quad \text{(Kullback-Leibler)} \end{aligned}$$

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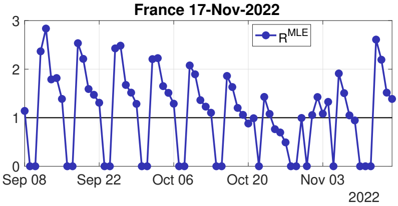

- huge variability along time/ no local trend
- not robust to pseudo-periodicity/ misreported counts

Penalized likelihood: regularization through nonlinear filtering

$$\widehat{\mathbf{R}}^{\mathsf{PKL}} = \underset{\mathbf{R}\in\mathbb{R}_+^T}{\operatorname{argmin}} \sum_{t=1}^{T} \mathsf{d}_{\mathsf{KL}}\left(\mathsf{Z}_t \,|\, \mathsf{R}_t\Phi_t\right) + \lambda\mathcal{R}(\mathbf{R}) \quad \text{(penalized Kullback-Leibler)}$$

with $\mathcal{R}(\mathbf{R})$ favoring some temporal regularity (Abry et al., 2020, *PLOSOne*)

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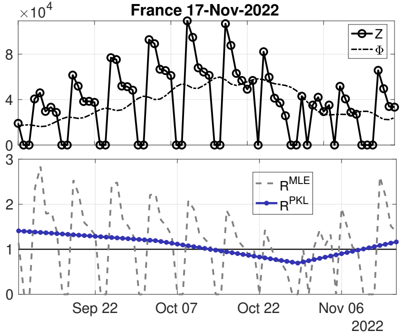
$$\mathcal{R}(\mathbf{R}) = \|\mathbf{D}_2\mathbf{R}\|_1$$

$$(\mathbf{D}_2\mathbf{R})_t = \mathrm{R}_{t+1} - 2\mathrm{R}_t + \mathrm{R}_{t-1}$$

2nd order derivative & ℓ_1-norm

$\Longrightarrow$ piecewise linearity



captures global **trend**, **smooth** temporal behavior, **no pseudo-oscillations**

Penalized Kullback-Leibler estimator:

$$\widehat{\mathbf{R}}(\mathbf{Z};\lambda) = \underset{\mathbf{R}\in\mathbb{R}_+^T}{\mathrm{argmin}} \sum_{t=1}^{T} \mathrm{d}_{\mathrm{KL}}\left(\mathrm{Z}_t \,|\, \mathrm{R}_t \Phi_t\right) + \lambda \|\mathbf{D}_2 \mathbf{R}\|_1$$

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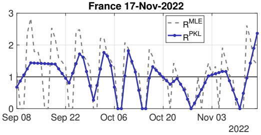
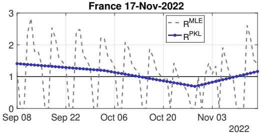
Fine tuning of the regularization parameter:

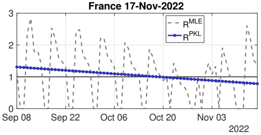
$\lambda = 3.5$



$\boldsymbol{\lambda^\dagger = 50}$



$\lambda = 250$



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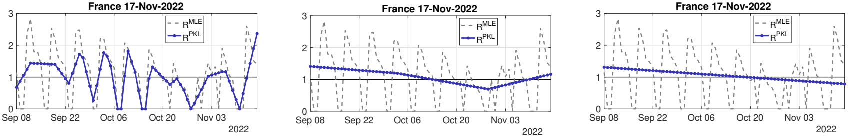

Data-driven oracle minimization

$$\lambda^\dagger \in \operatorname*{Argmin}_{\lambda\in\Lambda} \mathcal{O}(\mathbf{Z};\lambda)$$

$\Longrightarrow$ **Goal:** $\mathcal{O}$ data-driven proxy for $\|\widehat{\mathbf{R}}(\mathbf{Z};\lambda) - \overline{\mathbf{R}}\|_2^2$

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Reminder: $\mathbf{Z} \sim \mathcal{P}\left(\boldsymbol{\Phi}\mathbf{R}\right)$

- $\mathbf{Z} \in \mathbb{N}^T$: reported infection counts,
- $\mathbf{R} = (\mathsf{R}_1, \ldots, \mathsf{R}_T) \in \mathbb{R}_+^T$: daily unknown reproduction number,
- $\boldsymbol{\Phi} = \mathrm{diag}\left(\Phi_1, \ldots, \Phi_T\right)$: linear operator,
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- Poisson model: Stein lemma does not apply (Eldar, 2008, *IEEE Trans. Signal Process.*; Luisier et al., 2010, *IEEE Trans. Image Process.*; Li et al., 2017, *IEEE Trans. Image Process.*)

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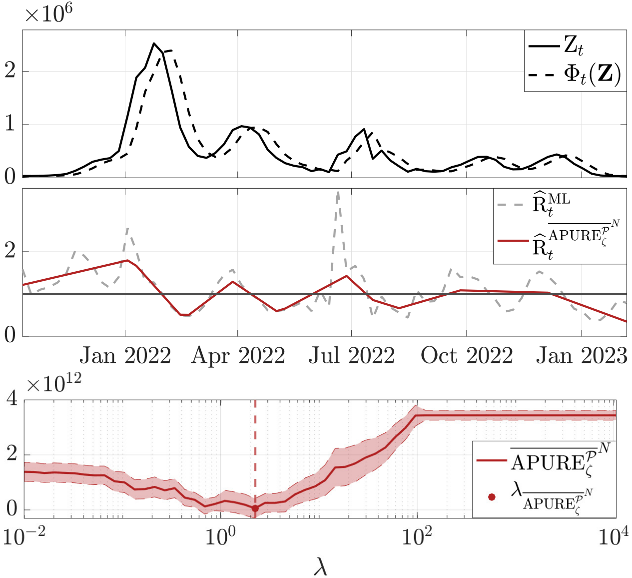
Autoregressive **P**oisson **U**nbiased **R**isk **E**stimate (APURE)



(Teaser, preprint available soon)

Inverse problem $\boldsymbol{y} \sim \mathcal{B}\left(\boldsymbol{\Phi}\overline{\boldsymbol{x}}\right)$

$$\lambda^\dagger \in \underset{\lambda \in \Lambda}{\mathrm{Argmin}}\, \mathcal{O}(\boldsymbol{y};\lambda), \quad \text{for} \quad \widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) \in \underset{\boldsymbol{x}\in\mathbb{R}^N}{\mathrm{Argmin}}\, \mathcal{D}(\boldsymbol{y},\boldsymbol{\Phi}\boldsymbol{x}) + \lambda\mathcal{R}(\boldsymbol{x})$$

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Data-driven parameter selection

$\Longrightarrow$ $\mathcal{O}$: Unbiased Risk Estimate (Stein, 1981, *Ann. Stat.*; Eldar, 2008, *IEEE Trans. Signal Process.*; Luisier et al., 2010, *IEEE Trans. Image Process.*; Deledalle et al., 2014, *SIAM J. Imaging Sci.*; Pascal et al., 2021, *J. Math. Imaging Vis.*; Lucas et al., 2023, *Signal, Image Video Process.*)

- Texture segmentation: additive correlated Gaussian noise;
- Epidemic monitoring: driven autoregressive data-dependent Poisson noise.

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Extensions and perspectives

- Efficient and robust scheme for nonconvex $\mathcal{R}(\boldsymbol{x})$;
- Generalization to other noise models: speckle noise in medical imaging;
- Unsupervised learning for $\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \mathbf{NN}_{\boldsymbol{\theta}}(\boldsymbol{y})$ with loss $\mathcal{O}(\boldsymbol{y}; \boldsymbol{\theta})$.