

## The Kravchuk transform:

A novel covariant representation for discrete signals  
amenable to zero-based detection tests

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Barbara Pascal, Rémi Bardenet

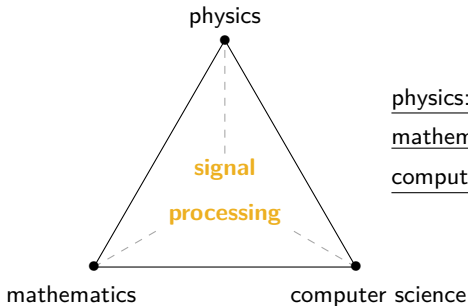
Sigma seminar  
CRIStAL, Lille

Signal processing aims to extract **information** from **data**.

Data of very diverse types:

- measurements of a physical quantity,
- biological or epidemiological indicators,
- data produced by human activities.

**The Golden triangle of signal processing**



physics: modeling of phenomena

mathematics: formalization & evaluation

computer science: efficient implementation

*inspired from P. Flandrin*

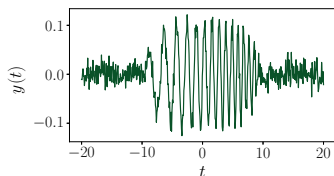
- Signal detection: the role of representations
- 

- Time-frequency analysis: the Short-Time Fourier Transform
  - Signal detection based on the spectrogram zeros - I
  - Covariance principle and stationary point processes
- 

- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

# Time and frequency: two dual descriptions of temporal signals

A continuous finite energy **signal** is a function of time  $y(t)$  with  $y \in L^2_{\mathbb{C}}(\mathbb{R})$ .



- electrical cardiac activity,
- audio recording,
- seismic activity,
- light intensity on a photosensor
- ...

## Information of interest:

- time events, e.g., an earthquake and its replica
- frequency content, e.g., monitoring of the heart beating rate

### time

ever-changing world  
marker of events and evolutions

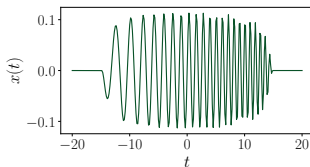
### frequency

waves, oscillations, rhythms  
intrinsic mechanisms

## Signal-plus-noise observation model

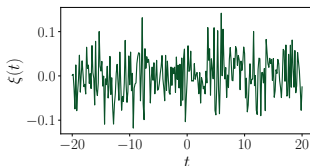
A **chirp** is a transient waveform modulated in amplitude and frequency:

$$x(t) = A_{\nu}(t) \sin \left( 2\pi \left( f_1 + (f_2 - f_1) \frac{t + \nu}{2\nu} \right) t \right)$$



**White noise** is a random variable  $\xi(t)$  such that

$$\mathbb{E}[\xi(t)] = 0 \quad \text{and} \quad \mathbb{E}[\overline{\xi(t)}\xi(t')] = \delta(t - t')$$

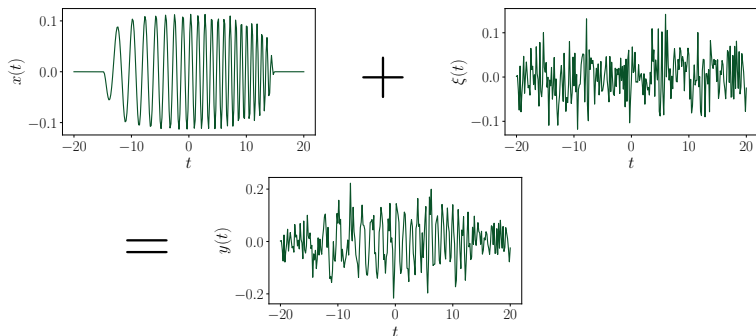


**P. Flandrin:** 'A signal is characterized by a structured organization.'

# Signal-plus-noise observation model

Noisy observations

$$y(t) = \text{snr} \times x(t) + \xi(t)$$

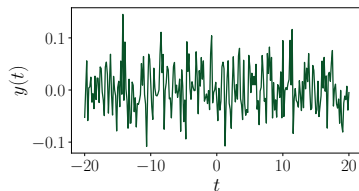
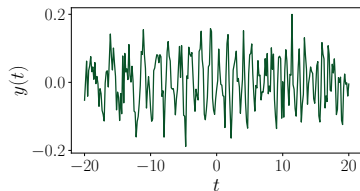


Signal processing task:

Given an observation  $y(t)$

**detection** decides whether there is an underlying signal or only noise.

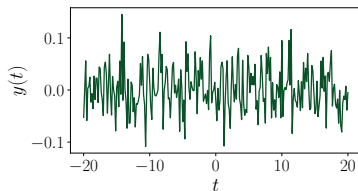
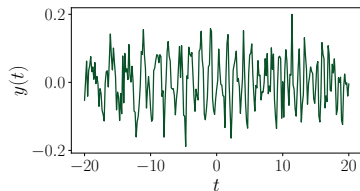
## Direct observation



**X** hard to distinguish between oscillations and fluctuations

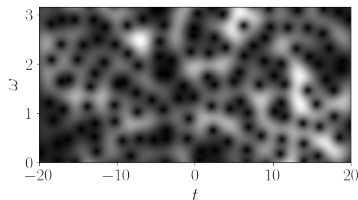
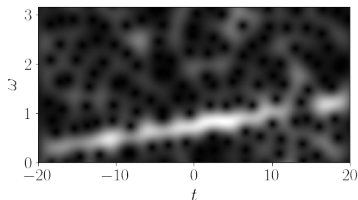
# The role of representations in signal processing

## Direct observation



✗ hard to distinguish between oscillations and fluctuations

## Time-frequency representation



✓ lines of local maxima: structures are simpler to capture



## Outline of the presentation

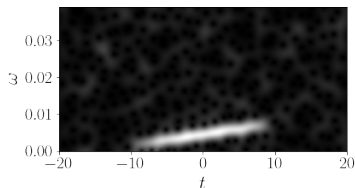
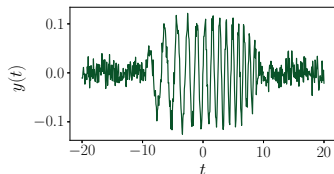
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Time and frequency Short-Time Fourier Transform with window  $h$ :

$$V_h y(t, \omega) \triangleq \int_{-\infty}^{\infty} \overline{y(u)} h(u - t) \exp(-i\omega u) du$$

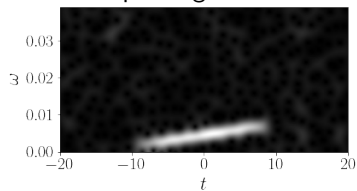


Energy density interpretation  $S_h y(t, \omega) = |V_h y(t, \omega)|^2$  the *spectrogram*

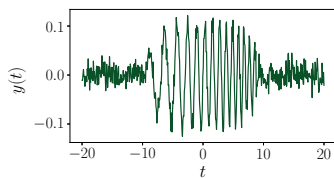
$$\int \int_{-\infty}^{+\infty} S_h y(t, \omega) dt \frac{d\omega}{2\pi} = \int_{-\infty}^{+\infty} |x(t)|^2 dt \quad \text{if } \|h\|_2^2 = 1$$

Signal, i.e., information of interest: regions of **maximal** energy.

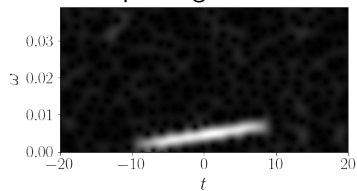
spectrogram



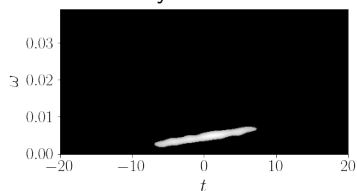
noisy observation



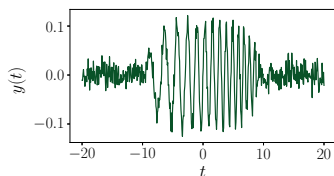
spectrogram



only maxima

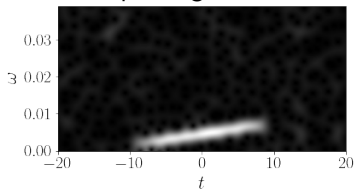


noisy observation

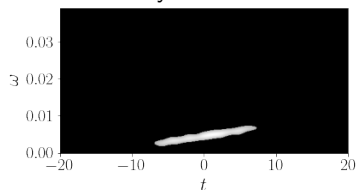


Inversion formula 
$$y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u, \omega)} h(t - u) \exp(i\omega u) du \frac{d\omega}{2\pi}$$

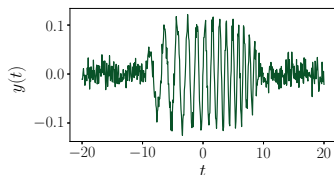
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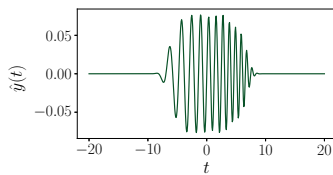
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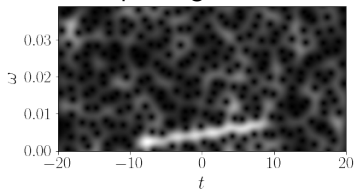


estimate

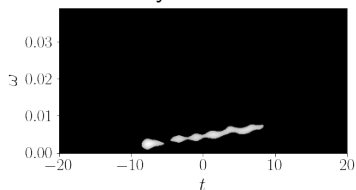


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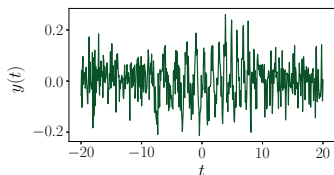
spectrogram



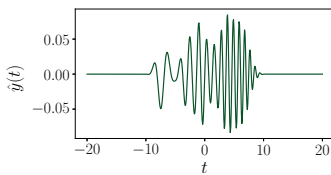
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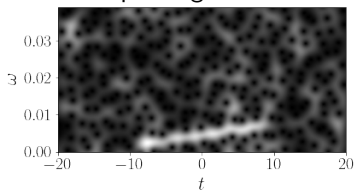
estimate



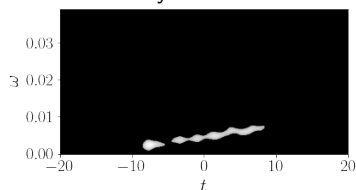
Denosing in the time-frequency plane:  $y = \text{snr} \times x + \xi$ ,  $\text{snr} = 0.5$

Inversion formula 
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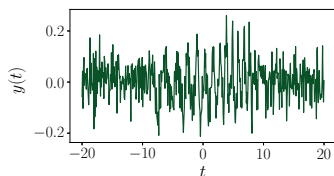
spectrogram



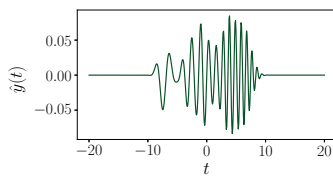
only maxima



noisy observation



estimate



**Maxima extraction:** *reassignment, synchrosqueezing, ridge extraction*

(Meignen et al., 2017)

# Outline of the presentation

- Signal detection: the role of representations
- 

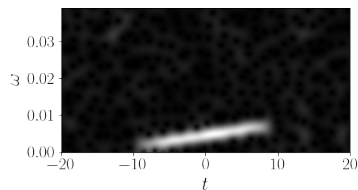
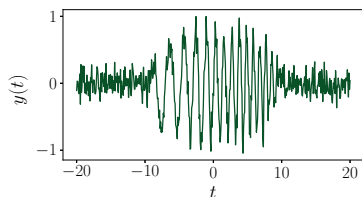
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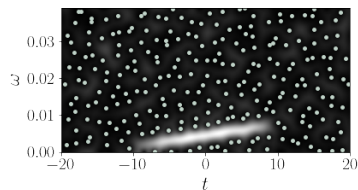
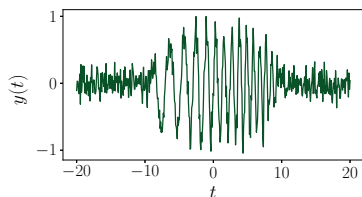
Restriction to the **circular Gaussian window**:  $g(t) = \pi^{-1/4} e^{-t^2/2}$

Look for the **zeros**, i.e., the points  $(t_i, \omega_i)$  such that  $|V_g y(t_i, \omega_i)|^2 = 0$ .



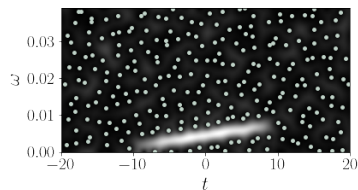
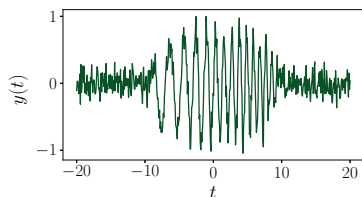
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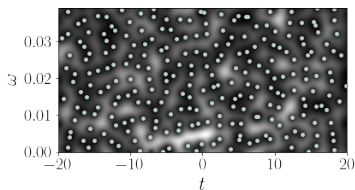
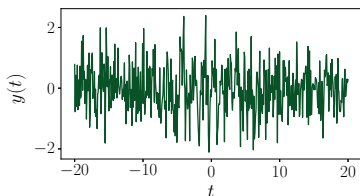


Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

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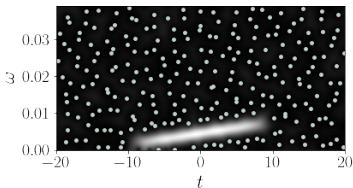
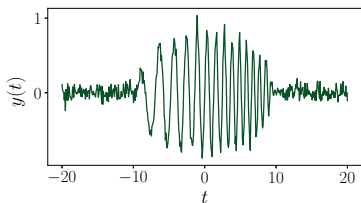


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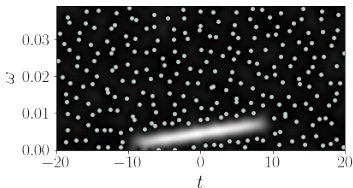
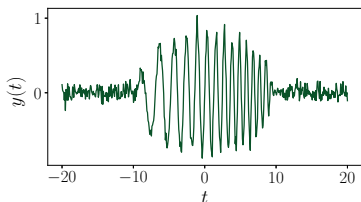


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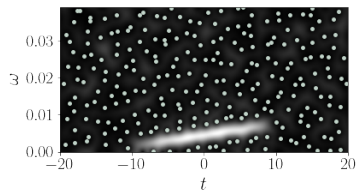


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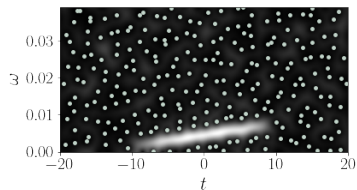
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**What can be said theoretically about the zeros of the spectrogram?**

**Idea** assimilate the time-frequency plane with  $\mathbb{C}$  through  $z = (\omega + it)/\sqrt{2}$



**Idea** assimilate the time-frequency plane with  $\mathbb{C}$  through  $z = (\omega + it)/\sqrt{2}$



Bargmann factorization

$$V_g y(t, \omega) = e^{-|z|^2/2} e^{-i\omega t/2} B_y(z)$$

$g$  the circular Gaussian window

Bargmann transform of the signal  $y$

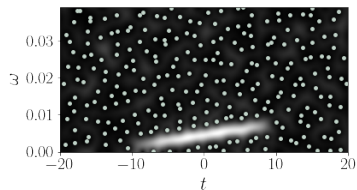
$$B_y(z) \triangleq \pi^{-1/4} e^{-z^2/2} \int_{\mathbb{R}} \overline{y(u)} \exp\left(\sqrt{2}uz - u^2/2\right) du,$$

$B_y$  is an **entire** function, almost characterized by its infinitely many zeros:

$$B_y(z) = z^m e^{C_0 + C_1 z + C_2 z^2} \prod_{n \in \mathbb{N}} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2} \left(\frac{z}{z_n}\right)^2\right).$$



**Idea** assimilate the time-frequency plane with  $\mathbb{C}$  through  $z = (\omega + it)/\sqrt{2}$



Bargmann factorization

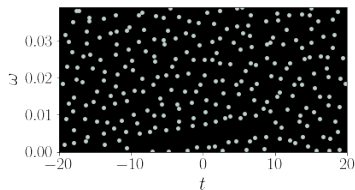
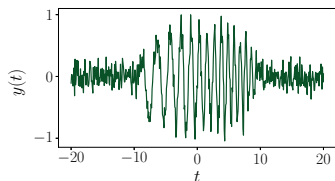
$$V_g y(t, \omega) = e^{-|z|^2/2} e^{-i\omega t/2} B_y(z)$$

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**Theorem** The zeros of the Gaussian spectrogram  $V_g y(t, \omega)$

- coincide with the zeros of the **entire** function  $B_y$ ,
- hence are **isolated** and constitute a **Point Process**,
- which almost completely **characterizes** the spectrogram.

(Flandrin, 2015)



## Advantages of working with the zeros

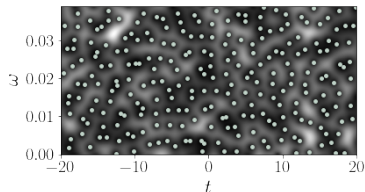
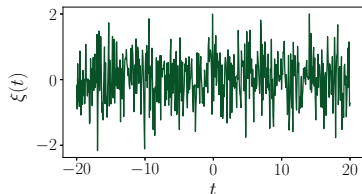
- Easy to find compared to relative maxima.
- Form a robust pattern in the time-frequency plane.
- Require little memory space for storage.
- Efficient tools were recently developed in **stochastic geometry**.

# Signal detection based on the spectrogram zeros

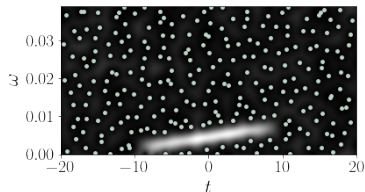
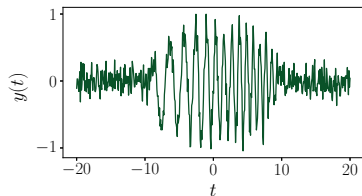
(Bardenet, Flamant & Chainais, 2020)

- $H_0$  white noisy only, i.e.,  $y(t) = \xi(t)$
- $H_1$  presence of a signal, i.e.,  $y(t) = \text{snr} \times x(t) + \xi(t)$ ,  $\text{snr} > 0$

null hypothesis



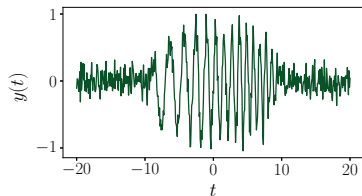
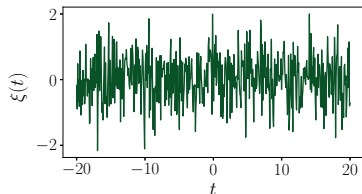
alternative hypothesis



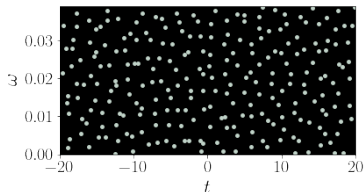
# Signal detection based on the spectrogram zeros

(Bardenet, Flamant & Chainais, 2020)

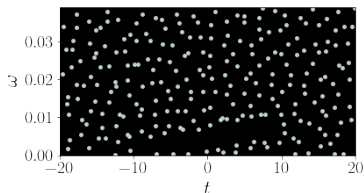
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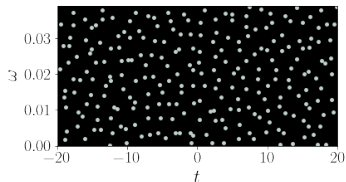
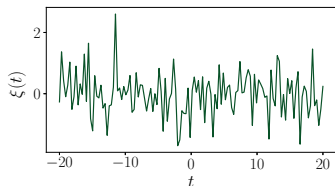


# The zeros of the spectrogram of white noise

## Continuous complex white Gaussian noise

(Bardenet et al., 2020), (Bardenet & Hardy, 2021)

$$\xi(t) = \sum_{n=0}^{\infty} \xi[n] h_n(t), \quad \xi[n] \sim \mathcal{N}_{\mathbb{C}}(0, 1), \quad \{h_n, k = 0, 1, \dots\} \text{ Hermite functions}$$

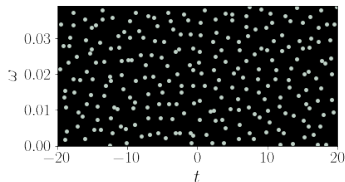
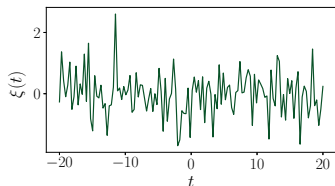


# The zeros of the spectrogram of white noise

## Continuous complex white Gaussian noise

(Bardenet et al., 2020), (Bardenet & Hardy, 2021)

$$\xi(t) = \sum_{n=0}^{\infty} \xi[n] h_n(t), \quad \xi[n] \sim \mathcal{N}_{\mathbb{C}}(0, 1), \quad \{h_n, k = 0, 1, \dots\} \text{ Hermite functions}$$

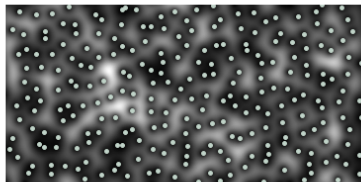


### Theorem

$$V_g \xi(t, \omega) = e^{-|z|^2/4} e^{-i\omega t/2} \text{GAF}_{\mathbb{C}}(z) \quad (\text{Bardenet \& Hardy, 2021})$$

$$\text{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}} \text{ the planar Gaussian Analytic Function and } z = \frac{\omega + it}{\sqrt{2}}.$$

# The zeros of the planar Gaussian Analytic Function



$$V_g \xi(t, \omega) \stackrel{\text{non-vanishing}}{\propto} \text{GAF}_{\mathbb{C}}(z)$$

$$z = (\omega + it)/\sqrt{2}$$

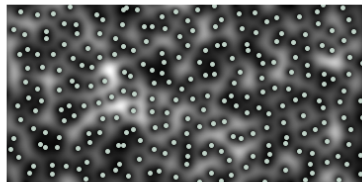
**Zeros of  $\text{GAF}_{\mathbb{C}}$ :** random set of points forming a **Point Process** characterized by a probability distribution on point configurations

Properties of the Point Process of the zeros of  $\text{GAF}_{\mathbb{C}}$ :

- invariant under the isometries of  $\mathbb{C}$ , i.e., **stationary**,
- has a uniform density  $\rho^{(1)}(z) = \rho^{(1)} = 1/\pi$ ,
- explicit two-point correlation function  $\rho^{(2)}(z, z') = \rho^{(2)}(|z - z'|)$ ,
- scaling of the *hole probability*:  $r^{-4} \log p_r \rightarrow -3e^2/4$ , as  $r \rightarrow \infty$

$$p_r = \mathbb{P}(\text{no point in the disk of center } 0 \text{ and radius } r)$$

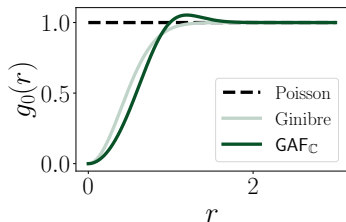
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Pair correlation  $\rho^{(2)}(z, z') dz dz' =$

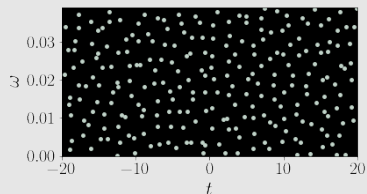
$$\mathbb{P}(\text{1 point in } B(z, dz) \text{ and 1 in } B(z', dz'))$$

The point process of the zeros of the spectrogram is not **determinantal**.



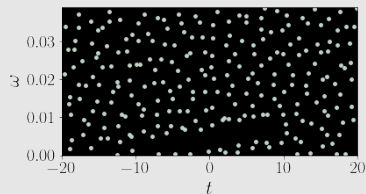
**Purpose:** summary statistic  $s$ , such that for  $y = \text{snr} \times x + \xi$

$\mathbb{E}[s(y)|\mathbf{H}_0]$  small



$\text{snr} = 0$

$\mathbb{E}[s(y)|\mathbf{H}_1]$  large

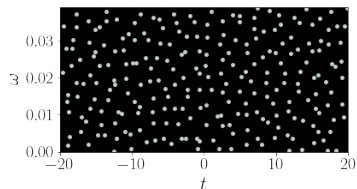


$\text{snr} > 0$

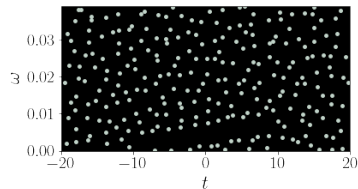
**'Large value of  $s(y)$  is a strong indication that there is a signal.'**

Tools from **stochastic geometry** to capture spatial statistics of the zeros.

## Unorthodox path: zeros of Gaussian Analytic Functions



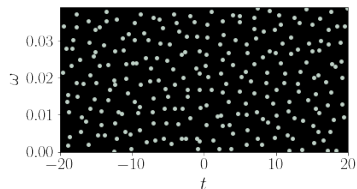
$\text{snr} = 0$



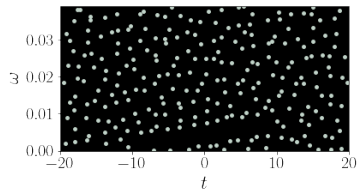
$\text{snr} > 0$

The signal creates **holes** in the zeros pattern: **second order** statistics.

# Unorthodox path: zeros of Gaussian Analytic Functions



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The signal creates **holes** in the zeros pattern: **second order** statistics.

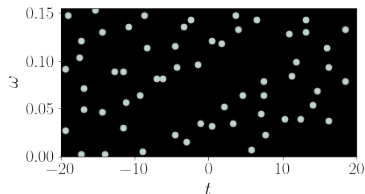
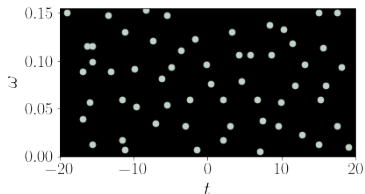
## A functional statistic: the empty space function

$Z$  a stationary point process,  $z_0$  any reference point

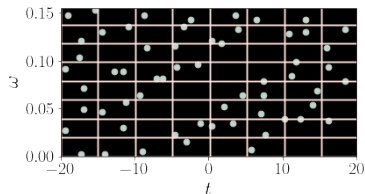
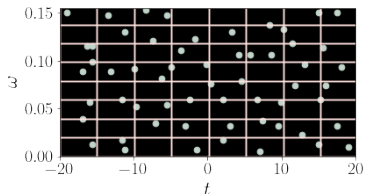
$$F(r) = \mathbb{P} \left( \inf_{z_i \in Z} d(z_0, z_i) < r \right)$$

→ probability to find a zero at distance less than  $r$  from  $z_0$

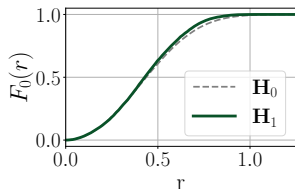
$$F(r) = \mathbb{P} \left( \inf_{z_i \in Z} d(z_0, z_i) < r \right) : \text{empty space function}$$



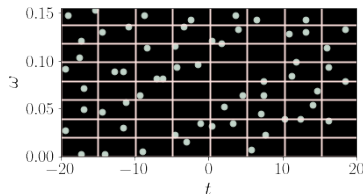
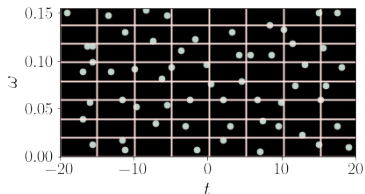
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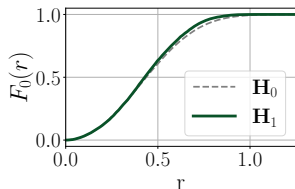
$$\hat{F}(r) = \frac{1}{N_{\#}} \sum_{j=1}^{N_{\#}} \mathbf{1} \left( \inf_{z \in \mathbf{Zeros}} d(z_j, z) < r \right)$$



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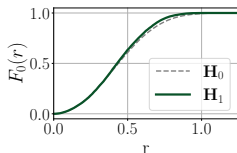
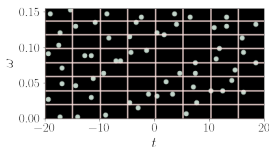
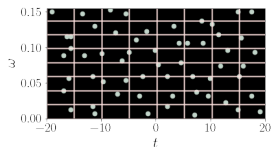


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- Monte Carlo envelope test based on the discrepancy between  $\hat{F}$  and  $F_0$

# Monte Carlo envelope test

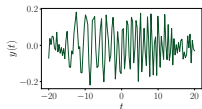
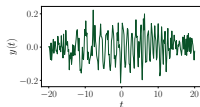
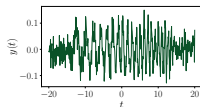
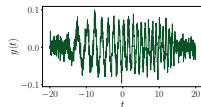


$$s(\mathbf{y}) = \sqrt{\int_0^{r_{\max}} |\hat{F}_{\mathbf{y}}(r) - F_0(r)|^2 dr},$$

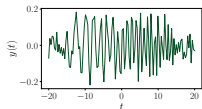
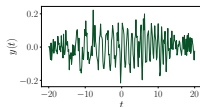
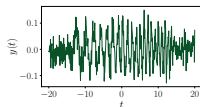
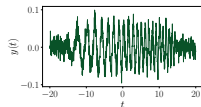
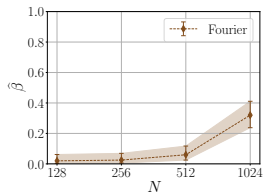
Test settings:  $\alpha$  level of significance,  $m$  number of samples under  $\mathbf{H}_0$

Index  $k$ , chosen so that  $\alpha = k/(m + 1)$

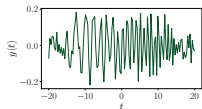
- (i) generate  $m$  independent samples of complex white Gaussian noise;
- (ii) compute their summary statistics  $s_1 \geq s_2 \geq \dots \geq s_m$ ;
- (iii) compute the summary statistic of the observation  $\mathbf{y}$  under concern;
- (iv) if  $s(\mathbf{y}) \geq s_k$ , then reject the null hypothesis with confidence  $1 - \alpha$ .

Detection of a noisy chirp of duration  $2\nu = 30$  s $N = 128$  $N = 256$  $N = 512$  $N = 1024$

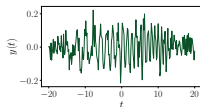


Detection of a noisy chirp of duration  $2\nu = 30$  s $N = 128$  $N = 256$  $N = 512$  $N = 1024$ **Performance:** power of the test computed over 200 samples

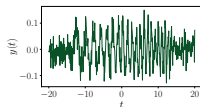
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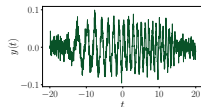
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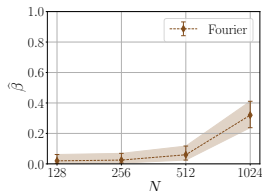


$N = 512$

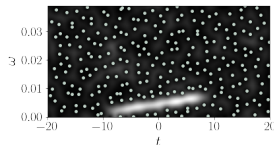


$N = 1024$

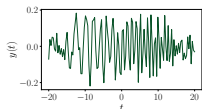
**Performance:** power of the test computed over 200 samples



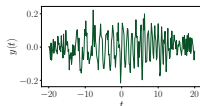
- ✓ Fast Fourier Transform ;
- ✗ Low detection power ;
- ✗ Requires large number of samples



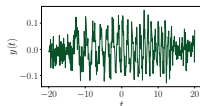
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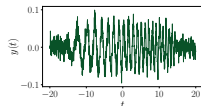
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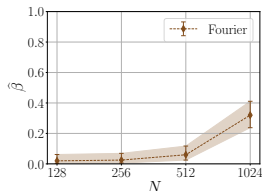


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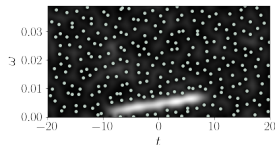
**Performance:** power of the test computed over 200 samples



- ✓ Fast Fourier Transform ;
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**Limitations:**

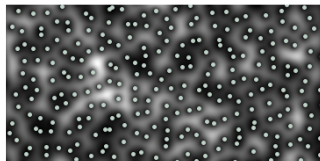
- Necessary discretization of the STFT: arbitrary resolution ;
- Observe only a bounded window: edge corrections to compute  $\hat{F}(r)$ .



## Short-Time Fourier Transform

$$V_g \xi(t, \omega) \propto \text{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$$

Unbounded phase space  $\mathbb{C}$



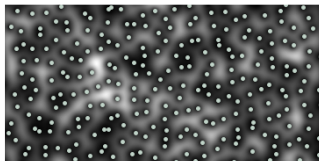
→ edge corrections

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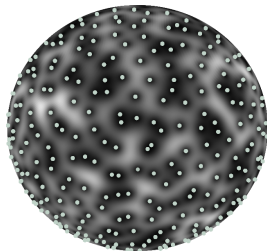
## New transform?

$$? \propto \text{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^N \xi[n] \sqrt{\binom{N}{n}} z^n$$

stereographic projection  $z = \cot(\vartheta/2)e^{i\varphi}$

→ spherical coordinates  $(\vartheta, \varphi) \in S^2$

Compact phase space  $S^2$ ?



→ no border!

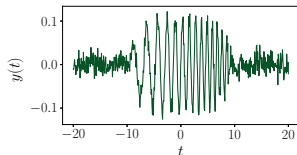
# Outline of the presentation

- Signal detection: the role of representations
- 

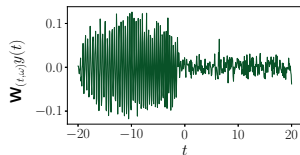
- Time-frequency analysis: the Short-Time Fourier Transform
  - Signal detection based on the spectrogram zeros - I
  - Covariance principle and stationary point processes
- 

- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

## Time and frequency shifts



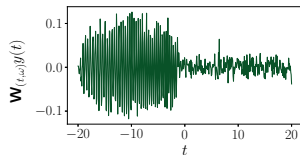
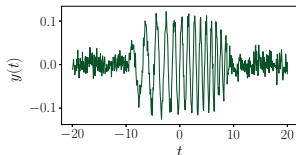
$$W_{(t,\omega)}y(u) = e^{-i\omega u}y(u - t)$$



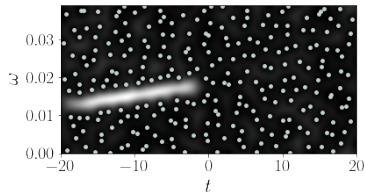
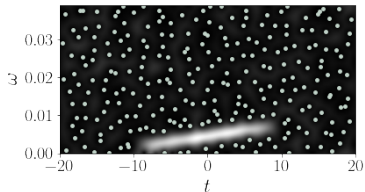
# Algebraic interpretation: covariance under a symmetry group

## Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u - t)$$



$$V_h[\mathbf{W}_{(t,\omega)}y](t', \omega') \stackrel{\text{(covariance)}}{=} e^{-i(\omega' - \omega)t} V_h y(t' - t, \omega' - \omega),$$





**Time and frequency shifts**

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u - t)$$

$$|V_h[\mathbf{W}_{(t,\omega)}y](t', \omega')|^2 \stackrel{\text{(covariance)}}{=} |V_hy(t' - t, \omega' - \omega)|^2,$$

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**Complex white Gaussian noise**

$$\tilde{\xi} = \mathbf{W}_{(t,\omega)}\xi$$

- $\mathbb{E}[\tilde{\xi}(u)] = e^{-i\omega u}\mathbb{E}[\xi(u - t)] = 0$

**Time and frequency shifts**

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## Time and frequency shifts

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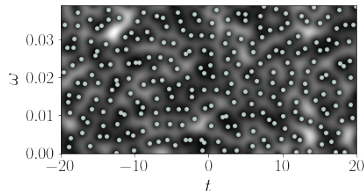
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## Invariance under time-frequency shifts:

$$\tilde{\xi} = \mathbf{W}_{(t,\omega)}\xi \stackrel{\text{(law)}}{=} \xi$$



### Time and frequency shifts

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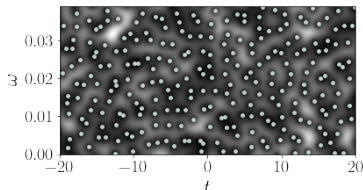
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Invariance under time-frequency shifts:

$$\tilde{\xi} = \mathbf{W}_{(t,\omega)}\xi \stackrel{\text{(law)}}{=} \xi$$



**Covariance** is the key to get **stationarity**: how to get covariant transforms?

**Time and frequency shifts**

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u - t)$$

$$|V_h[\mathbf{W}_{(t,\omega)}y](t', \omega')|^2 \stackrel{\text{(covariance)}}{=} |V_hy(t' - t, \omega' - \omega)|^2,$$

**Time and frequency shifts**

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$$|V_h[\mathbf{W}_{(t,\omega)}y](t',\omega')|^2 \stackrel{\text{(covariance)}}{=} |V_hy(t'-t,\omega'-\omega)|^2,$$

**Weyl-Heisenberg group**  $\{e^{i\gamma} \mathbf{W}_{(t,\omega)}, (\gamma, t, \omega) \in [0, 2\pi] \times \mathbb{R}^2\}$

$$\mathbf{W}_{(t',\omega')} \mathbf{W}_{(t,\omega)} = e^{i\omega t'} \mathbf{W}_{(t+t',\omega+\omega')}.$$

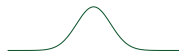
## Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u - t)$$

$$|V_h[\mathbf{W}_{(t,\omega)}y](t', \omega')|^2 \stackrel{\text{(covariance)}}{=} |V_hy(t' - t, \omega' - \omega)|^2,$$

**Weyl-Heisenberg group**  $\{e^{i\gamma} \mathbf{W}_{(t,\omega)}, (\gamma, t, \omega) \in [0, 2\pi] \times \mathbb{R}^2\}$

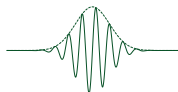
$$\mathbf{W}_{(t',\omega')} \mathbf{W}_{(t,\omega)} = e^{i\omega t'} \mathbf{W}_{(t+t', \omega+\omega')}.$$



$$g(t) = \pi^{-1/4} \exp(-t^2/2)$$



$$\mathbf{T}_u g(t) = g(t - u)$$



$$\mathbf{M}_\omega g(t) = g(t) \exp(-i\omega t)$$



# Algebraic interpretation: covariance under a symmetry group

## Time and frequency shifts

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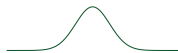
$$|V_h[\mathbf{W}_{(t,\omega)}y](t',\omega')|^2 \stackrel{\text{(covariance)}}{=} |V_hy(t'-t,\omega'-\omega)|^2,$$

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Coherent state interpretation  $\{\mathbf{W}_{(t,\omega)}h, t, \omega \in \mathbb{R}\}$  covariant family

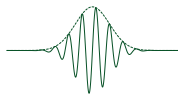
$$V_hy(t,\omega) = \int_{-\infty}^{\infty} \overline{y(u)} h(u-t) \exp(-i\omega u) du = \langle y, \mathbf{W}_{(t,\omega)}h \rangle$$



$$g(t) = \pi^{-1/4} \exp(-t^2/2)$$



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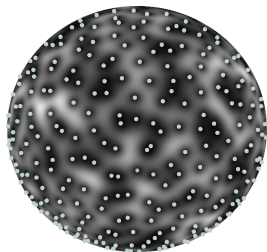
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# Outline of the presentation

- Signal detection: the role of representations
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- Time-frequency analysis: the Short-Time Fourier Transform
  - Signal detection based on the spectrogram zeros - I
  - Covariance principle and stationary point processes
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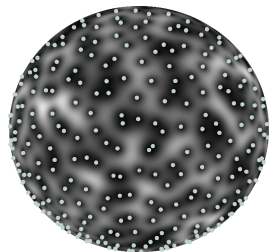


# The Kravchuk transform: covariance under $SO(3)$

Coherent state interpretation      $\mathbf{y} \in \mathbb{C}^{N+1}$

$$T\mathbf{y}(\vartheta, \varphi) = \langle \mathbf{y}, \Psi_{(\vartheta, \varphi)} \rangle$$

$$\vartheta \in [0, \pi], \varphi \in [0, 2\pi]$$

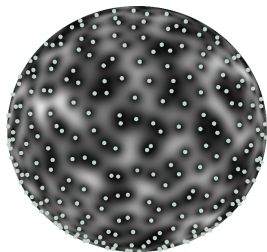


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$SO(3)$  **coherent states** (Gazeau, 2009)

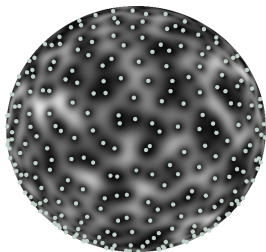
$$\Psi_{\vartheta, \varphi} = \sum_{n=0}^N \sqrt{\binom{N}{n}} \left( \cos \frac{\vartheta}{2} \right)^n \left( \sin \frac{\vartheta}{2} \right)^{N-n} e^{in\varphi} \mathbf{q}_n = \mathbf{R}_{\mathbf{u}(\vartheta, \varphi)} \Psi_{(0,0)},$$

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**Kravchuk transform**

$\{\mathbf{q}_n, n = 0, 1, \dots, N\}$  the *Kravchuk functions*

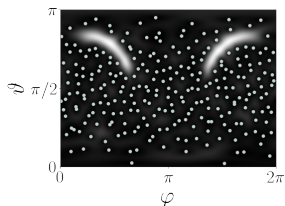
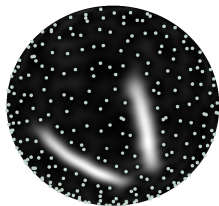
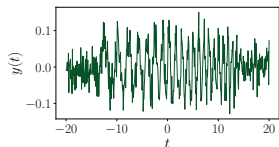
$$T\mathbf{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^N \langle \mathbf{y}, \mathbf{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2)e^{i\varphi}$$

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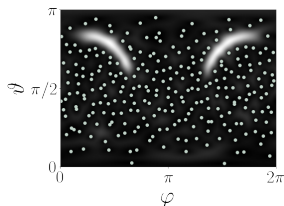
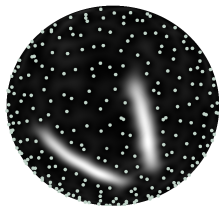
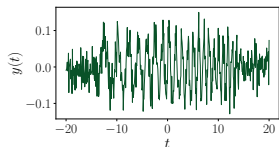


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**Theorem**

$$T\xi(\vartheta, \varphi) = \sqrt{(1+|z|^2)^{-N}} \text{GAF}_{\mathbb{S}}(z), \quad z = \cot(\vartheta/2) e^{i\varphi}$$

$$\text{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^N \xi[n] \sqrt{\binom{N}{n}} z^n \text{ the spherical Gaussian Analytic Function}$$

(Pascal & Bardenet, 2022)



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## Kravchuk transform

$\{\mathbf{q}_n, n = 0, 1, \dots, N\}$  the Kravchuk basis

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**Evaluation of Kravchuk functions**  $q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$

$$(N-n)Q_{n+1}(t; N) = (N-2t)Q_n(t; N) - nQ_{n-1}(t; N),$$

$\{Q_n(t; N), n = 0, 1, \dots, N\}$  orthogonal family of Kravchuk polynomials

$$\sum_{\ell=0}^N \binom{N}{\ell} Q_n(\ell; N) Q_{n'}(\ell; N) = 2^N \binom{N}{n}^{-1} \delta_{n,n'}$$

## Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N - n)Q_{n+1}(t; N) = (N - 2t)Q_n(t; N) - nQ_{n-1}(t; N),$$

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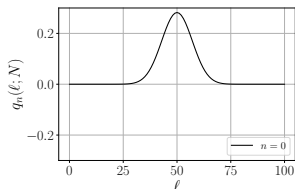
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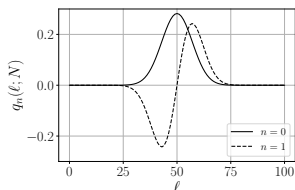
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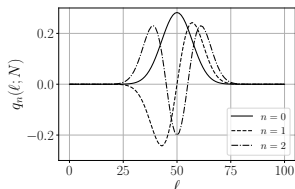
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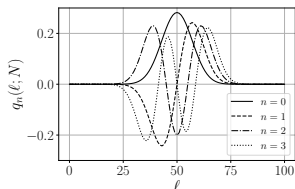
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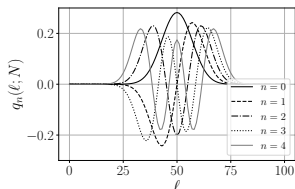
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# Instability of the computation of Kravchuk polynomials

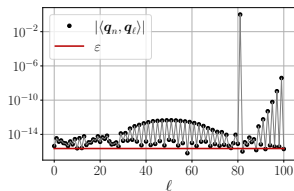
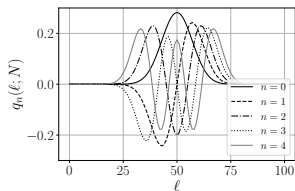
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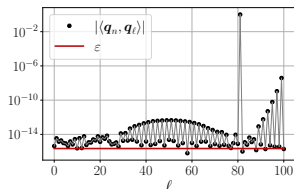
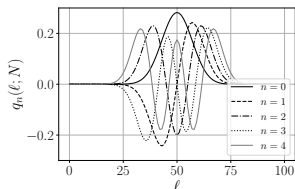
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→ estimated basis is **not orthogonal!** Not possible to compute  $\langle \mathbf{y}, \mathbf{q}_n \rangle$ .

## Kravchuk transform

$\{q_n, n = 0, 1, \dots, N\}$  the Kravchuk basis

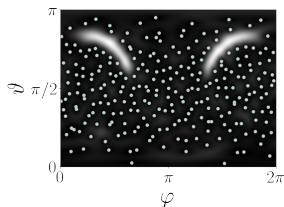
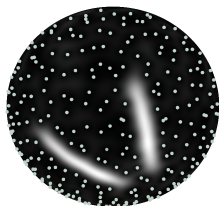
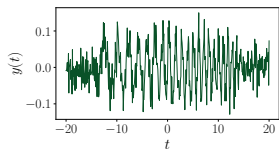
$$T\mathbf{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^N \left( \sum_{\ell=0}^N \overline{\mathbf{y}[\ell]} q_n(\ell; N) \right) \sqrt{\binom{N}{n}} z^n \rightarrow \text{intractable}$$

## A generative function for Kravchuk polynomials

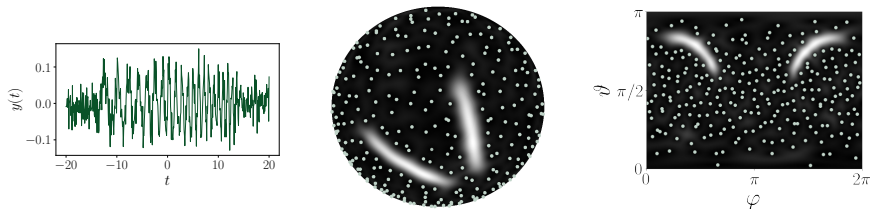
$$\begin{aligned} \sum_{n=0}^N \binom{N}{n} Q_n(\ell; N) z^n &= (1-z)^\ell (1+z)^{N-\ell} \\ \Rightarrow \sum_{n=0}^N \sqrt{\binom{N}{n}} q_n(\ell; N) z^n &= \sqrt{\binom{N}{\ell}} \frac{(1-z)^\ell (1+z)^{N-\ell}}{\sqrt{2^N}} \end{aligned}$$

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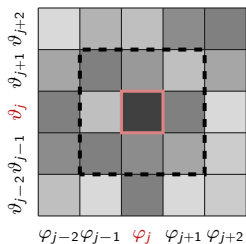
**X** no more Fast Fourier Transform algorithm using  $z^n = \cot(\vartheta/2)^n e^{in\varphi}$



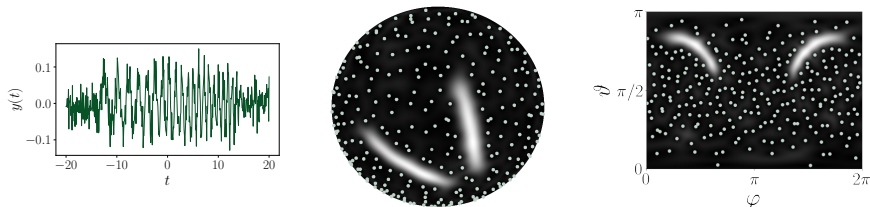
**Advantage compared to Fourier:** can tune the resolution of phase space.



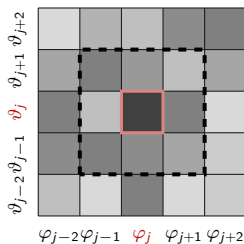
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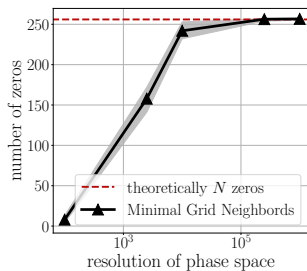
Minimal Grid Neighbors



**Advantage compared to Fourier:** can tune the resolution of phase space.



Minimal Grid Neighbors



**Proposition:** all local minima of  $|Ty(z)|^2$  are zeros.

## Outline of the presentation

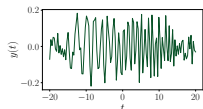
- Signal detection: the role of representations
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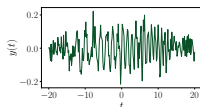
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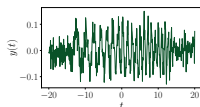
Detection of a noisy chirp of duration  $2\nu = 30$  s



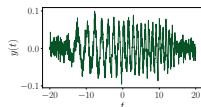
$N = 128$



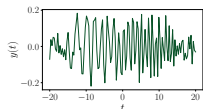
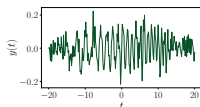
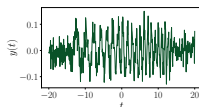
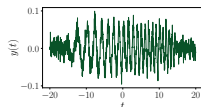
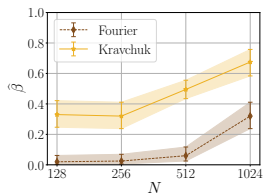
$N = 256$



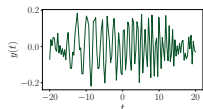
$N = 512$



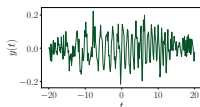
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Detection of a noisy chirp of duration  $2\nu = 30$  s $N = 128$  $N = 256$  $N = 512$  $N = 1024$ **Performance:** power of the test computed over 200 samples

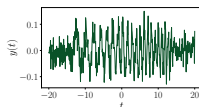
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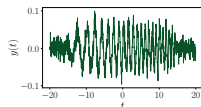
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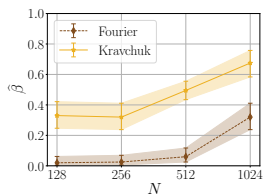


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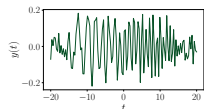
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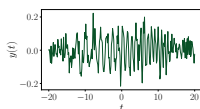


- ✓ higher detection power
- ✓ more robust to small  $N$
- ✗ no fast algorithm yet

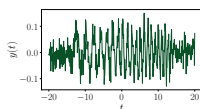
Detection of a noisy chirp of duration  $2\nu = 30$  s



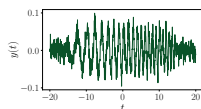
$N = 128$



$N = 256$

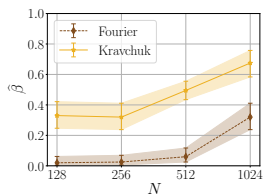


$N = 512$



$N = 1024$

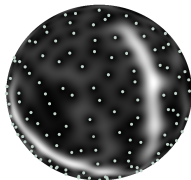
**Performance:** power of the test computed over 200 samples



- ✓ higher detection power
- ✓ more robust to small  $N$
- ✗ no fast algorithm yet

**Advantages of using Kravchuk vs. Fourier spectrogram**

- intrinsically encoded resolution: no need for prior knowledge
- compact phase space: no edge correction



## Take home messages

- Novel covariant discrete Kravchuk transform  $T_{\mathbf{y}}(\vartheta, \varphi)$ 
  - \* Interpreted as a coherent state decomposition,
  - \* Representation on a compact phase space,
  - \* Zeros of the Kravchuk spectrogram of white noise fully characterized.
- Signal detection based on spectrogram zeros
  - \* Preliminary work using the zeros of the Fourier spectrogram,
  - \* Significant improvement using the Kravchuk spectrogram.

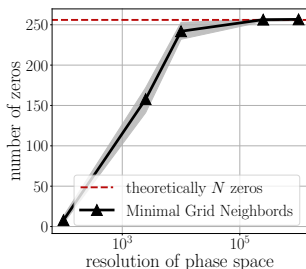
Pascal & Bardenet, 2022: [arxiv:2202.03835](https://arxiv.org/abs/2202.03835)

GitHub: [bpascal-fr/kravchuk-transform-and-its-zeros](https://github.com/bpascal-fr/kravchuk-transform-and-its-zeros)

## Work in progress and perspectives

- Interpretation of the action of  $SO(3)$  on  $\mathbb{C}^{N+1}$  ;
- Implementation of the inversion formula: denoising based on zeros ;
- Design of a Kravchuk FFT counterpart ;
- Convergence of Kravchuk toward the Fourier spectrogram as  $N \rightarrow \infty$ .

# Opening: can the Kravchuk spectrogram have multiple zeros?



## Spherical Gaussian Analytic Function

$$\text{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^N \xi[n] \sqrt{\binom{N}{n}} z^n$$

with  $\xi[n] \sim \mathcal{N}_{\mathbb{C}}(0, 1)$  i.i.d.

→ only **simple** zeros

.....

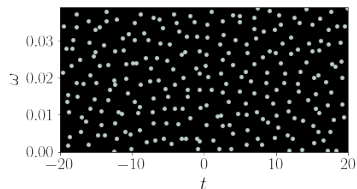
**General case**  $T\mathbf{y}(z) = \sqrt{(1 + |z|^2)}^{-N} \sum_{n=0}^N \sqrt{\binom{N}{n}} (\mathbf{Q}\mathbf{y})[n] z^n$

If  $\mathbf{y}$  deterministic, such that  $(\mathbf{Q}\mathbf{y})[n] = \sqrt{\binom{N}{n}} a^{N-n} b^n, a \in \mathbb{C}, b \in \mathbb{C}^*$ ,

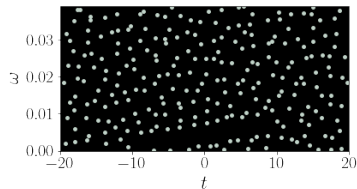
$$\sqrt{(1 + |z|^2)}^{-N} \sum_{n=0}^N \sqrt{\binom{N}{n}} (\mathbf{Q}\mathbf{y})[n] z^n = (a + bz)^N$$

→  $-a/b$  multiple root of order of degeneracy  $N$

# Unorthodox path: zeros of Gaussian Analytic Functions



$\text{snr} = 0$



$\text{snr} > 0$

The signal creates **holes** in the zeros pattern: **second order** statistics.

## Functional statistics:

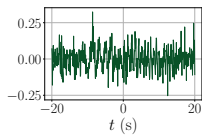
- the empty space function

$$F(r) = \mathbb{P} \left( \inf_{z_i \in Z} d(z_0, z_i) < r \right) : \text{probability to find a zero at less than } r$$

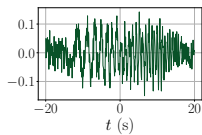
- Ripley's  $K$ -function

$$K(r) = 2\pi \int_0^r s g_0(s) ds : \text{expected \# of pairs at distance less than } r$$

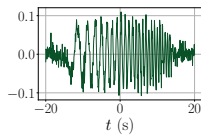
# Detection test: choice of the functional statistic



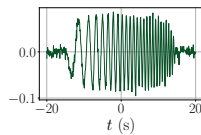
**snr = 0.5**



**snr = 1**



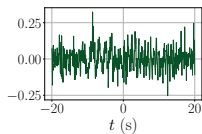
**snr = 2**



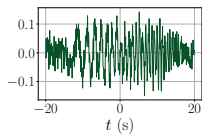
**snr = 5**



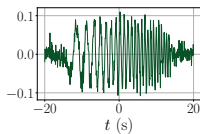
# Detection test: choice of the functional statistic



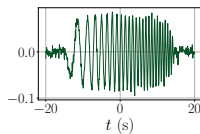
$\text{snr} = 0.5$



$\text{snr} = 1$

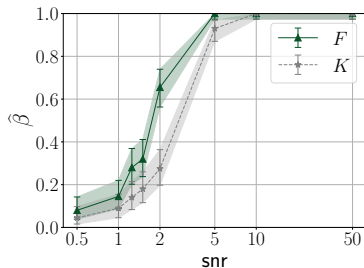


$\text{snr} = 2$

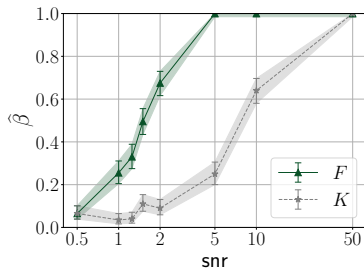


$\text{snr} = 5$

## Ripley's $K$ functional vs. empty space functional $F$



$N + 1 = 257$  points

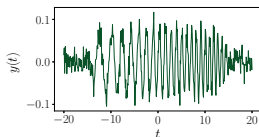


$N + 1 = 513$  points

# Detection test: snr and relative duration of the signal

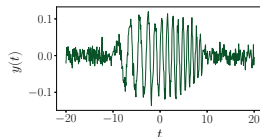
## Fixed observation window of 40 s

long time event



duration  $2\nu = 30$  s

short time event

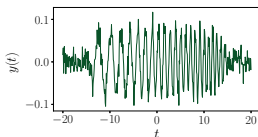


duration  $2\nu = 20$  s

# Detection test: snr and relative duration of the signal

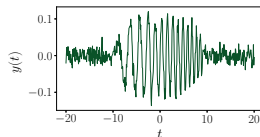
## Fixed observation window of 40 s

long time event



duration  $2\nu = 30$  s

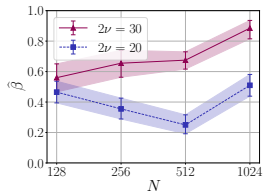
short time event



duration  $2\nu = 20$  s

## Robustness to small number of samples and short duration.

medium noise level

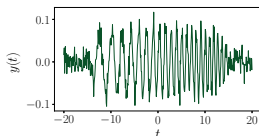


snr = 2

# Detection test: snr and relative duration of the signal

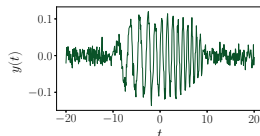
## Fixed observation window of 40 s

long time event



duration  $2\nu = 30$  s

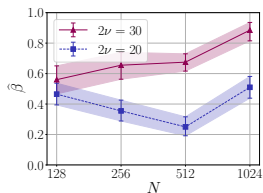
short time event



duration  $2\nu = 20$  s

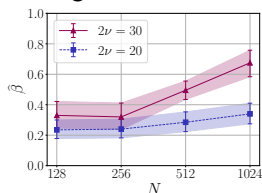
## Robustness to small number of samples and short duration.

medium noise level



snr = 2

high noise level



snr = 1.5