Texture segmentation based on fractal attributes

Barbara Pascal
(bpascal-fr.github.io)

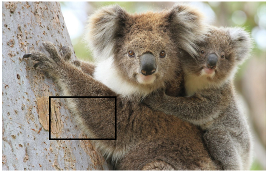
Joint work with Patrice Abry, Nelly Pustelnik, Valérie Vidal, and Samuel Vaiter

July 4th 2023

Harmonic and Multifractal Analyses: from Mathematics to Quantitative Neuroscience

Centre de Recherches Mathématiques (CRM), Montréal, Canada

Textures



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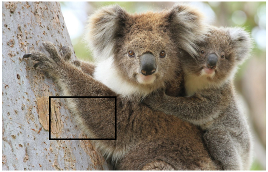

Textures

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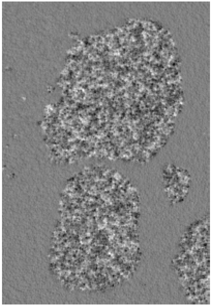
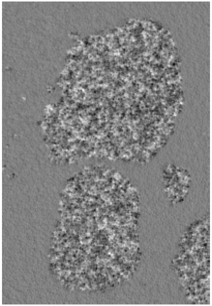
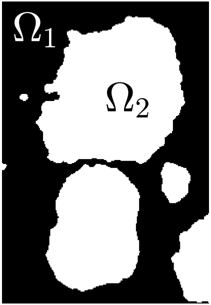
Crucial to describe real-world images



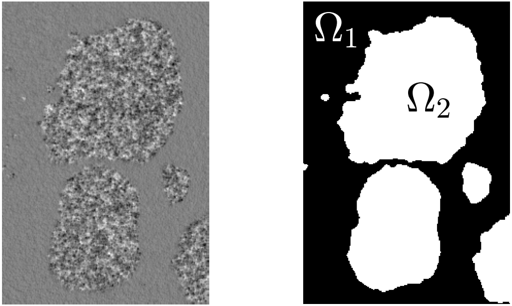
Textured image segmentation





Goal: obtain a partition of the image into K homogeneous textures

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_K$$



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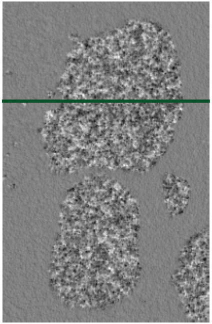
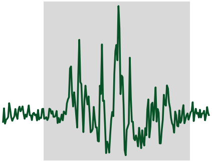
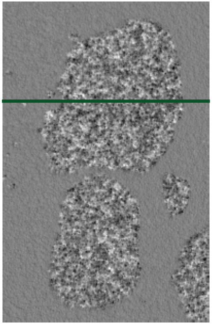
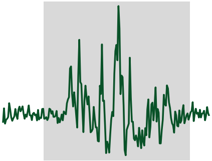




Fractal attributes

- variance σ^2 *amplitude of variations*





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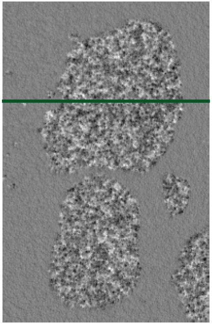

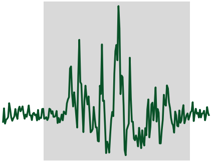

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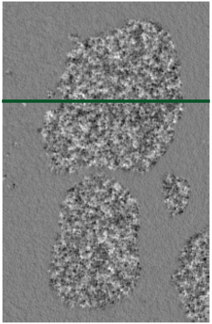

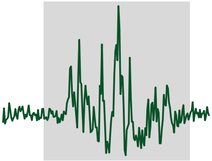

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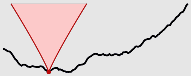
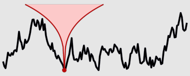


$h(x) \equiv h_1 = 0.9$



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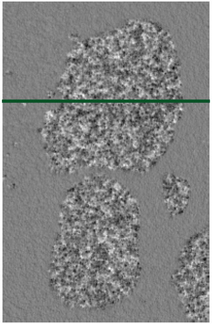

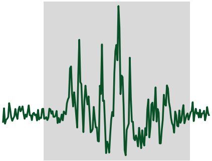

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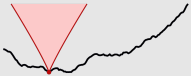

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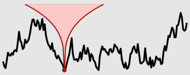

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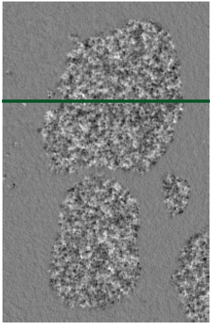
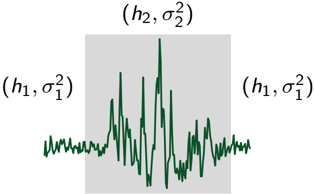
Segmentation

- ▶ σ^2 and h piecewise constant





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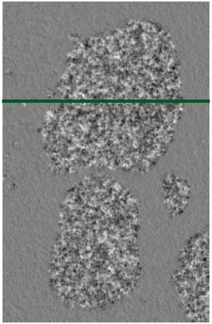
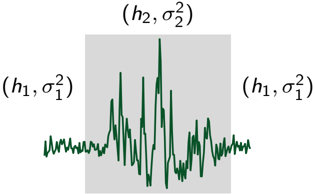
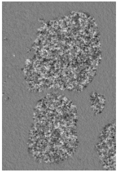
Segmentation

- σ^2 and h piecewise constant
- region Ω_k characterized by (σ_k^2, h_k)



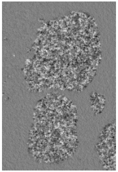


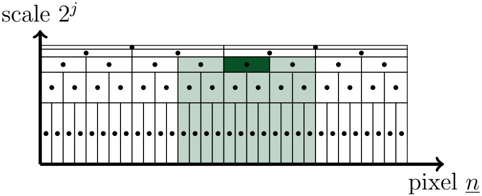
Textured image



Textured image

Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$





Multiscale analysis

Textured image

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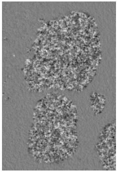
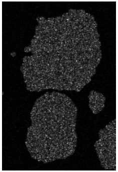
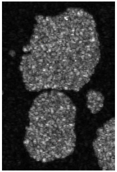
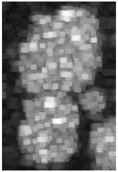
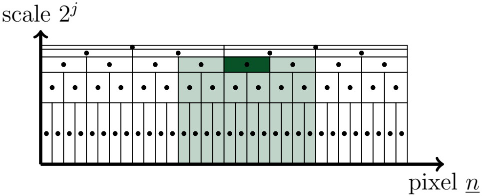
Scale $a = 2^1$ $a = 2^2$ … $a = 2^5$











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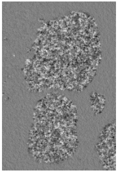

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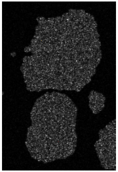

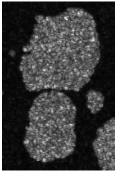

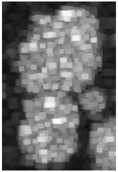

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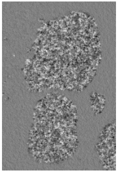

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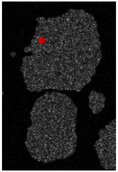

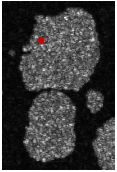

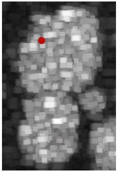

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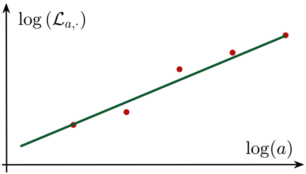

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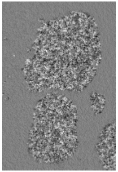

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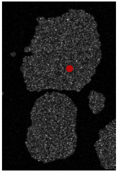

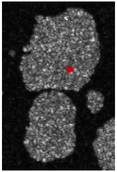

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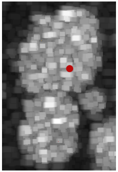

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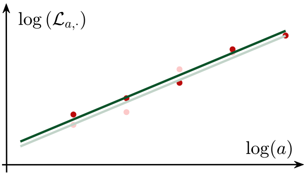

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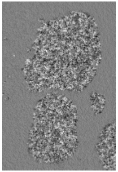

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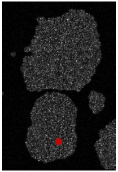

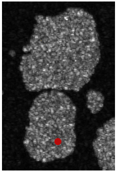

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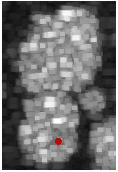

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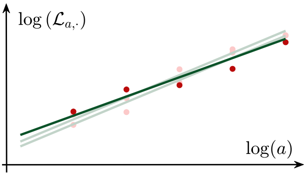

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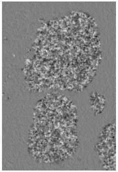

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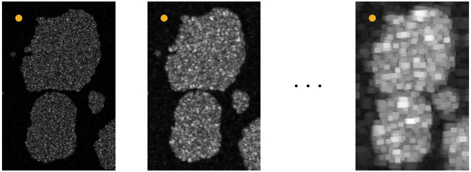

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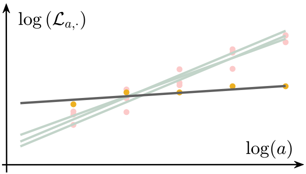

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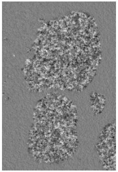

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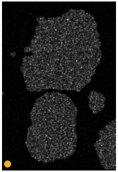

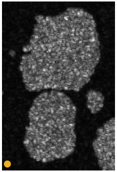

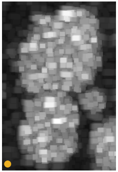

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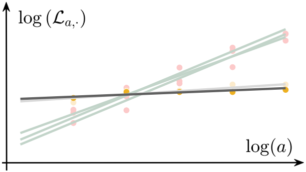

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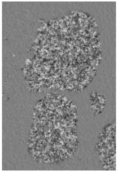
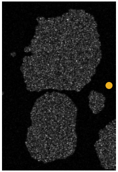
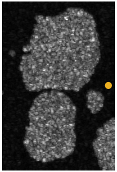
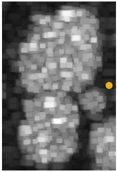


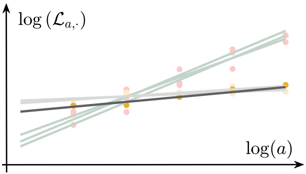






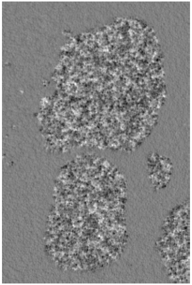
Proposition (Jaffard, 2004), (Wendt, 2008)

$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}$$



Linear regression $\log(\mathcal{L}_{a,\cdot}) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

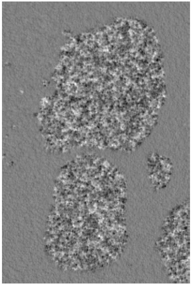
Textured image



Linear regression $\log(\mathcal{L}_{a,\cdot}) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\| \log(\mathcal{L}_{a,\cdot}) - \log(a)\boldsymbol{h} - \boldsymbol{v} \right\|^2$$

Textured image



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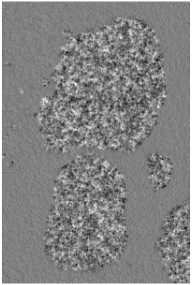
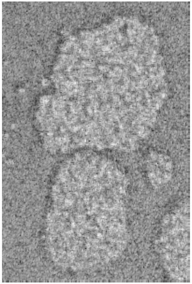
Textured image



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

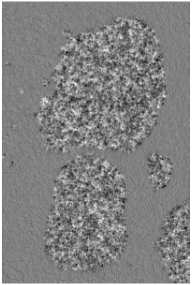
Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$



Linear regression $\underbrace{\mathbb{E}\log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{expected value}} = \log(a)\underbrace{\bar{\boldsymbol{h}}}_{\text{regularity}} + \underbrace{\bar{\boldsymbol{v}}}_{\propto\log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Textured image



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

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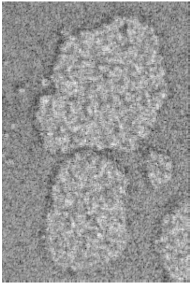


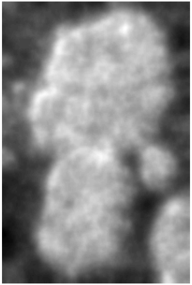
$\longrightarrow$ large estimation variance

Filter smoothing (linear)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Smoothing



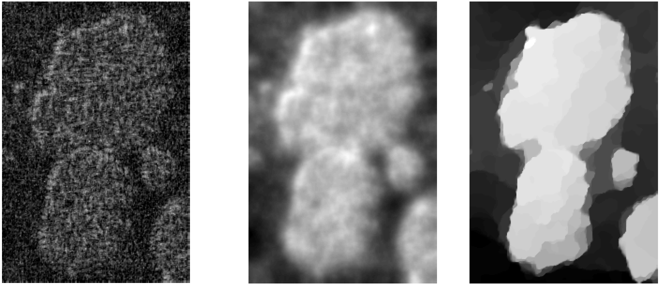
Filter smoothing (linear)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

ROF denoising (nonlinear)

$$\operatorname*{argmin}_{\boldsymbol{h}} \; \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

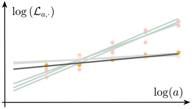
Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Smoothing | ROF



$\longrightarrow$ cumulative estimation variance and regularization bias

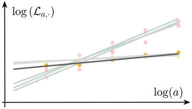
$$\underbrace{\sum_a \|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}}$$

$\rightarrow$ fidelity to the log-linear model

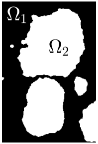


$$\underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to the log-linear model

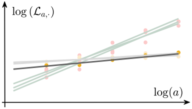


$\rightarrow$ favors piecewise constancy



$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model



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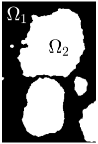

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

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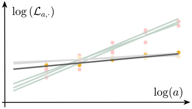
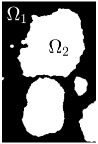
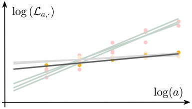




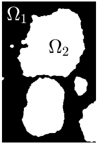
Finite differences $\mathbf{D}_1\boldsymbol{x}$ (horizontal), $\mathbf{D}_2\boldsymbol{x}$ (vertical) at each pixel

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model



→ favors piecewise constancy



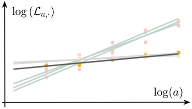
Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

<u>Free:</u> $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

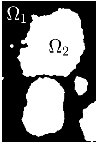
$$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model



→ favors piecewise constancy



Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

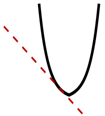
Free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

$$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

Co-localized: $\boldsymbol{h}$, $\boldsymbol{v}$ are **concomitantly** piecewise constant

$$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \|[\alpha\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}]\|_{2,1}$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

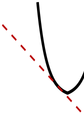


$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

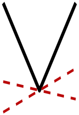
- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau\nabla\varphi(\boldsymbol{x}^n)$ $\quad\boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})$

Functionals minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \;\; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;\; + \;\; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



nonsmooth

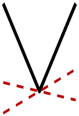


- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$ $\qquad \boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})$
- implicit subgradient descent: proximal point algorithm

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \boldsymbol{u}^n, \;\; \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \;\Leftrightarrow\; \boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



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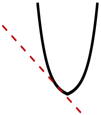
- splitting proximal algorithm (Chambolle, 2011)

$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau \mathbf{D}^\top \boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



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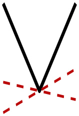

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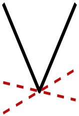
$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

Accelerated algorithm based on strong-convexity

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth

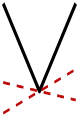


Primal-dual algorithm (Chambolle, 2011)

δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



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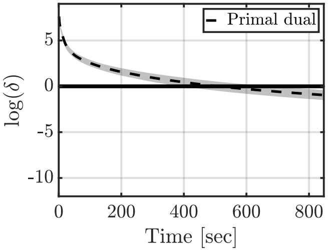
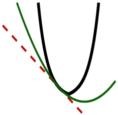


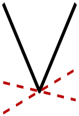
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



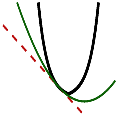
μ-strongly convex

nonsmooth



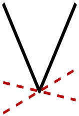
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



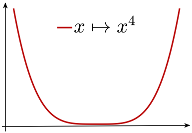
$\boldsymbol{\mu}$-strongly convex

nonsmooth

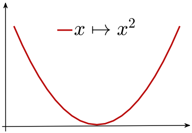


Strong-convexity

- φ $\boldsymbol{\mu}$-strongly convex iff $\varphi - \frac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convex

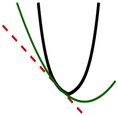


✓ strictly convex
✗ non strongly convex



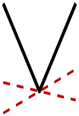
✓ strictly convex
✓ 1-strongly convex

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



$\boldsymbol{\mu}$-strongly convex

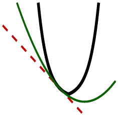
nonsmooth



Strong-convexity

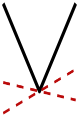
- φ $\boldsymbol{\mu}$-strongly convex iff $\varphi - \frac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convex
- φ $\mathcal{C}^2$ with Hessian matrix $\boldsymbol{H}\varphi \succeq 0 \implies \boldsymbol{\mu} = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



$\boldsymbol{\mu}$-strongly convex

nonsmooth



Strong-convexity

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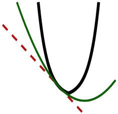
Proposition (Pascal, 2019)

$\sum_a \|\log \boldsymbol{\mathcal{L}} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2$ is $\boldsymbol{\mu}$-strongly convex.

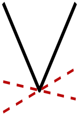
$a_{\min} = 2^1$, $a_{\max}$	2^2	2^3	2^4	2^5	2^6
$\boldsymbol{\mu} = \min \mathrm{Sp}\left(2\boldsymbol{\Phi}^\top\boldsymbol{\Phi}\right)$	0.29	**0.72**	1.20	1.69	2.20

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



μ-strongly convex

nonsmooth



Accelerated Primal-dual algorithm *(Chambolle, 2011)*

for $n = 0, 1, \ldots$ $\qquad \boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})$

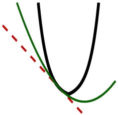
$$\boldsymbol{y}^{n+1} = \text{prox}_{\sigma_n(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma_n \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

$$\boldsymbol{x}^{n+1} = \text{prox}_{\tau_n\|\boldsymbol{\mathcal{L}} - \Phi\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau_n \mathbf{D}^\top \boldsymbol{y}^{n+1}\right)$$

$$\theta_n = \sqrt{1 + 2\mu\tau_n}, \quad \tau_{n+1} = \tau_n/\theta_n, \quad \sigma_{n+1} = \theta_n\sigma_n$$

$$\bar{\boldsymbol{x}}^{n+1} = \boldsymbol{x}^{n+1} + \theta_n^{-1}\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;+\; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



$\boldsymbol{\mu}$-strongly convex

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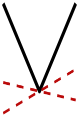
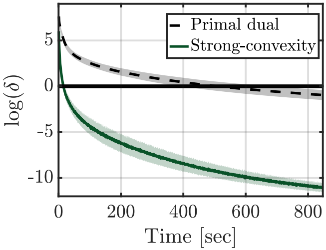


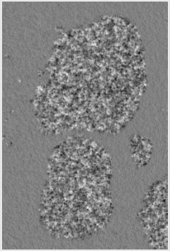
Accelerated Primal-dual algorithm *(Chambolle, 2011)*

δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$



$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \;\; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;\; + \;\; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

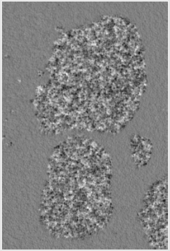
Textured image



Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

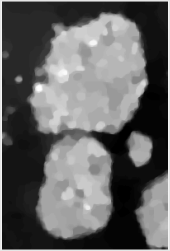
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \;\sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;+\; \lambda\, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image

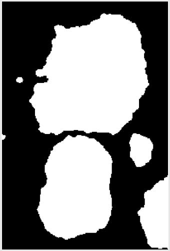


Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours $\widehat{\boldsymbol{h}}^{\mathrm{C}}$



Threshold estimate† $T\widehat{\boldsymbol{h}}^{\mathrm{C}}$



†(Cai, 2013)

Threshold-ROF on $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

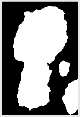
(Nafornita, 2014), (Pustelnik, 2016)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \ \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Lin. reg.

ROF

Threshold



Only based on regularity $\boldsymbol{h}$.

Threshold-ROF on $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

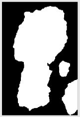
(Nafornita, 2014), (Pustelnik, 2016)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \; \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Lin. reg.

ROF

Threshold

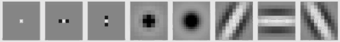


Only based on regularity $\boldsymbol{h}$.

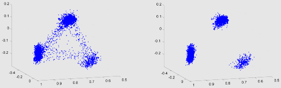
Factorization based segmentation[†]

(Yuan, 2015)

(i) local histograms



(ii) matrix factorization



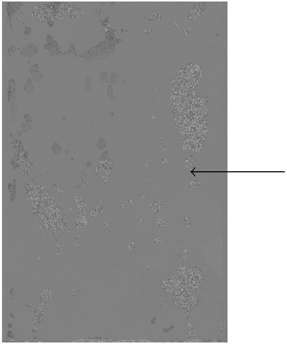
(a) (b)

Fig. 2. Scatterplot of features in subspace. (a) Scatterplot of features projected onto the 3-d subspace. (b) Scatterplot after removing features with high edgeness.

[†] https://sites.google.com/site/factorizationsegmentation/

Multiphase flow through porous media

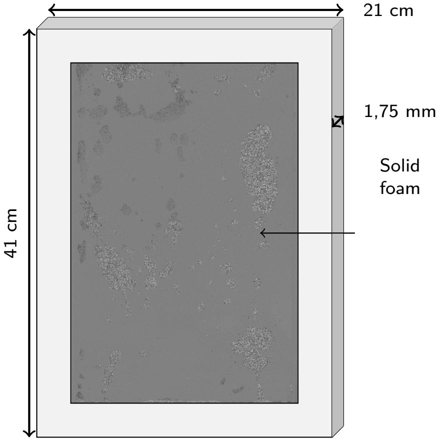
Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



Solid foam

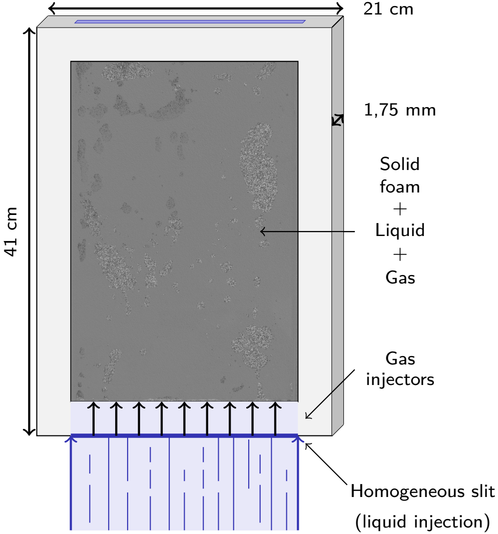
Multiphase flow through porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



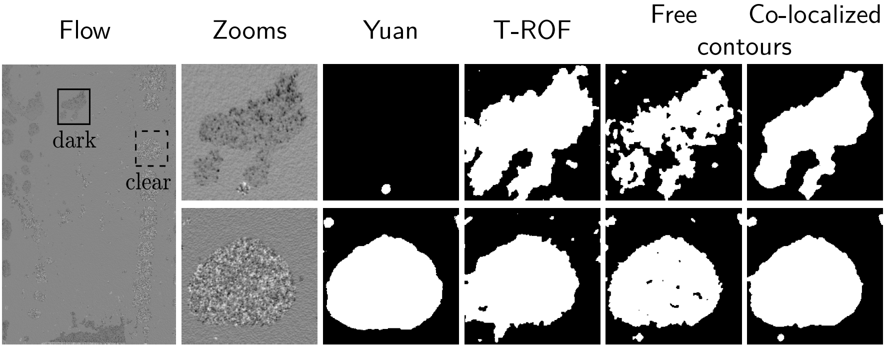
Multiphase flow through porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)

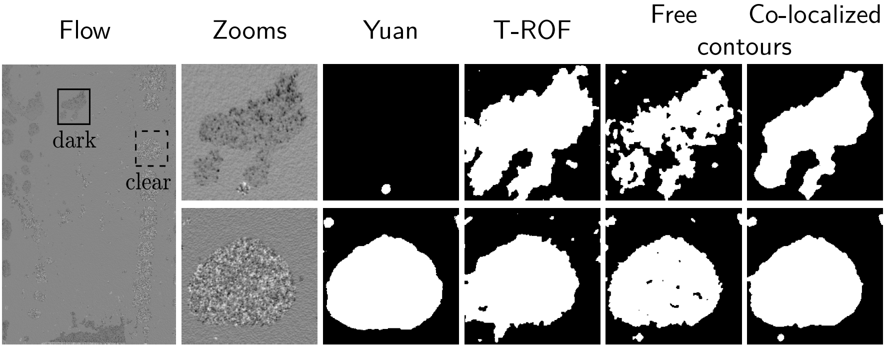


- → 1600 × 1100 pixels
- → video: ∼ 1000 images
- → phase diagram: ∼ 10 flow rates

Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$



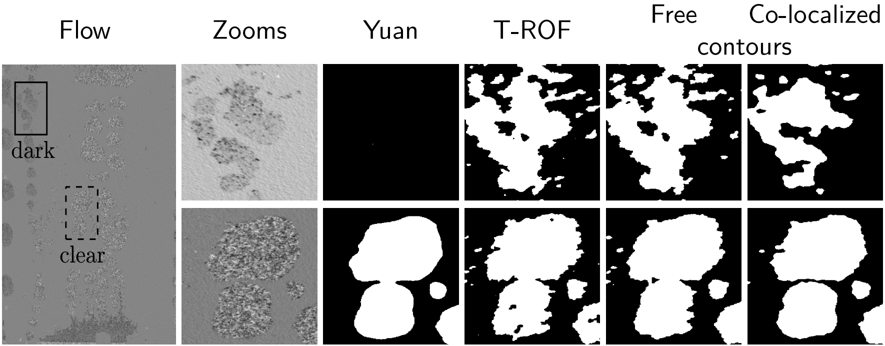
Low activity: $Q_{\mathrm{G}} = 300\text{mL/min}$ - $Q_{\mathrm{L}} = 300\text{mL/min}$



Liquid: $h_{\mathrm{L}} = 0.4$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$ (dark bubbles)
$\sigma^2_{\mathrm{clear}} = 10^{-1}$ (clear bubbles).

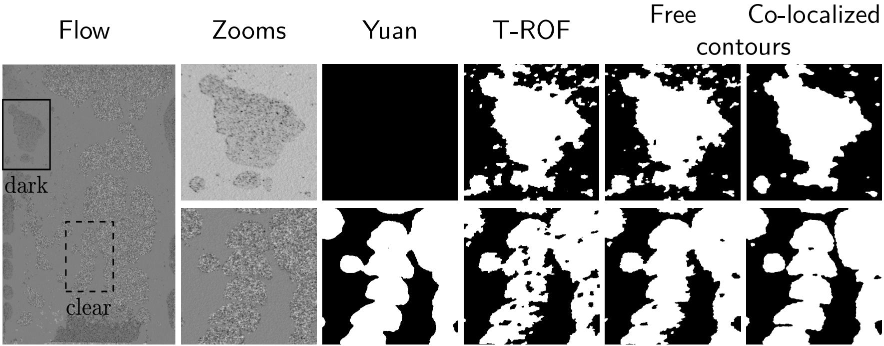
Transition: $Q_{\mathrm{G}} = 400$mL/min - $Q_{\mathrm{L}} = 700$mL/min



Liquid: $h_{\mathrm{L}} = 0.4 \qquad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9 \quad \left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

High activity: $Q_{\mathrm{G}} = 1200\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$

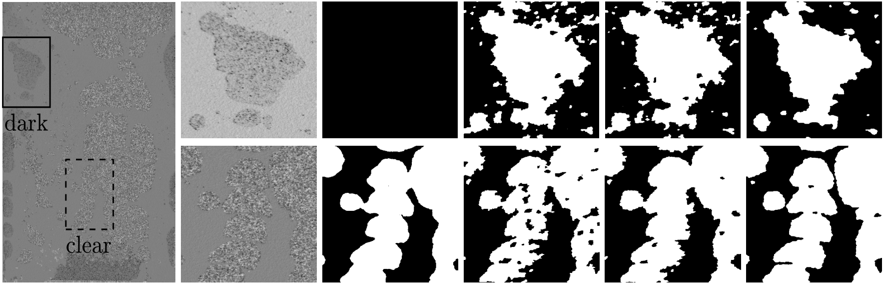


Liquid: $h_{\mathrm{L}} = 0.4$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

High activity: $Q_{\mathrm{G}} = 1200\text{mL/min}$ - $Q_{\mathrm{L}} = 300\text{mL/min}$

Flow	Zooms	Yuan	T-ROF	Free contours	Co-localized contours



Computational load		1s	12s	700s	**2100s**

Liquid: $h_{\mathrm{L}} = 0.4 \qquad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9 \quad \left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

$(\lambda, \alpha) = (0, 0)$

Regularization parameters selection

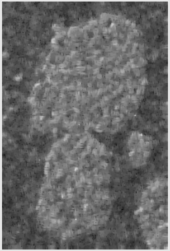
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

Regularization parameters selection

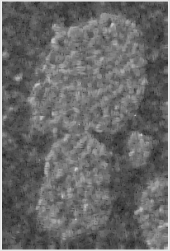
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

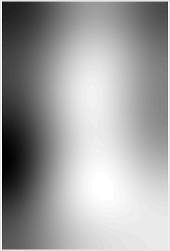
$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

$(\lambda, \alpha) = (500, 500)$



too large

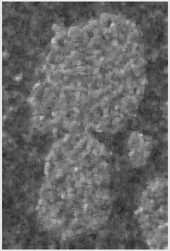
Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ — Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

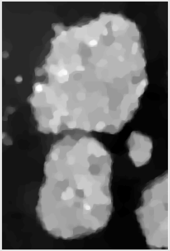
$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



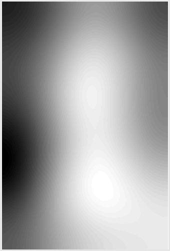
too small

$(\lambda^\dagger, \alpha^\dagger) = (11.5, 0.8)$



optimal

$(\lambda, \alpha) = (500, 500)$



too large

What *optimal* means? How to determine $\lambda^\dagger$ and $\alpha^\dagger$?

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \operatorname*{argmin}_{\boldsymbol{h},\boldsymbol{v}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$

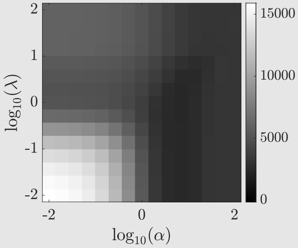
Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\text{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$

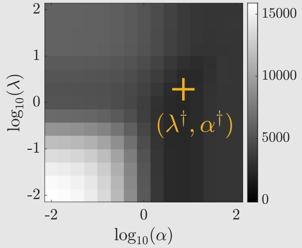


$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$

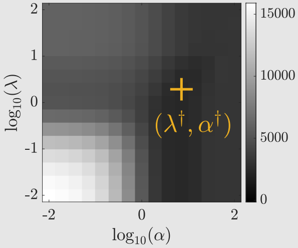


$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

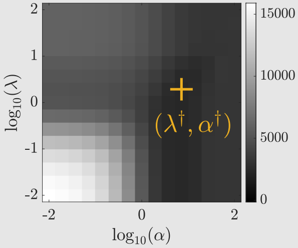
?

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

?

Stein Unbiased Risk Estimate
(SURE)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

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Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

$$\textbf{Ex.} \quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \, \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

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Quadratic error $R(\lambda) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \stackrel{?}{=} \mathbb{E}_{\boldsymbol{\zeta}} \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ unknown

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Theorem (Stein, 1981)

Let $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\boldsymbol{\zeta} \mapsto \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \boldsymbol{\zeta} \rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$.

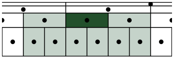
$$\widehat{R}(\boldsymbol{y}; \lambda) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\rho^2 \operatorname{tr}\left(\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)\right) - \rho^2 P$$
$$\Longrightarrow R(\lambda) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{y}; \lambda)].$$

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\quad$ $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$ $\quad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\quad$ $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$ $\quad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

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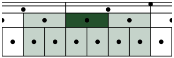


$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Theorem (Pascal, 2020)

Let $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\boldsymbol{\zeta} \mapsto \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \mathbf{A}\boldsymbol{\zeta} \rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$.

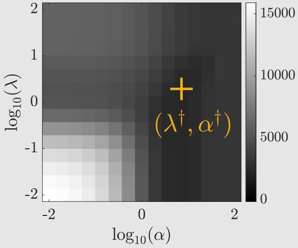
$$\widehat{R}(\boldsymbol{\Lambda}) \triangleq \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\operatorname{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi} \partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right) - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

$$\Longrightarrow R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{\Lambda})].$$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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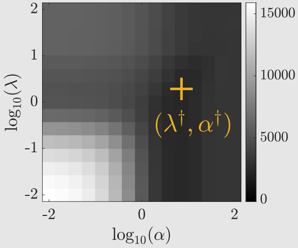

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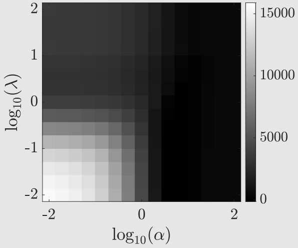

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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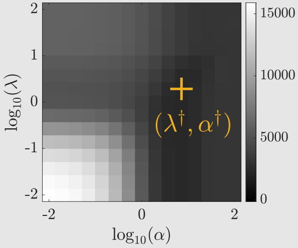

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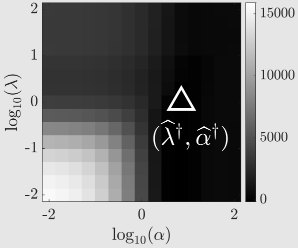

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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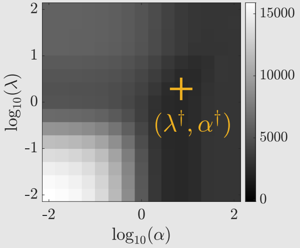

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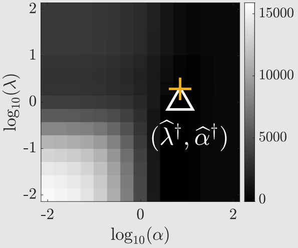

Parameter tuning (Automatic selection)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\text{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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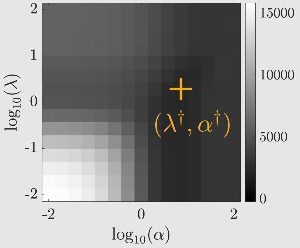

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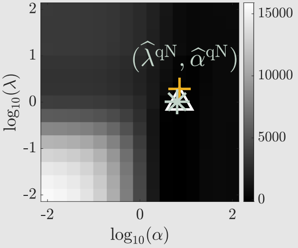
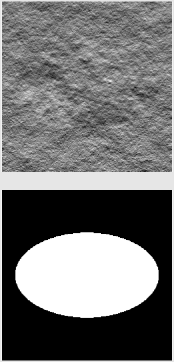


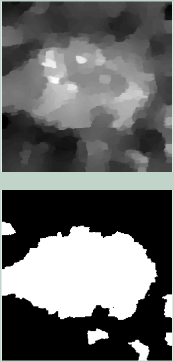
Automated selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \operatorname*{argmin}_{\boldsymbol{h},\boldsymbol{v}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v};\alpha)$$

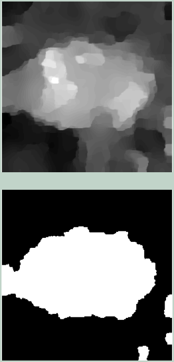
Example



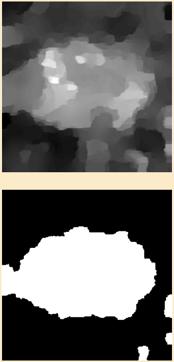
$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}};\lambda^\dagger,\alpha^\dagger)$ (grid)



$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}};\widehat{\lambda}^\dagger,\widehat{\alpha}^\dagger)$ (grid)



$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}};\widehat{\lambda}^{\mathrm{qN}},\widehat{\alpha}^{\mathrm{qN}})$ (quasi-Newton)



225 calls of the estimator over the grid v.s. **40** for quasi-Newton

Take home messages

- Fractal texture model based on local *regularity* and *variance*
    - ∗ appropriate for real-world texture characterization
    - ∗ complementary attributes able to finely discriminate

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  - significant decrease of the estimation error
  - accurate and regular contours thanks to co-localized penalization

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  - ∗ possibility to manage huge amount of data
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MATLAB toolbox available: **github.com/bpascal-fr/gsugar**