How fractal texture segmentation turns out to be a strongly convex optimization problem ?[†].

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Laboratoire CRIStAL, Lille
March, 3rd 2020

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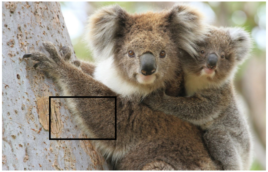
[†] Supported by Defi Imag'in SIROCCO and ANR-16-CE33-0020 MultiFracs, France.

Describing and interpreting real-world images

Texture as a discriminating feature



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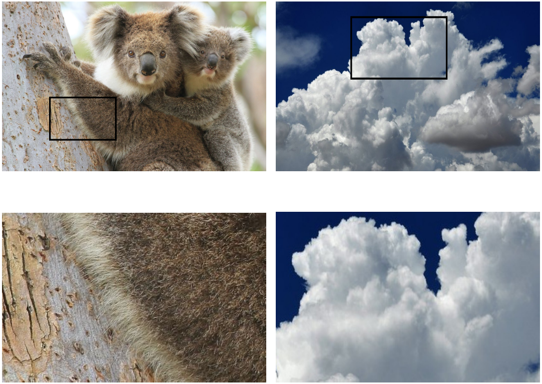

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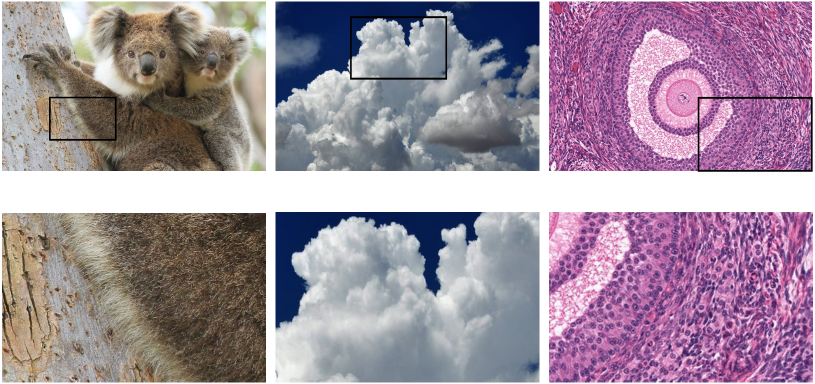

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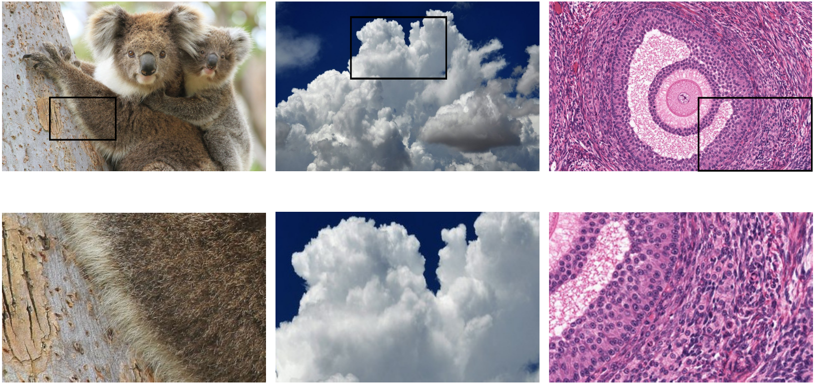

Texture is of utmost importance in complex computer vision tasks.

Describing and interpreting real-world images

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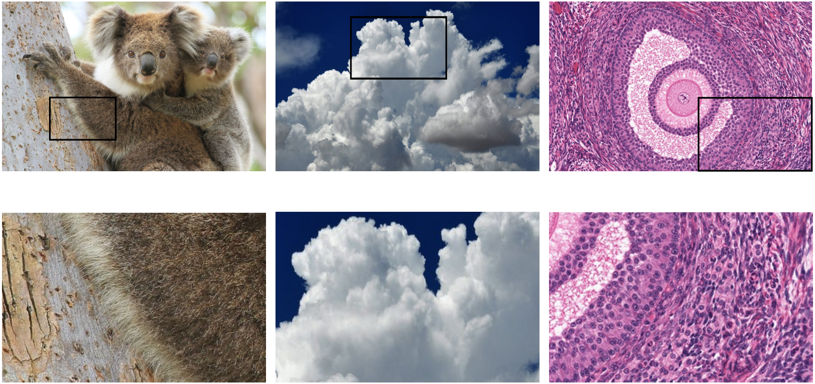

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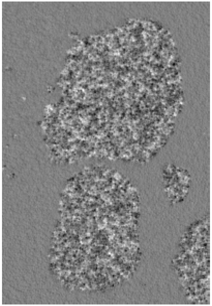
fractal segmentation

Formulation of the texture segmentation problem



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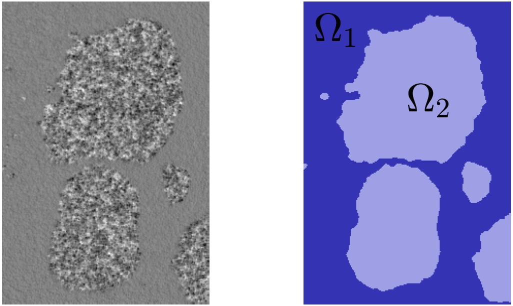


Purpose: obtaining a partition of the image into κ homogeneous regions

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_\kappa$$

Ω_k: pixels corresponding to texture k.

Outline – Fractal texture segmentation

1. Fractal texture model

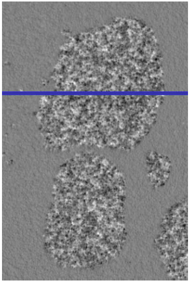
2. Attributes estimation and segmentation

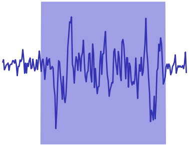
3. Multiphasic flow segmentation

4. Regularization parameters tuning

Texture's attributes definition

Piecewise monofractal model





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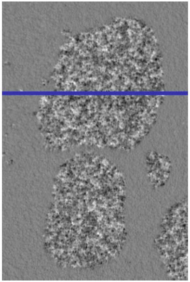
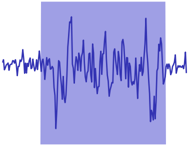
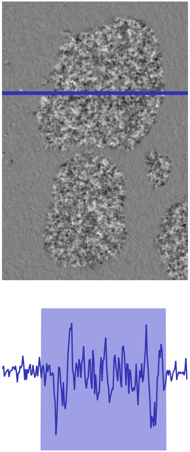




Variance σ^2 *amplitude of variations*

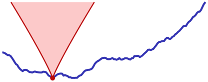
Texture's attributes definition

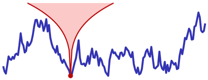
Piecewise monofractal model



Variance σ^2 — *amplitude of variations*

Local regularity h — *scale-free behavior*





$h(x) \equiv h_1 = 0.9$ $\qquad$ $h(x) \equiv h_2 = 0.3$

Fit local behavior with power law functions

$$|f(x) - f(y)| \leq C|x - y|^{h(x)}$$

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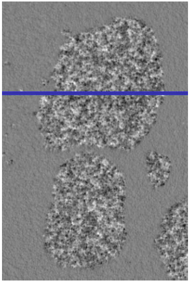
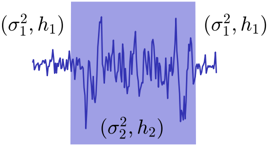
Piecewise monofractal model





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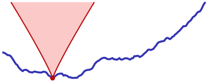

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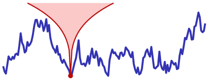

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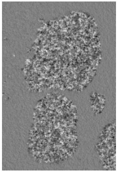
Segmentation requires <u>local</u> measurement of σ^2 and h.

Textured image

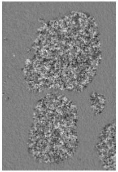


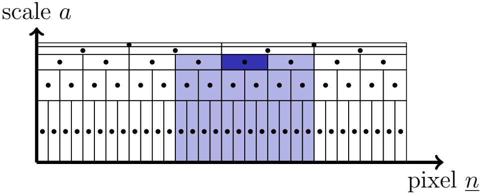
Texture's attributes estimation

Multiscale analysis

Textured image

Local supremum of wavelet coefficients: *leaders* $\mathcal{L}_{a,\cdot}$





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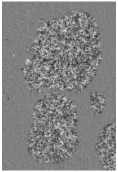
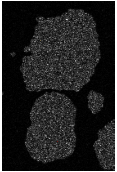


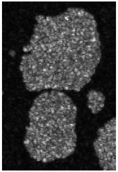
Local supremum of wavelet coefficients: *leaders* $\mathcal{L}_{a,\cdot}$

Scale

$a = 2^1$

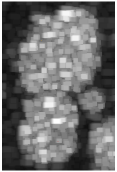


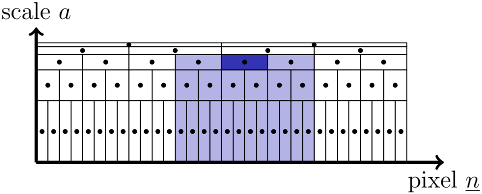
$a = 2^2$



...

$a = 2^5$





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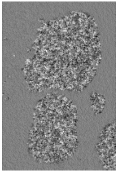

Local supremum of wavelet coefficients: *leaders* $\mathcal{L}_{a,\cdot}$

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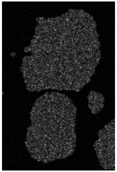

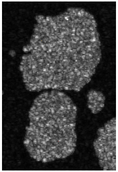

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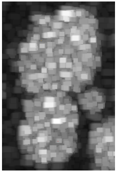

Log-log linear behavior

$$\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \underbrace{\boldsymbol{v}}_{\substack{\sim\log(\boldsymbol{\sigma}^2)\\ \text{(variance)}}} + \log(a) \underbrace{\boldsymbol{h}}_{\textit{regularity}}$$

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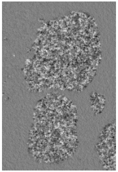

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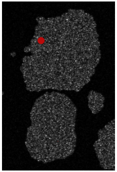

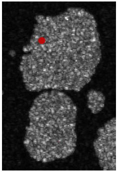

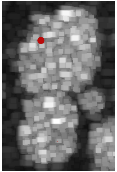

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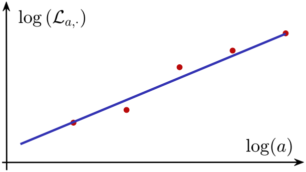

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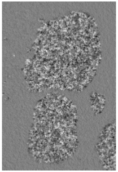

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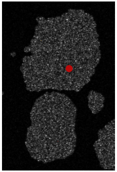

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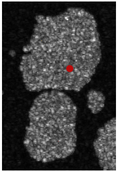

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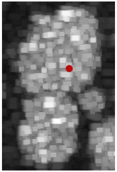

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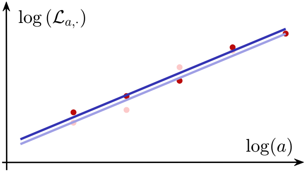
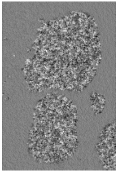


Textured image



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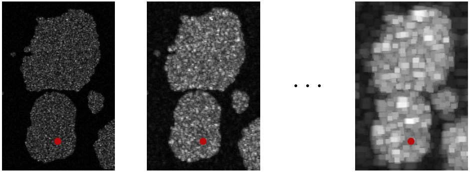

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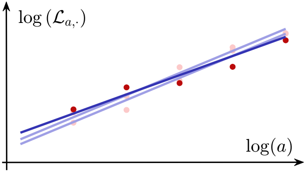

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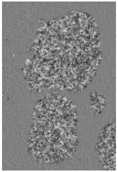

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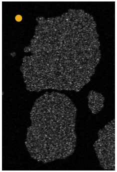

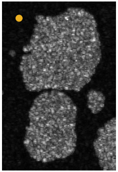

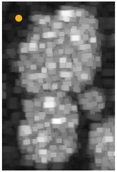

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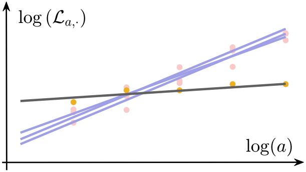

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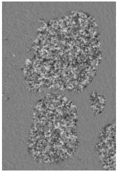

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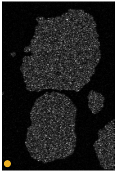

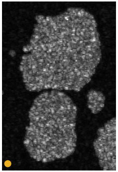

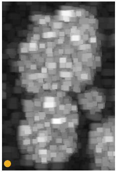

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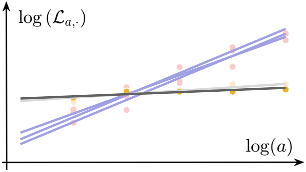
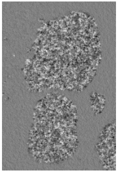


Texture's attributes estimation

Multiscale analysis

Textured image



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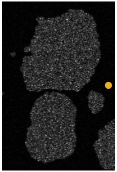

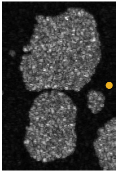

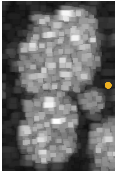

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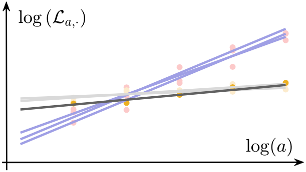

Texture's attributes estimation

Pointwise linear regression

$$\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \underbrace{\boldsymbol{v}}_{\sim \log(\boldsymbol{\sigma}^2)} + \log(a) \underbrace{\boldsymbol{h}}_{regularity}$$

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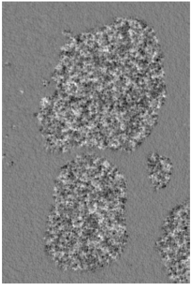
Pointwise linear regression

$$\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \underbrace{\boldsymbol{v}}_{\sim\log(\boldsymbol{\sigma}^2)} + \log(a)\underbrace{\boldsymbol{h}}_{\textit{regularity}}$$

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \boldsymbol{v} - \log(a)\boldsymbol{h}\right\|^2$$

Textured image



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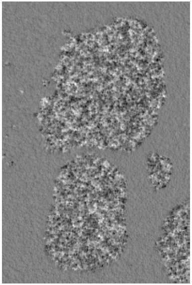
Pointwise linear regression

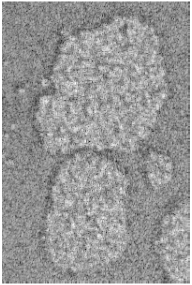
$$\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \underset{\sim\log(\boldsymbol{\sigma}^2)}{\underline{\boldsymbol{v}}} + \log(a) \underset{\textit{regularity}}{\underline{\boldsymbol{h}}}$$

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_{a} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \boldsymbol{v} - \log(a)\boldsymbol{h}\right\|^2$$

Textured image



Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

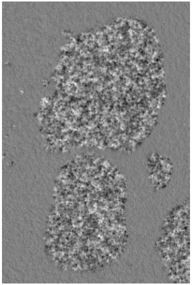
Texture's attributes estimation

Pointwise linear regression

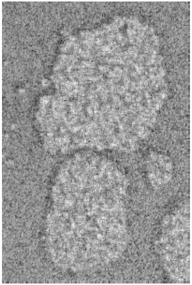
$$\underbrace{\mathbb{E}\log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{expected value}} \simeq \underbrace{\boldsymbol{v}}_{\sim\log(\boldsymbol{\sigma}^2)} + \log(a)\underbrace{\boldsymbol{h}}_{\textit{regularity}}$$

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \boldsymbol{v} - \log(a)\boldsymbol{h}\right\|^2$$

Textured image



Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Pointwise linear regression is an estimation from **one** sample!

Outline – Fractal texture segmentation

1. Fractal texture model

$$\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \underbrace{\boldsymbol{v}}_{\sim\log(\boldsymbol{\sigma}^2)} + \log(a) \underbrace{\boldsymbol{h}}_{regularity}$$

2. Attributes estimation and segmentation

3. Multiphasic flow segmentation

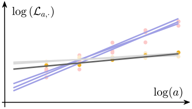
4. Regularization parameters tuning

Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\sum_{a} \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}}$$

$\rightarrow$ fidelity to log-linear model



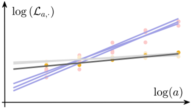
Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to log-linear model

$\rightarrow$ enforce piecewise constancy



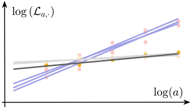
Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \; \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to log-linear model

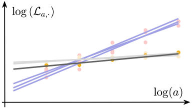
$\rightarrow$ enforce piecewise constancy



Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\substack{\textbf{Least-Squares} \\ \rightarrow \text{ fidelity to log-linear model}}} \quad + \quad \underbrace{\lambda\, \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\substack{\textbf{Total Variation} \\ \rightarrow \text{ enforce piecewise constancy}}}$$



joint: $\boldsymbol{v}$, $\boldsymbol{h}$ are **independently** piecewise constant

coupled: $\boldsymbol{v}$, $\boldsymbol{h}$ are **concomitantly** piecewise constant

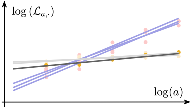
Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \; \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \, \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to log-linear model

→ enforce piecewise constancy



Discrete differences $\mathbf{H}\boldsymbol{x}$ (horizontal), $\mathbf{V}\boldsymbol{x}$ (vertical) at each pixel

joint: $\boldsymbol{v}$, $\boldsymbol{h}$ are **independently** piecewise constant

$$\mathcal{N}_{\mathrm{J}}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha) = \left(\sum_{\text{pixels}} \sqrt{(\mathbf{H}\boldsymbol{v})^2 + (\mathbf{V}\boldsymbol{v})^2} + \alpha \sum_{\text{pixels}} \sqrt{(\mathbf{H}\boldsymbol{h})^2 + (\mathbf{V}\boldsymbol{h})^2} \right)$$

coupled: $\boldsymbol{v}$, $\boldsymbol{h}$ are **concomitantly** piecewise constant

$$\mathcal{N}_{\mathrm{C}}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha) = \sum_{\text{pixels}} \sqrt{(\mathbf{H}\boldsymbol{v})^2 + (\mathbf{V}\boldsymbol{v})^2 + \alpha^2(\mathbf{H}\boldsymbol{h})^2 + \alpha^2(\mathbf{V}\boldsymbol{h})^2}$$

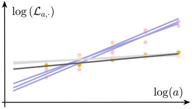
Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} + \underbrace{\lambda\, \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to log-linear model

$\rightarrow$ enforce piecewise constancy



Discrete differences $\mathbf{D}\boldsymbol{x} = [\mathbf{H}\boldsymbol{x}, \mathbf{V}\boldsymbol{x}]$

joint: $\boldsymbol{v}$, $\boldsymbol{h}$ are **independently** piecewise constant

$$\mathcal{N}_{\mathrm{J}}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha) = \|\mathbf{D}\boldsymbol{v}\|_{2,1} + \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

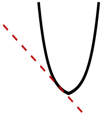
coupled: $\boldsymbol{v}$, $\boldsymbol{h}$ are **concomitantly** piecewise constant

$$\mathcal{N}_{\mathrm{C}}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha) = \|[\mathbf{D}\boldsymbol{v}, \alpha\mathbf{D}\boldsymbol{h}]\|_{2,1}$$

Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

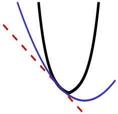


convex

Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \ \sum_a \underbrace{\|\log \mathcal{L}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \ + \ \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$



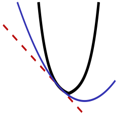
$\boldsymbol{\mu}$-strongly convex

convex

Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$



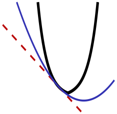
$\boldsymbol{\mu}$-strongly convex — convex

φ is $\boldsymbol{\mu}$-strongly convex iff $\varphi - \frac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ is convex.

Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

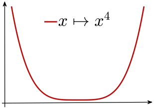
$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \; \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \; + \; \underbrace{\lambda\, \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$



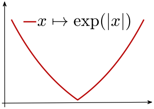
$\boldsymbol{\mu}$-strongly convex

convex

φ is $\boldsymbol{\mu}$-strongly convex iff $\varphi - \frac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ is convex.



✓ strictly convex
✗ not strongly convex

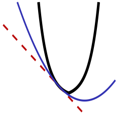


✓ strictly convex
✓ 1-strongly convex

Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$



$\boldsymbol{\mu}$-strongly convex

convex

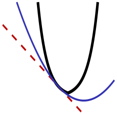
φ is $\boldsymbol{\mu}$-strongly convex iff $\varphi - \dfrac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ is convex.

If φ is $\mathcal{C}^2$ with Hessian $\boldsymbol{H}\varphi$, φ is $\min \mathrm{Sp}(\boldsymbol{H}\varphi)$-strongly convex.

Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \underbrace{\lambda\, \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$



$\boldsymbol{\mu}$-strongly convex

convex

$$\varphi \text{ is } \boldsymbol{\mu}\text{-strongly convex iff } \varphi - \frac{\boldsymbol{\mu}}{2}\|\cdot\|^2 \text{ is convex.}$$

If φ is $\mathcal{C}^2$ with Hessian $\boldsymbol{H}\varphi$, φ is $\min \mathrm{Sp}(\boldsymbol{H}\varphi)$-strongly convex.

$$\mathrm{LS}(\boldsymbol{v},\boldsymbol{h}) = \sum_{a=a_{\min}}^{a_{\max}} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2 = \|\log \boldsymbol{\mathcal{L}} - \mathbf{A}(\boldsymbol{v},\boldsymbol{h})\|^2$$

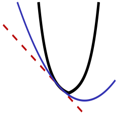
$$\boldsymbol{H}\mathrm{LS} = 2\mathbf{A}^*\mathbf{A}, \quad \text{with} \quad \mathbf{A} : (\boldsymbol{v},\boldsymbol{h}) \mapsto \{\boldsymbol{v} + \log(a)\boldsymbol{h}\}_{a=a_{\min}}^{a_{\max}}$$

$\mathrm{LS}(\boldsymbol{v},\boldsymbol{h})$ is $\boldsymbol{\mu}$-strongly convex, $\boldsymbol{\mu}$ the smallest eigenvalue of $2\mathbf{A}^*\mathbf{A}$

Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

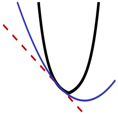


μ-strongly convex

Texture's attributes estimation

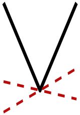
One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \mathcal{L}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$



$\boldsymbol{\mu}$-strongly convex

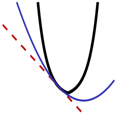
nonsmooth



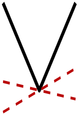
Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \underbrace{\lambda\, \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$



$\boldsymbol{\mu}$-strongly convex nonsmooth



Accelerated primal-dual algorithm (Chambolle, Pock 11')

$$\boldsymbol{x}^n = (\boldsymbol{v}^n, \boldsymbol{h}^n), \quad \boldsymbol{y}^n = (\boldsymbol{u}^n, \boldsymbol{\ell}^n)$$

$$\boldsymbol{y}^{n+1} = \text{prox}_{\sigma_n\|\cdot\|_{2,1}} (\boldsymbol{y}^n + \sigma_n \mathbf{D}\bar{\boldsymbol{x}}^n)$$

$$\boldsymbol{x}^{n+1} = \text{prox}_{\tau_n \text{LS}} \left(\boldsymbol{x}^n - \tau_n \mathbf{D}^* \boldsymbol{y}^{n+1}\right)$$

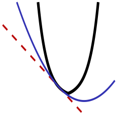
$$\theta_n = \sqrt{1 + 2\boldsymbol{\mu}\tau_n}, \quad \tau_{n+1} = \tau_n/\theta_n, \quad \sigma_{n+1} = \theta_n\sigma_n$$

$$\bar{\boldsymbol{x}}^{n+1} = \boldsymbol{x}^{n+1} + \theta_n^{-1}\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

Texture's attributes estimation

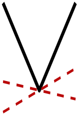
One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$



$\boldsymbol{\mu}$-strongly convex

nonsmooth



Accelerated primal-dual algorithm (Chambolle, Pock 11')

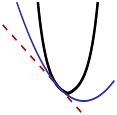
$$\boldsymbol{x}^n = (\boldsymbol{v}^n, \boldsymbol{h}^n), \qquad \boldsymbol{y}^n = (\boldsymbol{u}^n, \boldsymbol{\ell}^n)$$

$$\delta(\boldsymbol{x}^n;\boldsymbol{y}^n) = \mathrm{LS}(\boldsymbol{x}^n) + \mathcal{N}(\mathbf{D}\boldsymbol{x}^n) - (-\mathrm{LS}^*(-\mathbf{D}^*\boldsymbol{y}^n) - \mathcal{N}^*(\boldsymbol{y}^n)) \underset{n\to\infty}{\longrightarrow} 0$$

Texture's attributes estimation

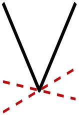
One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda\, \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$



μ-strongly convex

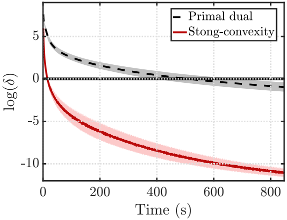
nonsmooth



Accelerated primal-dual algorithm (Chambolle, Pock 11')

$$\boldsymbol{x}^n = (\boldsymbol{v}^n, \boldsymbol{h}^n), \qquad \boldsymbol{y}^n = (\boldsymbol{u}^n, \boldsymbol{\ell}^n)$$

$$\delta(\boldsymbol{x}^n;\boldsymbol{y}^n) = \mathrm{LS}(\boldsymbol{x}^n) + \mathcal{N}(\mathbf{D}\boldsymbol{x}^n) - (-\mathrm{LS}^*(-\mathbf{D}^*\boldsymbol{y}^n) - \mathcal{N}^*(\boldsymbol{y}^n)) \underset{n\to\infty}{\longrightarrow} 0$$

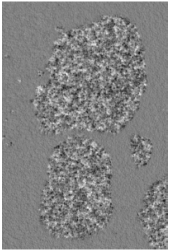


Texture's attributes estimation

Iterated thresholding for segmentation

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_{a} \underbrace{\|\log \mathcal{L}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Textured image

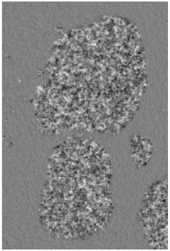


Texture's attributes estimation

Iterated thresholding for segmentation

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \; \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Textured image



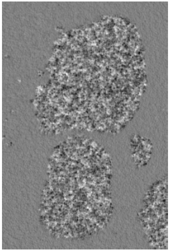
Lin. Reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Texture's attributes estimation

Iterated thresholding for segmentation

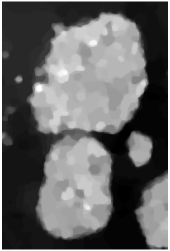
$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \ \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \ + \ \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Textured image



Lin. Reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Coupled $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

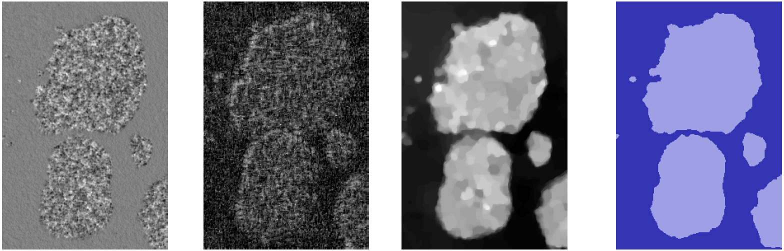


Texture's attributes estimation

Iterated thresholding for segmentation

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Textured image | Lin. Reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Coupled $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ | Thresholded $T\widehat{\boldsymbol{h}}^{\mathrm{C}}$

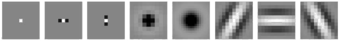


X.Cai, *et al.*, *Multiclass segmentation by iterated ROF thresholding* (2013)

State-of-the-art two-step texture segmentation

Factorization-based segmentation [Yuan *et al.* 15']†

(i) local spectral histograms



(ii) matrix factorization

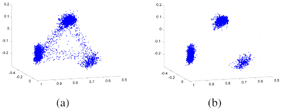


Fig. 2. Scatterplot of features in subspace. (a) Scatterplot of features projected onto the 3-d subspace. (b) Scatterplot after removing features with high edgeness.

† `https://sites.google.com/site/factorizationsegmentation/`

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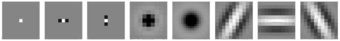

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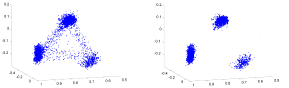


(a) (b)

Fig. 2. Scatterplot of features in subspace. (a) Scatterplot of features projected onto the 3-d subspace. (b) Scatterplot after removing features with high edgeness.

Threshold-ROF on $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ [Pustelnik 16']

$$\min_{\boldsymbol{h}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Lin. Reg.

ROF

Threshold

Based on regularity $\boldsymbol{h}$ only.

† https://sites.google.com/site/factorizationsegmentation/

Outline – Fractal texture segmentation

1. Fractal texture model

$$\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \underbrace{\boldsymbol{v}}_{\sim\log(\boldsymbol{\sigma}^2)} + \log(a) \underbrace{\boldsymbol{h}}_{\textit{regularity}}$$

2. Attributes estimation and segmentation

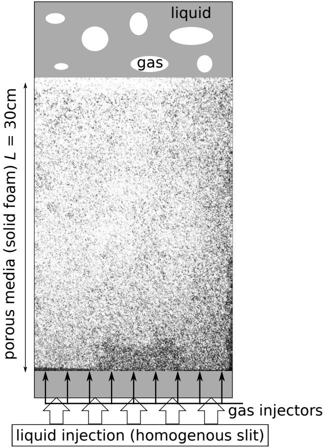
$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

3. Multiphasic flow segmentation

4. Regularization parameters tuning

Multiphasic (quasi 2D) flow in a porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



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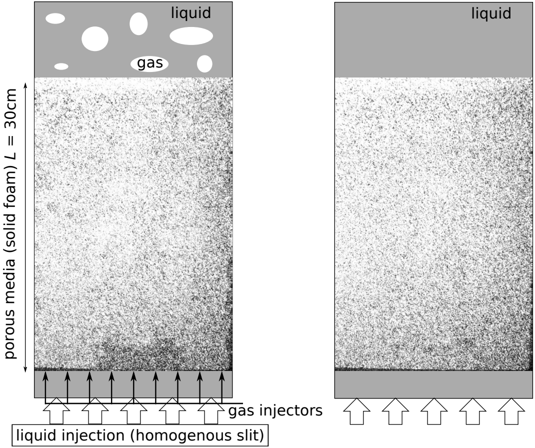

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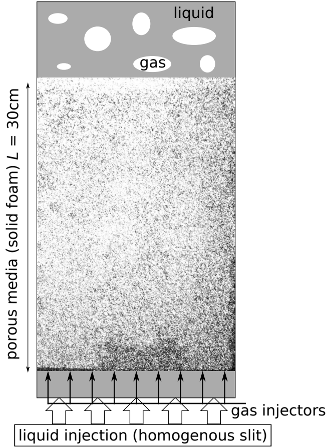
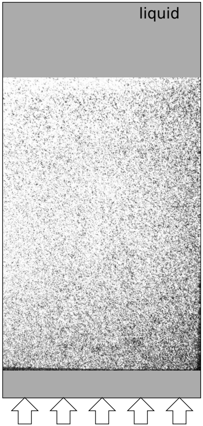
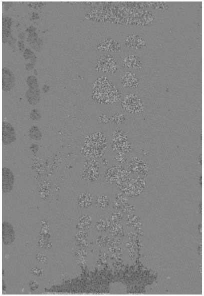




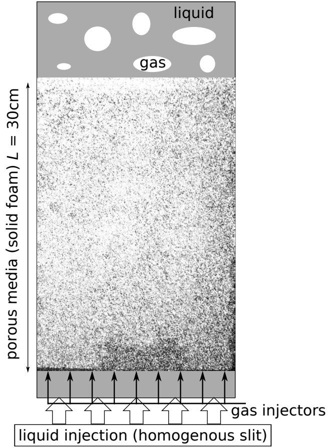
Normalized image

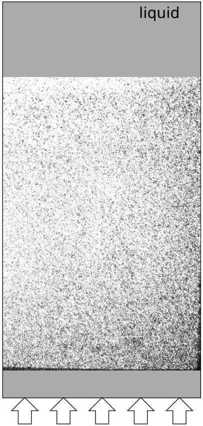


Textured

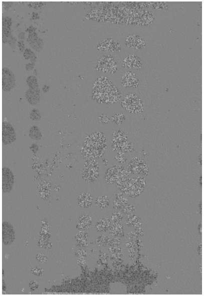
Multiphasic (quasi 2D) flow in a porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)





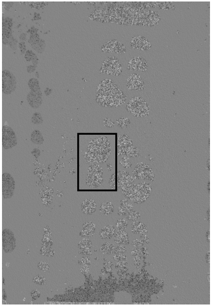
Normalized image



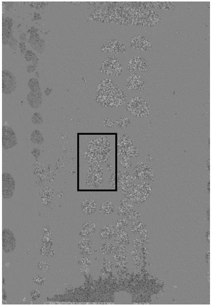
Textured

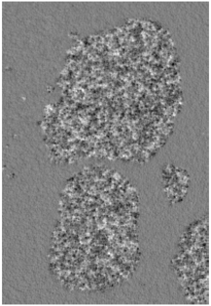
Physical quantities: $\underbrace{\text{gas volume}}_{\text{area}}$ & $\underbrace{\text{contact surface}}_{\text{perimeter}}$.

Texture segmentation

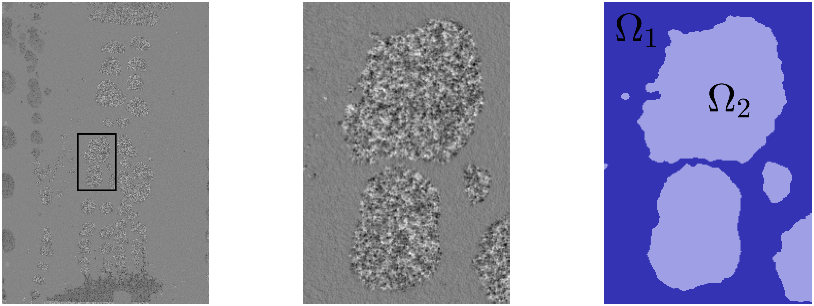


Texture segmentation





Texture segmentation

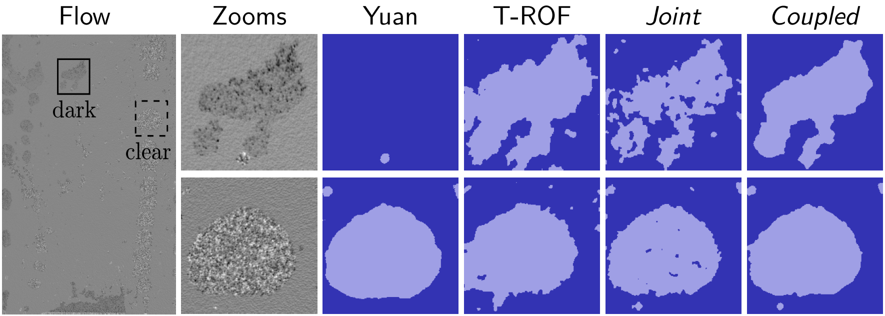


Purpose: obtaining a partition of the image into two regions

$$\Omega = \Omega_1 \bigsqcup \Omega_2$$

Ω_1: liquid, Ω_2: gas.

Multiphasic flow. $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$: low activity



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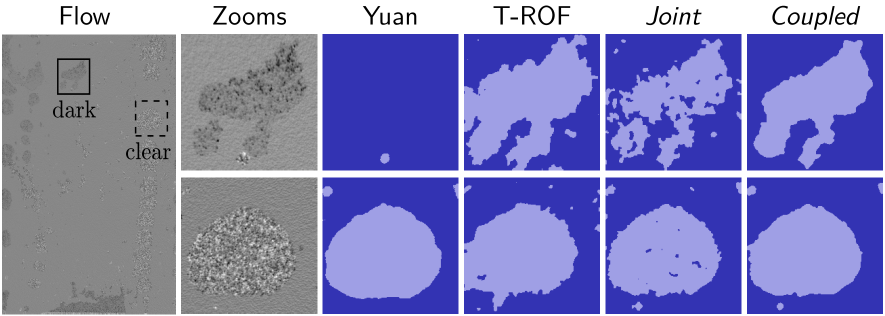
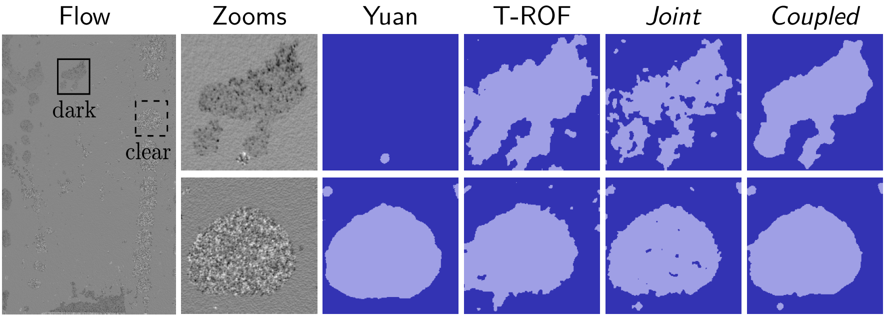


Liquid: $h_{\mathrm{L}} = 0.4$

Gas: $h_{\mathrm{G}} = 0.9$

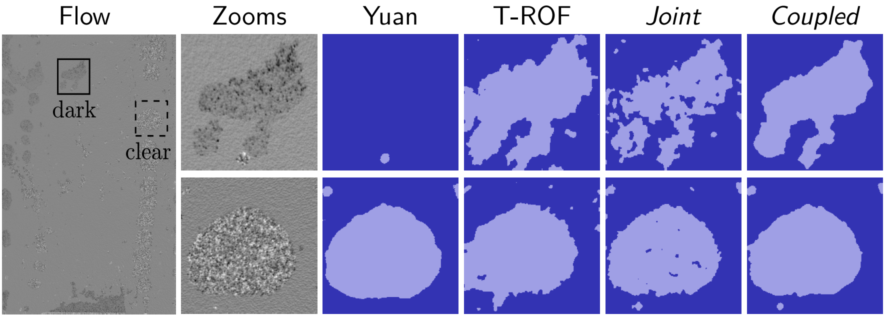
Multiphasic flow. $Q_{\mathrm{G}} = 300$mL/min - $Q_{\mathrm{L}} = 300$mL/min: low activity



Liquid: $h_{\mathrm{L}} = 0.4$ $\quad$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\quad$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$(dark bubbles)

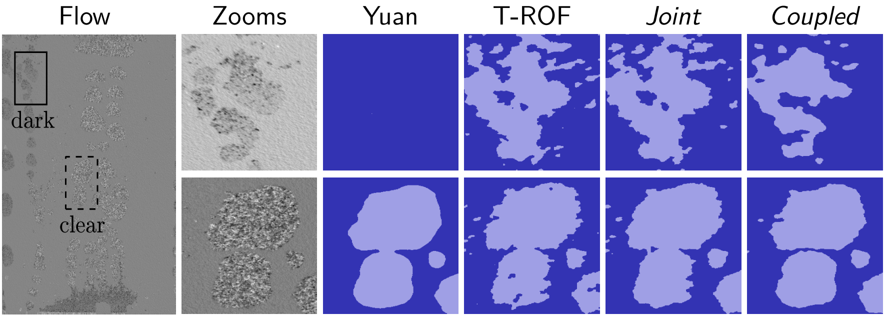
Multiphasic flow. $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$: low activity



Liquid: $h_{\mathrm{L}} = 0.4$ $\qquad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\qquad \left| \begin{array}{l} \sigma^2_{\mathrm{dark}} = 10^{-2} \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} \text{(clear bubbles)} \end{array} \right.$

Multiphasic flow. $Q_{\mathrm{G}} = 400$mL/min - $Q_{\mathrm{L}} = 700$mL/min: transition



Liquid: $h_{\mathrm{L}} = 0.4$

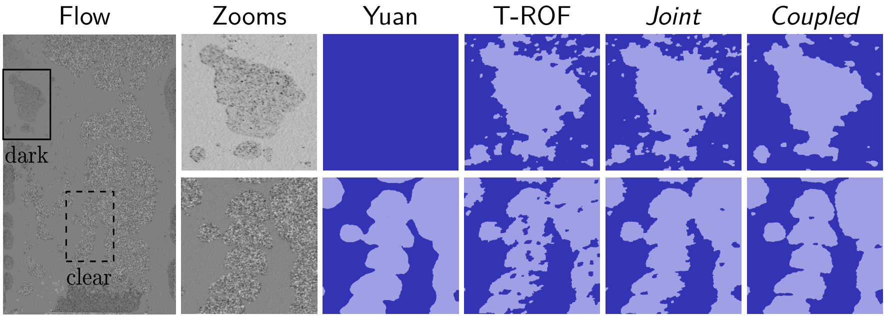
Gas: $h_{\mathrm{G}} = 0.9$

$\sigma^2_{\mathrm{dark}} = 10^{-2}$

$\sigma^2_{\mathrm{dark}} = 10^{-2}$(dark bubbles)

$\sigma^2_{\mathrm{clear}} = 10^{-1}$(clear bubbles)

Multiphasic flow. $Q_{\mathrm{G}} = 1200$mL/min - $Q_{\mathrm{L}} = 300$mL/min: high activity



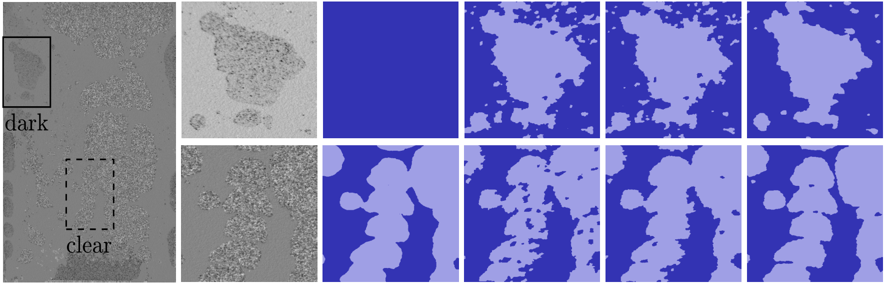
Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$(dark bubbles)

$\sigma^2_{\mathrm{clear}} = 10^{-1}$(clear bubbles)

Multiphasic flow. $Q_{\rm G} = 1200$mL/min - $Q_{\rm L} = 300$mL/min: high activity

Flow	Zooms	Yuan	T-ROF	*Joint*	*Coupled*



Computational time	1s	12s	700s	**2100s**

Liquid: $h_{\rm L} = 0.4$ $\quad\sigma^2_{\rm dark} = 10^{-2}$

Gas: $h_{\rm G} = 0.9$ $\quad\left|\begin{array}{l}\sigma^2_{\rm dark} = 10^{-2}\text{(dark bubbles)}\\ \sigma^2_{\rm clear} = 10^{-1}\text{(clear bubbles)}\end{array}\right.$

Outline – Fractal texture segmentation

1. Fractal texture model

$$\log\left(\boldsymbol{\mathcal{L}}_{a,\cdot}\right) \simeq \underbrace{\boldsymbol{v}}_{\sim \log(\boldsymbol{\sigma}^2)} + \log(a) \underbrace{\boldsymbol{h}}_{\textit{regularity}}$$

2. Attributes estimation and segmentation

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

3. Multiphasic flow segmentation

4. Regularization parameters tuning

Texture's attributes estimation

Hyperparameters tuning

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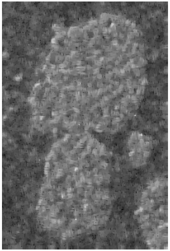
Lin. Reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_{a} \underbrace{\|\log \mathcal{L}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Lin. Reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ Too small



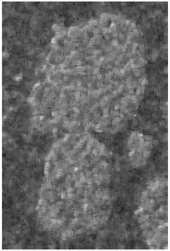
Texture's attributes estimation

Hyperparameters tuning

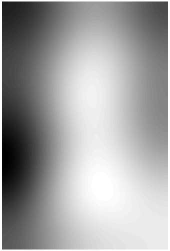
$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \;\; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \, \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Lin. Reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Too small



Too large

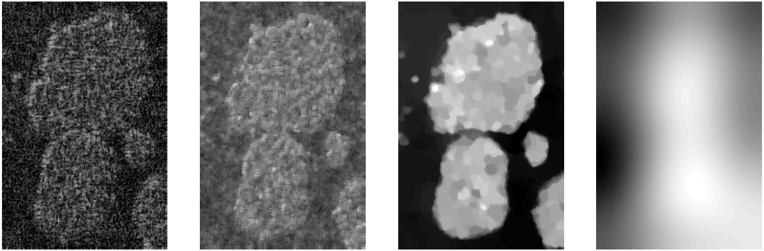


Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underset{\textbf{Least-Squares}}{\underline{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}} \quad + \quad \lambda\, \underset{\textbf{Total Variation}}{\underline{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}}$$

Lin. Reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Too small | Optimal | Too large



Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_{a} \underbrace{\|\log \mathcal{L}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

$\bar{\boldsymbol{h}}$: true regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}_{\lambda,\alpha} - \bar{\boldsymbol{h}} \right\|^2$$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

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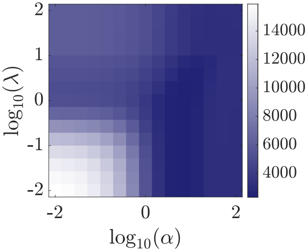

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

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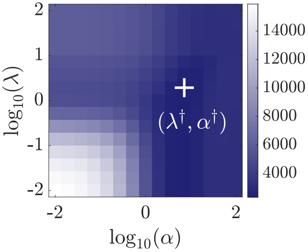
$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}_{\lambda,\alpha} - \bar{\boldsymbol{h}} \right\|^2$$



Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

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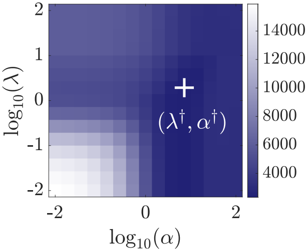


$\bar{\boldsymbol{h}}$: unknown!

?

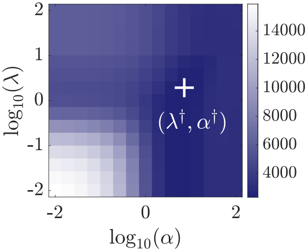
Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \underbrace{\lambda\, \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

$\bar{\boldsymbol{h}}$: true regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}_{\lambda,\alpha} - \bar{\boldsymbol{h}} \right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

?

Stein Unbiased Risk Estimate

$$\mathbb{E}[\underbrace{\widehat{\mathcal{R}}(\lambda, \alpha)}_{\text{ind. of } \bar{\boldsymbol{h}}}] = \mathcal{R}(\lambda, \alpha)$$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \; \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Fractal model $\quad \log \mathcal{L}_{a,\cdot} = \bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} + \varepsilon_{a,\cdot}$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \;\; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \;\; + \;\; \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Fractal model $\quad \log \boldsymbol{\mathcal{L}}_{a,.} = \bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} + \boldsymbol{\varepsilon}_{a,.}$

<u>Stein Unbiased Risk Estimator</u> (SURE) $\quad \widehat{R}(\lambda)$

Observation model $\quad \boldsymbol{y} = \underset{\text{truth}}{\bar{\boldsymbol{x}}} + \underset{\text{noise}}{\boldsymbol{n}} \in \mathbb{R}^N, \qquad$ Estimator $\quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \; + \; \underbrace{\lambda\, \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Fractal model $\quad \log \boldsymbol{\mathcal{L}}_{a,\cdot} = \bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} + \boldsymbol{\varepsilon}_{a,\cdot}$

<u>Stein Unbiased Risk Estimator</u> (SURE) $\quad \widehat{R}(\lambda)$

Observation model $\quad \boldsymbol{y} = \underset{\text{truth}}{\bar{\boldsymbol{x}}} + \underset{\text{noise}}{\boldsymbol{n}} \in \mathbb{R}^N,$ $\qquad$ Estimator $\quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

- If $\boldsymbol{n}$ is i.i.d. Gaussian noise of variance ρ^2

$$R(\lambda) \triangleq \mathbb{E}\, \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2$$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

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<u>Stein Unbiased Risk Estimator</u> (SURE) $\quad \widehat{R}(\lambda)$

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$$R(\lambda) \triangleq \mathbb{E}\,\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \bar{\boldsymbol{x}}\|^2 = \mathbb{E}\,\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y} + \boldsymbol{y} - \bar{\boldsymbol{x}}\|^2$$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Fractal model $\quad \log \mathcal{L}_{a,\cdot} = \bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} + \boldsymbol{\varepsilon}_{a,\cdot}$

Stein Unbiased Risk Estimator (SURE) $\quad \widehat{R}(\lambda)$

Observation model $\quad \boldsymbol{y} = \underset{\text{truth}}{\bar{\boldsymbol{x}}} + \underset{\text{noise}}{\boldsymbol{n}} \in \mathbb{R}^N, \qquad$ Estimator $\quad \widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda)$

- If $\boldsymbol{n}$ is i.i.d. Gaussian noise of variance ρ^2

$$\begin{aligned} R(\lambda) &\triangleq \mathbb{E}\left\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \bar{\boldsymbol{x}}\right\|^2 = \mathbb{E}\left\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y} + \boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2 \\ &= \mathbb{E}\left\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y}\right\|^2 + 2\mathbb{E}\left\langle \widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y}, \boldsymbol{y} - \bar{\boldsymbol{x}}\right\rangle + \mathbb{E}\left\|\boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2 \end{aligned}$$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \underbrace{\lambda\, \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Fractal model $\quad \log \boldsymbol{\mathcal{L}}_{a,\cdot} = \bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} + \boldsymbol{\varepsilon}_{a,\cdot}$

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$$\begin{aligned}
R(\lambda) \triangleq \mathbb{E}\left\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \bar{\boldsymbol{x}}\right\|^2 &= \mathbb{E}\left\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y} + \boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2 \\
&= \mathbb{E}\left\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y}\right\|^2 + 2\mathbb{E}\left\langle\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y}, \boldsymbol{y} - \bar{\boldsymbol{x}}\right\rangle + \mathbb{E}\left\|\boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2 \\
&= \underline{\mathbb{E}\left\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y}\right\|^2} + 2 \quad \underline{\mathbb{E}\left\langle\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda), \boldsymbol{n}\right\rangle} \quad - \underline{\mathbb{E}\left\|\boldsymbol{n}\right\|^2}
\end{aligned}$$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

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$$\begin{aligned} R(\lambda) &\triangleq \mathbb{E}\,\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \bar{\boldsymbol{x}}\|^2 = \mathbb{E}\,\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y} + \boldsymbol{y} - \bar{\boldsymbol{x}}\|^2 \\ &= \mathbb{E}\,\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y}\|^2 + 2\mathbb{E}\,\langle \widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y}, \boldsymbol{y} - \bar{\boldsymbol{x}}\rangle + \mathbb{E}\,\|\boldsymbol{y} - \bar{\boldsymbol{x}}\|^2 \\ &= \underbrace{\mathbb{E}\,\|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y}\|^2}_{\text{known}} + 2 \quad \underline{\mathbb{E}\,\langle \widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda), \boldsymbol{n}\rangle} \quad - \underline{\mathbb{E}\,\|\boldsymbol{n}\|^2} \end{aligned}$$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

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Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \ \ \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \ + \ \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Fractal model $\quad \log \boldsymbol{\mathcal{L}}_{a,.} = \bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} + \boldsymbol{\varepsilon}_{a,.}$

Stein Unbiased Risk Estimator (SURE) $\quad \widehat{R}(\lambda)$

Observation model $\quad \boldsymbol{y} = \underset{\text{truth}}{\bar{\boldsymbol{x}}} + \underset{\text{noise}}{\boldsymbol{n}} \in \mathbb{R}^N, \qquad$ Estimator $\quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

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Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \underbrace{\lambda\, \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

$$\text{Fractal model} \quad \log \boldsymbol{\mathcal{L}}_{a,\cdot} = \bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} + \boldsymbol{\varepsilon}_{a,\cdot}$$

Stein Unbiased Risk Estimator (SURE) $\quad \widehat{R}(\lambda)$

$$\text{Observation model} \quad \boldsymbol{y} = \underset{\text{truth}}{\bar{\boldsymbol{x}}} + \underset{\text{noise}}{\boldsymbol{n}} \in \mathbb{R}^N, \qquad \text{Estimator} \quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$$

- If $\boldsymbol{n}$ is i.i.d. Gaussian noise of variance ρ^2

$$\widehat{R}(\lambda|\rho^2) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\mathrm{tr}\left(\frac{\partial \widehat{\boldsymbol{x}}}{\partial \boldsymbol{y}}(\boldsymbol{y}; \lambda)\right) - \rho^2 N$$

Texture's attributes estimation

Hyperparameters tuning

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \underbrace{\lambda\, \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

Fractal model $\quad \log \boldsymbol{\mathcal{L}}_{a,\cdot} = \bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} + \boldsymbol{\varepsilon}_{a,\cdot}$

Stein Unbiased Risk Estimator (SURE) $\quad \widehat{R}(\lambda)$

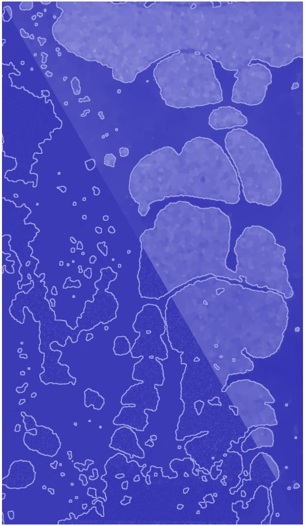
Observation model $\quad \boldsymbol{y} = \underset{\text{truth}}{\bar{\boldsymbol{x}}} + \underset{\text{noise}}{\boldsymbol{n}} \in \mathbb{R}^N, \qquad$ Estimator $\quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

- If $\boldsymbol{n}$ is i.i.d. Gaussian noise of variance ρ^2

$$\widehat{R}(\lambda|\rho^2) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) - \boldsymbol{y}\|^2 + 2\mathrm{tr}\left(\frac{\partial \widehat{\boldsymbol{x}}}{\partial \boldsymbol{y}}(\boldsymbol{y};\lambda)\right) - \rho^2 N$$

- Fractal model: $\boldsymbol{\varepsilon}$ Gaussian with covariance matrix $\boldsymbol{\mathcal{S}}$

$$\widehat{R}(\lambda|\boldsymbol{\mathcal{S}}) \quad \nabla_\lambda \widehat{R}(\lambda|\boldsymbol{\mathcal{S}})$$



Thank you for your attention!

Outline – Fractal texture segmentation

1. Fractal texture model

$$\log\left(\boldsymbol{\mathcal{L}}_{a,\cdot}\right) \simeq \underbrace{\boldsymbol{v}}_{\sim\log(\boldsymbol{\sigma}^2)} + \log(a) \underbrace{\boldsymbol{h}}_{\textit{regularity}}$$

2. Attributes estimation and segmentation

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

3. Multiphasic flow segmentation

4. Regularization parameters tuning

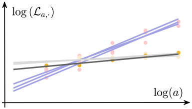
Texture's attributes estimation

Fine tuning of regularization parameters

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} \quad + \quad \underbrace{\color{red}{\lambda}\, \mathcal{N}(\boldsymbol{v}, \boldsymbol{h}; \color{red}{\alpha})}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to log-linear model

$\rightarrow$ enforce piecewise constancy



Texture's attributes estimation

Fine tuning of regularization parameters

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} + \underbrace{{\color{red}\lambda}\, \mathcal{N}(\boldsymbol{v},\boldsymbol{h};{\color{red}\alpha})}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to log-linear model

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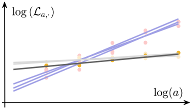
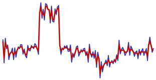
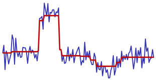


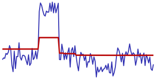
<u>Fine tuning</u> of regularization parameters (λ, α) is necessary ...



too small



optimal



too large

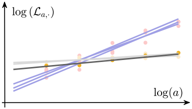
Texture's attributes estimation

Fine tuning of regularization parameters

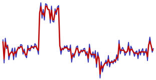
$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{Least-Squares}} + \underbrace{\lambda\, \mathcal{N}(\boldsymbol{v}, \boldsymbol{h}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to log-linear model

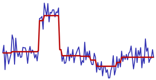
→ enforce piecewise constancy



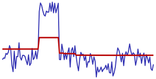
<u>Fine tuning</u> of regularization parameters (λ, α) is necessary ... but **costly**!



too small



optimal



too large

In practice, we explore a log-spaced grid of <u>$15 \times 15 = \mathbf{225}$</u> hyperparameters (λ, α).

Ongoing work and perspectives

- Video analysis (temporal series of hundreds of images)

Intership of L. Helmlinger

Ongoing work and perspectives

- Video analysis (temporal series of hundreds of images)

  Intership of L. Helmlinger

  ✓ Best (λ, α) tuned on 1^{st} image is sufficiently robust for the entire series.

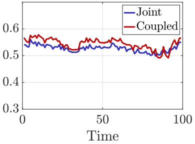
Ongoing work and perspectives

- Video analysis (temporal series of hundreds of images)

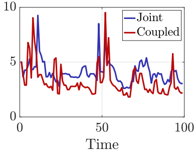
Intership of L. Helmlinger

✓ Best (λ, α) tuned on 1^{st} image is sufficiently robust for the entire series.
✓ Time evolution of physical quantities can be assessed.

Fraction of gas *(area)*



Liquid/gas contact *perimeter*



Ongoing work and perspectives

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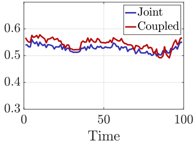
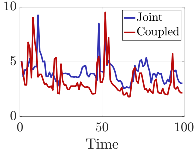


Liquid/gas contact *perimeter*

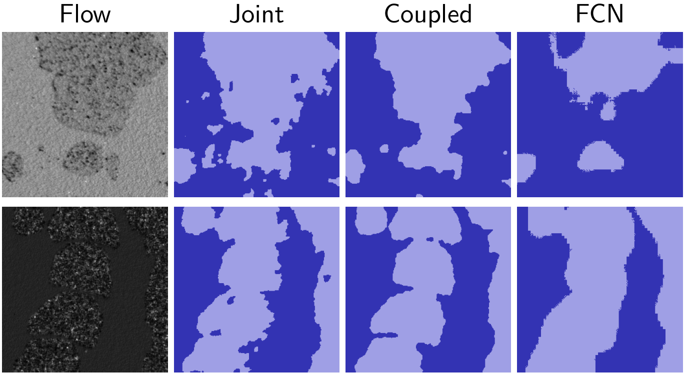


- Automatic tuning of hyperparameters

Stein's **U**nbiased **R**isk **E**stimate $\widehat{\mathcal{R}}(\lambda, \alpha)$

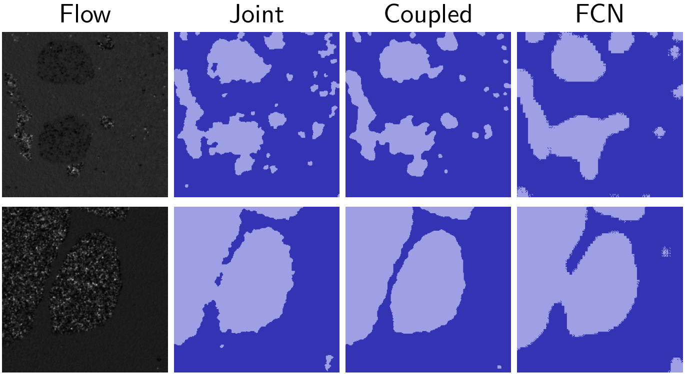
Stein **U**nbiased **GrA**dient estimator of the **R**isk $\nabla_\lambda \widehat{\mathcal{R}}(\lambda, \alpha)$

Fully Convolutional Neural Networks†



† V. Andrearczyk, `https://arxiv.org/abs/1703.05230`

Fully Convolutional Neural Networks[†]



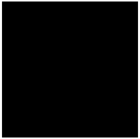
[†] V. Andrearczyk, https://arxiv.org/abs/1703.05230

Gas/liquid flow modeled by piecewise monofractal textures

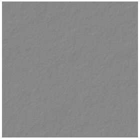
Synthetic textures

Liquid: $h_1 = 0.4$, $\sigma_1^2 = 10^{-2}$

Mask



Texture



Gas/liquid flow modeled by piecewise monofractal textures

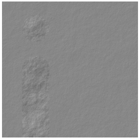
Synthetic textures

Liquid: $h_1 = 0.4$, $\sigma_1^2 = 10^{-2}$

Gas: $h_2 = 0.9$, $\sigma_1^2 = 10^{-2}$ (dark bubbles)

Mask

Texture



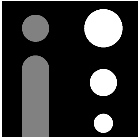
Gas/liquid flow modeled by piecewise monofractal textures

Synthetic textures

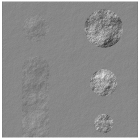
Liquid: $h_1 = 0.4,\ \sigma_1^2 = 10^{-2}$

Gas: $\left|\begin{array}{l} h_2 = 0.9,\ \sigma_1^2 = 10^{-2} \text{ (dark bubbles)} \\ h_2 = 0.9,\ \sigma_2^2 = 10^{-1} \text{(clear bubbles)} \end{array}\right.$

Mask



Texture



Gas/liquid flow modeled by piecewise monofractal textures

Synthetic textures

Liquid: $h_1 = 0.4,\ \sigma_1^2 = 10^{-2}$

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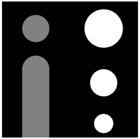
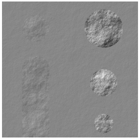
Mask



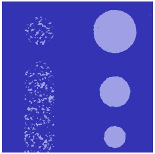
Texture



Segmentation performance

'Gas/Liquid'

Yuan 88%



T-ROF 88%

Joint 95%

Coupled 95%

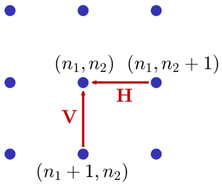
Optimization scheme - Monofractal model and piecewise constancy

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2 + \lambda \mathcal{N}(\boldsymbol{v}, \boldsymbol{h}; \alpha)$$

Optimization scheme - Monofractal model and piecewise constancy

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2 + \lambda \mathcal{N}(\boldsymbol{v}, \boldsymbol{h}; \alpha)$$

aim: enforce piecewise behavior of estimate

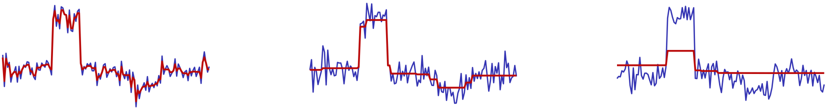


Discrete difference operator

$$(\mathbf{D}x)_{n_1,n_2} := \begin{pmatrix} x_{n_1,n_2+1} - x_{n_1,n_2} \\ x_{n_1+1,n_2} - x_{n_1,n_2} \end{pmatrix} := \begin{bmatrix} \mathbf{H}\boldsymbol{x} \\ \mathbf{V}\boldsymbol{x} \end{bmatrix}_{n_1,n_2}$$

Total Variation penalization

$$\mathcal{N}(\boldsymbol{x}) = \|\mathbf{D}\boldsymbol{x}\|_{2,1} = \sum_{n_1=1}^{N-1} \sum_{n_2=1}^{N-1} \sqrt{(\mathbf{H}\boldsymbol{x})^2_{n_1,n_2} + (\mathbf{V}\boldsymbol{x})^2_{n_1,n_2}}$$



Too small | **Optimal** | Too large

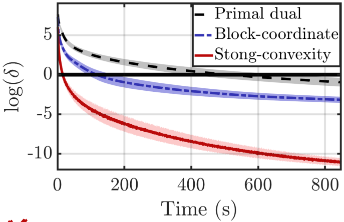
Optimization scheme - Monofractal model and piecewise constancy

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2 + \lambda\mathcal{N}(\boldsymbol{v},\boldsymbol{h};\alpha)$$

State-of-the-art - Segmentation on h only

$$\underset{\boldsymbol{h}}{\text{minimize}}\, \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|_2^2 + \lambda\mathcal{N}(\boldsymbol{h})$$

- ✓ only one parameter λ
- ✓ fast algorithms **[Pascal2018]**



- ✗ poor segmentation performance

$$\underset{\boldsymbol{h},\boldsymbol{\omega}}{\text{minimize}}\, \|\boldsymbol{h} - \textstyle\sum_a \boldsymbol{\omega}_a \boldsymbol{\mathcal{L}}_{a,.}\|_2^2 + \lambda\mathcal{N}(\boldsymbol{h},\boldsymbol{\omega};\alpha_a)$$

- ✗ additional constraints on $\{\omega\}_a$
- ✗ time and memory consuming

- ✓ very good accuracy **[Pustelnik2016]**

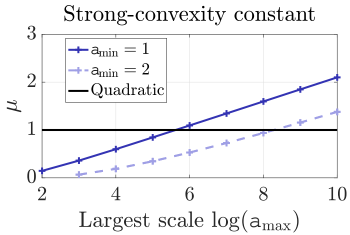
Strong convexity of data fidelity term

$$\mathrm{LS}(\boldsymbol{v},\boldsymbol{h}) = \sum_{a=a_{\min}}^{a_{\max}} \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2 = \|\log \mathcal{L} - \mathbf{A}(\boldsymbol{v},\boldsymbol{h})\|^2$$

where $\mathbf{A} : (\boldsymbol{v},\boldsymbol{h}) \mapsto \{\boldsymbol{v} + \log(a)\boldsymbol{h}\}_{a=a_{\min}}^{a_{\max}}$ is linear.

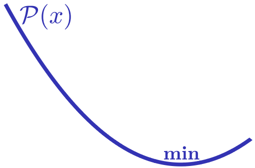
$$\boldsymbol{H}\mathrm{LS}\,(\boldsymbol{v},\boldsymbol{h}) = \mathbf{A}^*\mathbf{A} = \begin{pmatrix} \mathrm{A}_0\boldsymbol{I} & \mathrm{A}_1\boldsymbol{I} \\ \mathrm{A}_1\boldsymbol{I} & \mathrm{A}_2\boldsymbol{I} \end{pmatrix}, \; \mathrm{A}_m = \sum_{a=a_{\min}}^{a_{\max}} (\log a)^m, \; \forall m \in \{0,1,2\}.$$

Prop: $\mathrm{LS}(\boldsymbol{v},\boldsymbol{h})$ is μ-strongly convex, μ the smallest eigenvalue of $\mathbf{A}^*\mathbf{A}$.



Primal problem

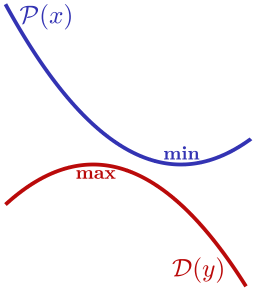
$$\widehat{x} = \underset{x}{\operatorname{argmin}}\ \mathrm{LS}(x) + \mathcal{N}(\mathbf{D}x)$$



Convergence speed and stopping criterion

Duality gap

Primal problem	**Dual problem**
$\widehat{x} = \underset{x}{\mathrm{argmin}}\ \mathrm{LS}(x) + \mathcal{N}(\mathbf{D}x)$	$\widehat{y} = \underset{y}{\mathrm{argmax}}\ -\mathrm{LS}^*(-\mathbf{D}^* y) - \mathcal{N}^*(y)$



Convergence speed and stopping criterion
Duality gap

Primal problem

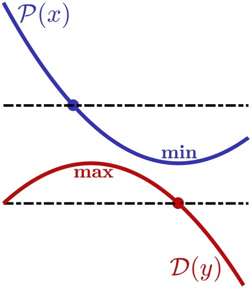
$$\widehat{x} = \underset{x}{\operatorname{argmin}}\ \mathrm{LS}(x) + \mathcal{N}(\mathbf{D}x)$$

Dual problem

$$\widehat{y} = \underset{y}{\operatorname{argmax}}\ -\mathrm{LS}^*(-\mathbf{D}^* y) - \mathcal{N}^*(y)$$

Duality gap $\delta(x; y)$

$$\underset{\text{def.}}{=} \mathrm{LS}(x) + \mathcal{N}(\mathbf{D}x) + \mathrm{LS}^*(-\mathbf{D}^* y) + \mathcal{N}^*(y)$$



Primal problem

$$\widehat{x} = \operatorname*{argmin}_x \mathrm{LS}(x) + \mathcal{N}(\mathbf{D}x)$$

Dual problem

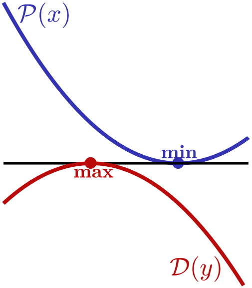
$$\widehat{y} = \operatorname*{argmax}_y -\mathrm{LS}^*(-\mathbf{D}^* y) - \mathcal{N}^*(y)$$

Duality gap $\delta(x; y)$

$$\underset{\text{def.}}{=} \mathrm{LS}(x) + \mathcal{N}(\mathbf{D}x) + \mathrm{LS}^*(-\mathbf{D}^* y) + \mathcal{N}^*(y)$$

Characterization of the solution

$$\delta(\widehat{x}; \widehat{y}) \underset{\text{prop.}}{=} 0$$



Computing the duality gap

For Joint penalization

$$\delta(\quad ; \quad)$$

$$= \quad + $$

Computing the duality gap

For Joint penalization

$$\delta(\underbrace{\boldsymbol{v}, \boldsymbol{h}}_{\text{primal}}; \quad)$$

$$= \mathrm{LS}(\boldsymbol{v}, \boldsymbol{h}) + \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}) +$$

Data fidelity

$$\mathrm{LS}(\boldsymbol{v}, \boldsymbol{h}) = \sum_a \|\boldsymbol{v} + \log(a)\boldsymbol{h} - \boldsymbol{\mathcal{L}}_{a,.}\|_2^2$$

Penalization

$$\mathcal{N}(\boldsymbol{u}, \boldsymbol{\ell}) = \lambda \left(\|\boldsymbol{u}\|_{2,1} + \alpha \|\boldsymbol{\ell}\|_{2,1}\right)$$

Computing the duality gap

For Joint penalization

$$\delta(\underbrace{\boldsymbol{v}, \boldsymbol{h}}_{\text{primal}}; \underbrace{\boldsymbol{u}, \boldsymbol{\ell}}_{\text{dual}})$$

$$= \mathrm{LS}(\boldsymbol{v}, \boldsymbol{h}) + \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}) + \mathrm{LS}^*(-\mathbf{D}^*\boldsymbol{u}, -\mathbf{D}^*\boldsymbol{\ell}) + \mathcal{N}^*(\boldsymbol{u}, \boldsymbol{\ell})$$

Data fidelity

$$\mathrm{LS}(\boldsymbol{v}, \boldsymbol{h}) = \sum_a \|\boldsymbol{v} + \log(a)\boldsymbol{h} - \boldsymbol{\mathcal{L}}_{a,.}\|_2^2$$

$$\mathrm{LS}^*(\boldsymbol{v}, \boldsymbol{h})$$
$$= \frac{1}{4}\langle(\boldsymbol{v}, \boldsymbol{h}), (\mathbf{A}^*\mathbf{A})^{-1}(\boldsymbol{v}, \boldsymbol{h})\rangle$$
$$+ \langle(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{T}}), (\mathbf{A}^*\mathbf{A})^{-1}(\boldsymbol{v}, \boldsymbol{h})\rangle$$
$$+ \mathcal{C}$$

where $\mathcal{C}$ constant term only depending on $\boldsymbol{\mathcal{L}}_{a,.}$.

Penalization

$$\mathcal{N}(\boldsymbol{u}, \boldsymbol{\ell}) = \lambda\left(\|\boldsymbol{u}\|_{2,1} + \alpha\|\boldsymbol{\ell}\|_{2,1}\right)$$

$$\mathcal{N}^*(\boldsymbol{u}, \boldsymbol{\ell}) = \iota_{\mathrm{B}_{2,\infty}(\lambda)}(\boldsymbol{u}) + \iota_{\mathrm{B}_{2,\infty}(\lambda\alpha)}(\boldsymbol{\ell})$$

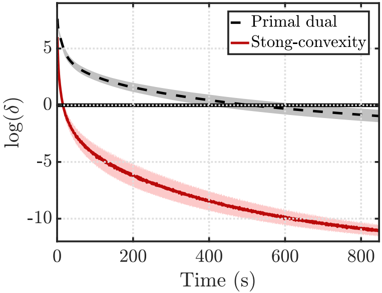
$\mathrm{B}_{2,\infty}(\lambda)$: ball of radius λ w.r.t. $\|.\|_{2,\infty}$.

Computing the duality gap

For Joint penalization

$$\delta(\underbrace{\boldsymbol{v}, \boldsymbol{h}}_{\text{primal}}; \underbrace{\boldsymbol{u}, \boldsymbol{\ell}}_{\text{dual}})$$

$$= \mathrm{LS}(\boldsymbol{v}, \boldsymbol{h}) + \mathcal{N}(\mathbf{D}\boldsymbol{v}, \mathbf{D}\boldsymbol{h}) + \mathrm{LS}^*(-\mathbf{D}^*\boldsymbol{u}, -\mathbf{D}^*\boldsymbol{\ell}) + \mathcal{N}^*(\boldsymbol{u}, \boldsymbol{\ell})$$



✓ Significant convergence acceleration

✓ Good stopping criterion: $\delta(\boldsymbol{v}^n, \boldsymbol{h}^n; \boldsymbol{u}^n, \boldsymbol{\ell}^n) \leq 10^{-3}$

Convex conjugate of data fidelity term

$$\mathrm{LS}^*(\boldsymbol{v},\boldsymbol{h}) = \sup_{\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{h}}\in\mathbb{R}^{|\Omega|}} \langle \widetilde{\boldsymbol{v}},\boldsymbol{v}\rangle + \langle \widetilde{\boldsymbol{h}},\boldsymbol{h}\rangle - \mathrm{LS}(\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{h}}) \underset{\text{(if sup is reached)}}{=} \langle \bar{\boldsymbol{v}},\boldsymbol{v}\rangle + \langle \bar{\boldsymbol{h}},\boldsymbol{h}\rangle - \mathrm{LS}(\bar{\boldsymbol{v}},\bar{\boldsymbol{h}}).$$

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Euler condition

$$\begin{cases} \boldsymbol{v} - 2\sum_a \left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.} \right) = 0 \\ \boldsymbol{h} - 2\sum_a \log(a) \left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.} \right) = 0 \end{cases}$$

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$$\mathcal{S} = \sum_a \log \mathcal{L}_{a,.} \quad \text{and} \quad \mathcal{T} = \sum_a \log(a) \log \mathcal{L}_{a,.},$$

Convex conjugate of data fidelity term

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where $\mathcal{C}$ constant term only depending on $\boldsymbol{\mathcal{L}}(X)$.

Conclusion

Comparison of the different methods

	Liquid/Gas (regularity change)	Clear/Dark bubbles (variance change)	Smooth contours

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Yuan	✗	✓	✓

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T-ROF	✓	✓	✗

Conclusion

Comparison of the different methods

	Liquid/Gas (regularity change)	Clear/Dark bubbles (variance change)	Smooth contours
Yuan	✗	✓	✓
T-ROF	✓	✓	✗
Joint	✓	✓	∼

Conclusion

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T-ROF	✓	✓	✗
Joint	✓	✓	∼
Coupled	✓	✓	✓

Conclusion

Comparison of the different methods

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Yuan	✗	✓	✓
T-ROF	✓	✓	✗
Joint	✓	✓	$\sim$
Coupled	✓	✓	✓

Coupled is the most satisfactory in term of segmentation quality . . .

Conclusion

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Yuan	✗	✓	✓
T-ROF	✓	✓	✗
Joint	✓	✓	$\sim$
Coupled	✓	✓	✓

Coupled is the most satisfactory in term of segmentation quality ...

... but it is the most time consuming (2100s)
Yuan(1s), T-ROF (12s), Joint (700s)