The Kravchuk transform:

A novel covariant representation for discrete signals amenable to zero-based detection tests

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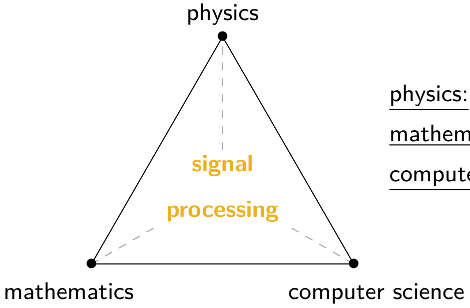
Probability and Statistics Seminar,
Laboratoire J. A. Dieudonné, Université Côte d'Azur

Signal processing aims to extract **information** from **data**.

Data of very diverse types:

- measurements of a physical quantity,
- biological or epidemiological indicators,
- data produced by human activities.

The Golden triangle of signal processing



physics: modeling of phenomena

mathematics: formalization & evaluation

computer science: efficient implementation

inspired from P. Flandrin

Outline of the presentation

- Signal detection: the role of representations

. . . .

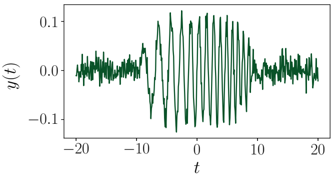
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- Covariance principle and stationary point processes

. . . .

- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

Time and frequency: two dual descriptions of temporal signals

A continuous finite energy **signal** is a function of time $y(t)$ with $y \in L^2_{\mathbb{C}}(\mathbb{R})$.



- electrical cardiac activity,
- audio recording,
- seismic activity,
- light intensity on a photosensor
- ...

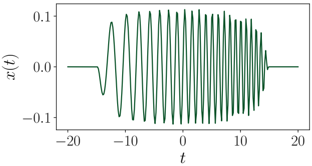
Information of interest:

– time events, e.g., an earthquake and its replica
– frequency content, e.g., monitoring of the heart beating rate

time	**frequency**
ever-changing world	waves, oscillations, rhythms
marker of events and evolutions	intrinsic mechanisms

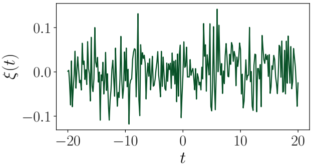
A **chirp** is a transient waveform modulated in amplitude and frequency:

$$x(t) = A_\nu(t) \sin\left(2\pi\left(f_1 + (f_2 - f_1)\frac{t+\nu}{2\nu}\right)t\right)$$



White noise is a random variable $\xi(t)$ such that

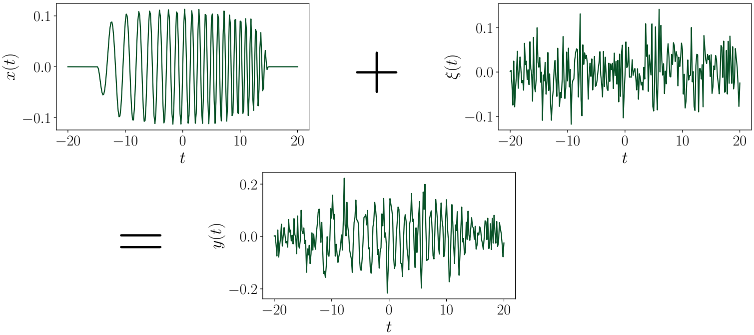
$$\mathbb{E}[\xi(t)] = 0 \quad \text{and} \quad \mathbb{E}[\overline{\xi(t)}\xi(t')] = \delta(t - t')$$



P. Flandrin: ‘A signal is characterized by a structured organization.’

Signal-plus-noise observation model

Noisy observations $y(t) = \mathrm{snr} \times x(t) + \xi(t)$

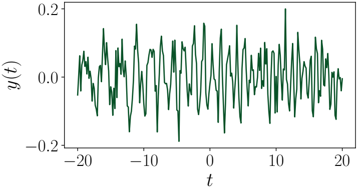


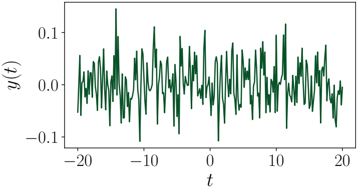
Signal processing task:

Given an observation $y(t)$

detection decides whether there is an underlying signal or only noise.

Direct observation





✗ hard to distinguish between oscillations and fluctuations

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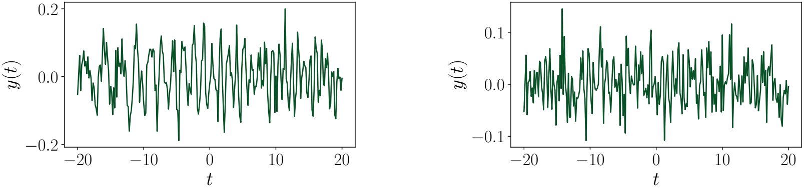

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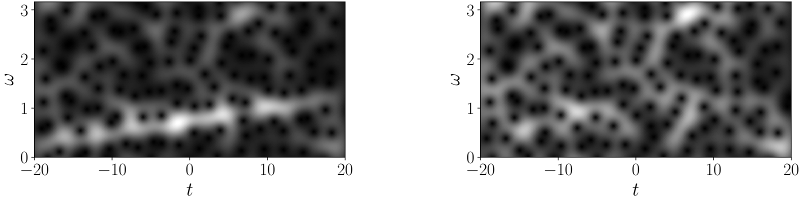
Time-frequency representation



✓ lines of local maxima: structures are simpler to capture

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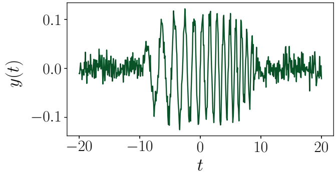
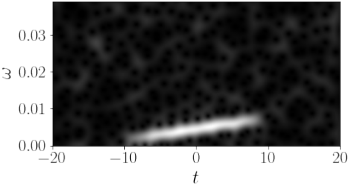
Time *and* frequency Short-Time Fourier Transform with window h:

$$V_h y(t,\omega) \triangleq \int_{-\infty}^{\infty} \overline{y(u)} h(u-t) \exp(-\mathrm{i}\omega u) \, \mathrm{d}u$$



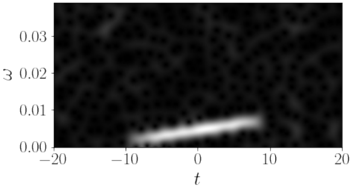


Energy density interpretation $S_h y(t,\omega) = |V_h y(t,\omega)|^2$ the *spectrogram*

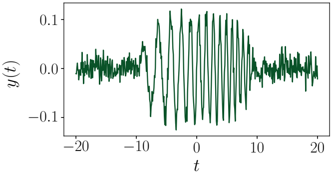
$$\int \int_{-\infty}^{+\infty} S_h y(t,\omega) \, \mathrm{d}t \frac{\mathrm{d}\omega}{2\pi} = \int_{-\infty}^{+\infty} |x(t)|^2 \, \mathrm{d}t \quad \text{if} \quad \|h\|_2^2 = 1$$

Signal, i.e., information of interest: regions of **maximal** energy.

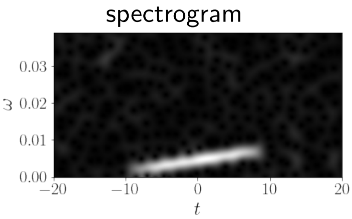
spectrogram

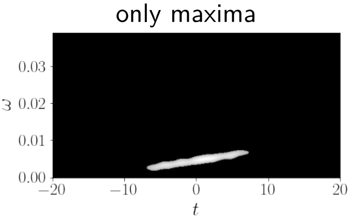


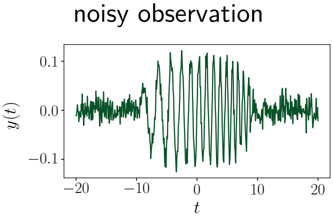
noisy observation



Denoising in the time-frequency plane: $y = \text{snr} \times x + \xi$, $\quad$ $\text{snr} = 2$

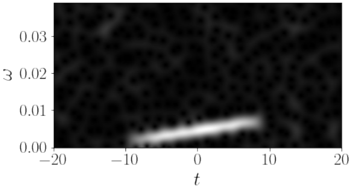




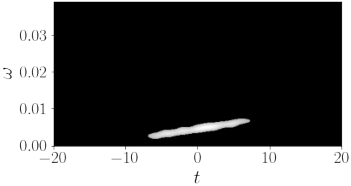


Inversion formula $\quad y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u,\omega)} h(t-u) \exp(\mathrm{i}\omega u) \, \mathrm{d}u \frac{\mathrm{d}\omega}{2\pi}$

spectrogram



only maxima



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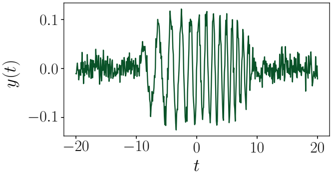
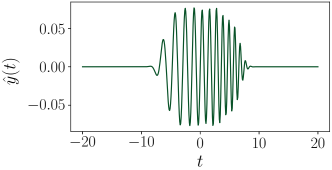


estimate



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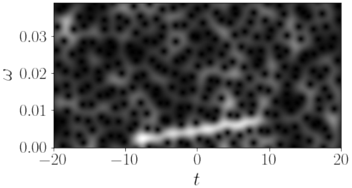
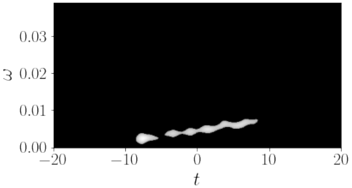
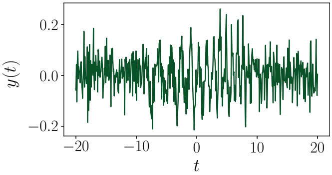


only maxima



noisy observation



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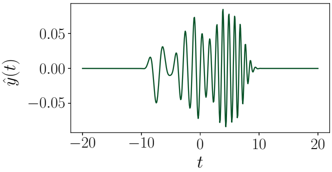

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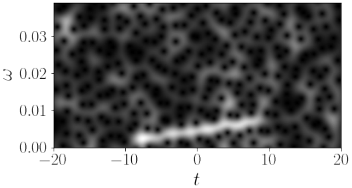

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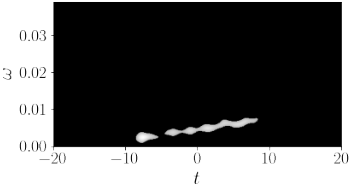

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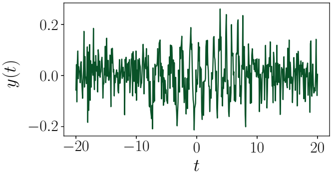

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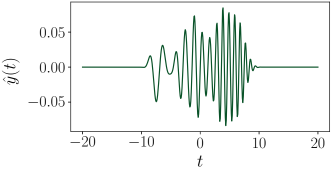

Maxima extraction: *reassignment, synchrosqueezing, ridge extraction*

(Meignen et al., 2017)

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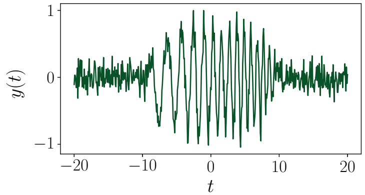
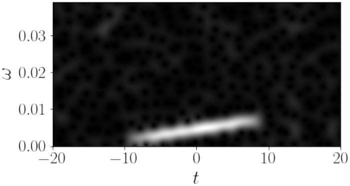
Restriction to the **circular Gaussian window**: $g(t) = \pi^{-1/4} \mathrm{e}^{-t^2/2}$

Look for the zeros, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.





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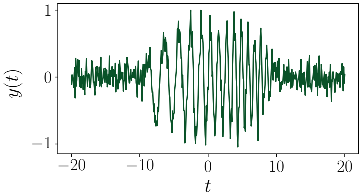

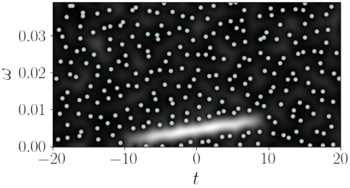

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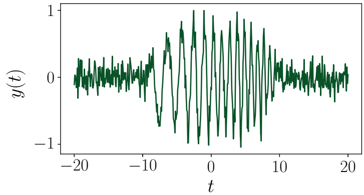

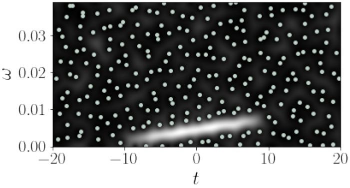

Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

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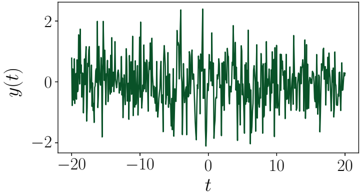

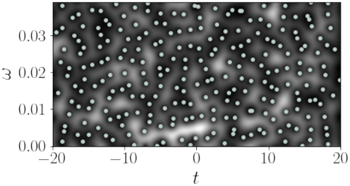

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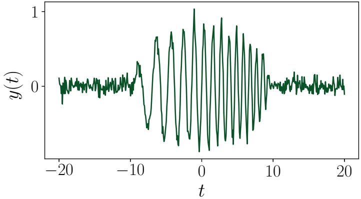

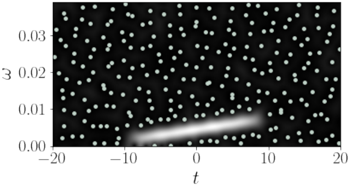

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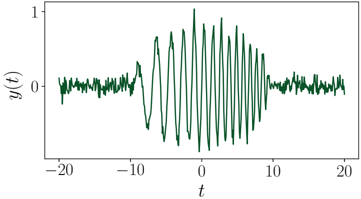

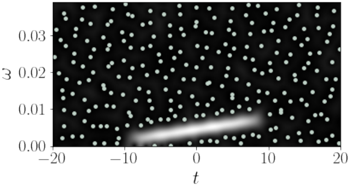

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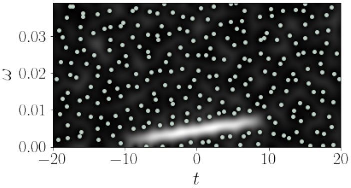
What can be said theoretically about the zeros of the spectrogram?

Idea assimilate the time-frequency plane with $\mathbb{C}$ through $z = (\omega + \mathrm{i}t)/\sqrt{2}$



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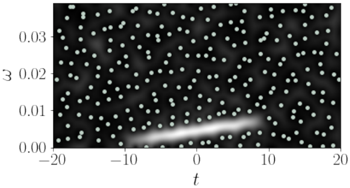

Bargmann factorization

$$V_g y(t, \omega) = \mathrm{e}^{-|z|^2/2} \mathrm{e}^{-\mathrm{i}\omega t/2} By(z)$$

g the circular Gaussian window

Bargmann transform of the signal y

$$By(z) \triangleq \pi^{-1/4} \mathrm{e}^{-z^2/2} \int_{\mathbb{R}} \overline{y(u)} \exp\left(\sqrt{2}uz - u^2/2\right) \mathrm{d}u,$$

By is an **entire** function, almost characterized by its infinitely many zeros:

$$By(z) = z^m \mathrm{e}^{C_0 + C_1 z + C_2 z^2} \prod_{n \in \mathbb{N}} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2\right).$$

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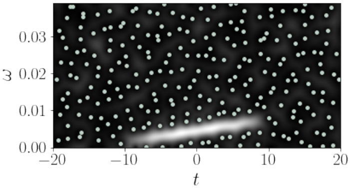

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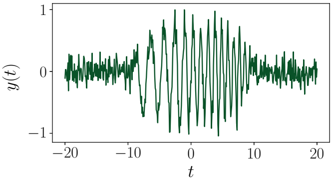
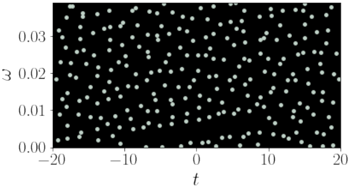
Theorem The zeros of the Gaussian spectrogram $V_g y(t, \omega)$

- coincide with the zeros of the **entire** function By,
- hence are **isolated** and constitute a **Point Process**,
- which almost completely **characterizes** the spectrogram.

(Flandrin, 2015)





Advantages of working with the zeros

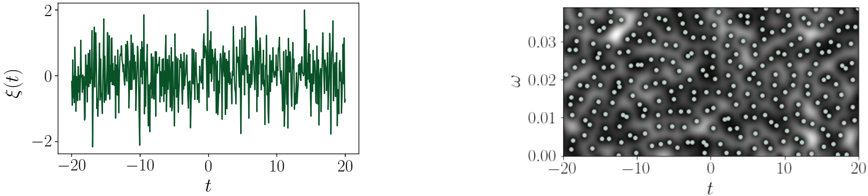
- Easy to find compared to relative maxima.
- Form a robust pattern in the time-frequency plane.
- Require little memory space for storage.
- Efficient tools were recently developed in **stochastic geometry**.

Signal detection based on the spectrogram zeros

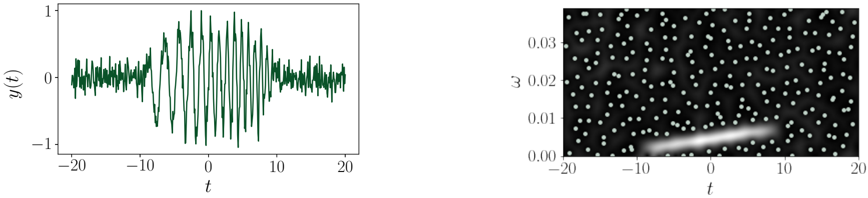
(Bardenet, Flamant & Chainais, 2020)

- $\mathbf{H}_0$ white noisy only, i.e., $y(t) = \xi(t)$
- $\mathbf{H}_1$ presence of a signal, i.e., $y(t) = \text{snr} \times x(t) + \xi(t)$, $\text{snr} > 0$

null hypothesis



alternative hypothesis



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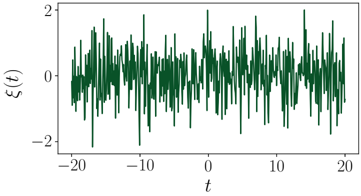
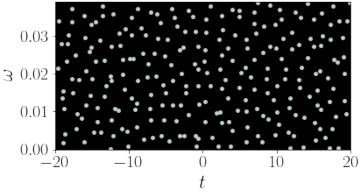
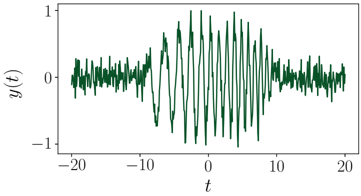
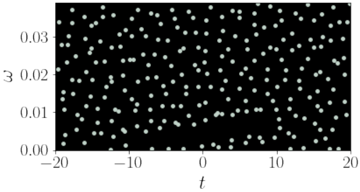




alternative hypothesis

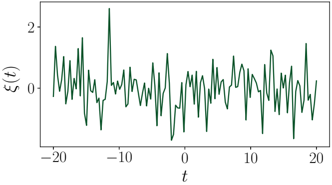


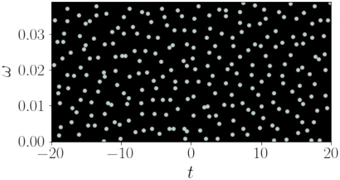


Continuous complex white Gaussian noise

(Bardenet et al., 2020), (Bardenet & Hardy, 2020)

$$\xi(t) = \sum_{n=0}^{\infty} \xi[n] h_n(t),\ \xi[n] \sim \mathcal{N}_{\mathbb{C}}(0,1), \quad \{h_n, k = 0, 1, \ldots\} \text{ Hermite functions}$$





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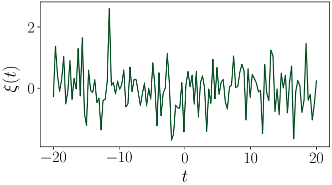
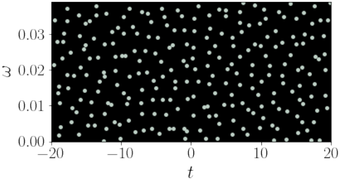
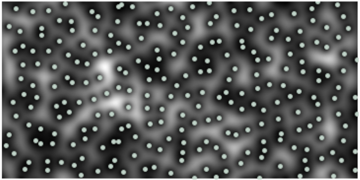




Theorem $V_g\xi(t,\omega) = \mathrm{e}^{-|z|^2/4}\mathrm{e}^{-\mathrm{i}\omega t/2}\,\mathrm{GAF}_{\mathbb{C}}(z)$ (Bardenet & Hardy, 2021)

$\mathrm{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$ the *planar Gaussian Analytic Function* and $z = \frac{\omega + \mathrm{i}t}{\sqrt{2}}$.



$$V_g\xi(t,\omega) \overset{\text{non-vanishing}}{\propto} \mathrm{GAF}_{\mathbb{C}}(z)$$

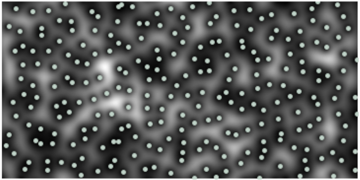
$$z = (\omega + \mathrm{i}t)/\sqrt{2}$$

Zeros of $\mathrm{GAF}_{\mathbb{C}}$: random set of points forming a **Point Process** characterized by a probability distribution on point configurations

Properties of the Point Process of the zeros of $\mathrm{GAF}_{\mathbb{C}}$:

- invariant under the isometries of $\mathbb{C}$, i.e., **stationary**,
- has a uniform density $\rho^{(1)}(z) = \rho^{(1)} = 1/\pi$,
- explicit two-point correlation function $\rho^{(2)}(z,z') = \rho^{(2)}(|z-z'|)$,
- scaling of the *hole probability*: $r^{-4}\log p_r \to -3\mathrm{e}^2/4$, as $r \to \infty$

$$p_r = \mathbb{P}\,(\text{no point in the disk of center 0 and radius } r)$$



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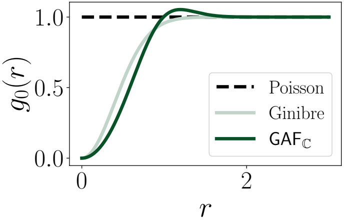
Zeros of $\mathrm{GAF}_{\mathbb{C}}$: random set of points forming a **Point Process** characterized by a probability distribution on point configurations



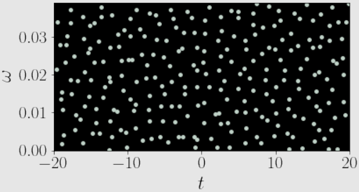
Pair correlation $\rho^{(2)}(z,z')\mathrm{d}z\mathrm{d}z' =$

$$\mathbb{P}\left(1 \text{ point in } B(z,\mathrm{d}z) \text{ and } 1 \text{ in } B(z',\mathrm{d}z')\right)$$

The point process of the zeros of the spectrogram is not **determinantal**.

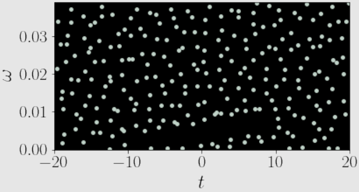
Purpose: summary statistic s, such that for $y = \text{snr} \times x + \xi$

$\mathbb{E}[s(y)|\mathbf{H}_0]$ small



snr $= 0$

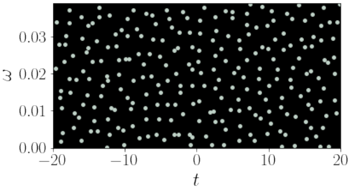
$\mathbb{E}[s(y)|\mathbf{H}_1]$ large



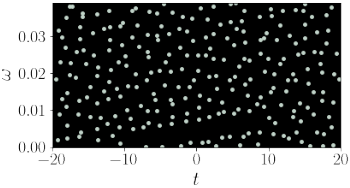
snr > 0

'**Large value of $s(y)$ is a strong indication that there is a signal.**'

Tools from **stochastic geometry** to capture spatial statistics of the zeros.

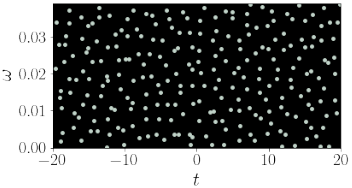


snr $= 0$

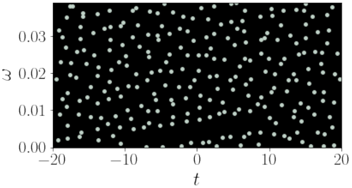


snr > 0

The signal creates **holes** in the zeros pattern: **sedond order** statistics.



snr $= 0$



snr > 0

The signal creates **holes** in the zeros pattern: **sedond order** statistics.

A functional statistic: the empty space function

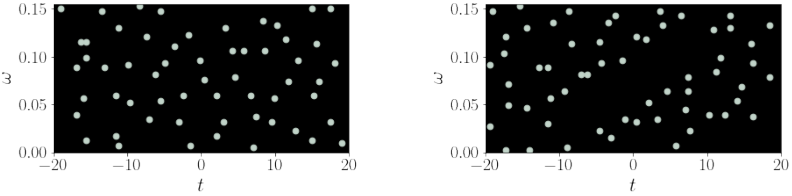
Z a stationary point process, z_0 any reference point

$$F(r) = \mathbb{P}\left(\inf_{z_i \in Z} \mathrm{d}(z_0, z_i) < r\right)$$

$\rightarrow$ probability to find a zero at distance less than r from z_0

Estimation of the F-function of a **stationary** Point Process (Møller, 2007)

$$F(r) = \mathbb{P}\left(\inf_{z_i \in Z} \mathrm{d}(z_0, z_i) < r\right) : \textbf{ empty space function}$$



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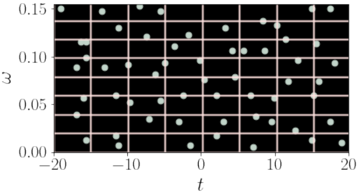
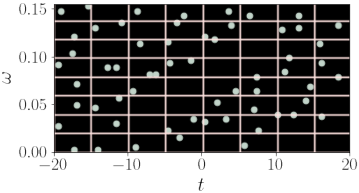
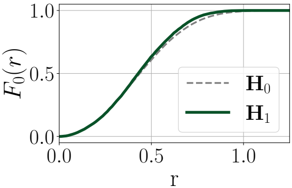




$$\widehat{F}(r) = \frac{1}{N_\#} \sum_{j=1}^{N_\#} \mathbf{1}\left(\inf_{z \in \mathbf{Zeros}} \mathrm{d}\left(z_j, z\right) < r\right)$$



Signal detection based on the spectrogram zeros

Estimation of the F-function of a **stationary** Point Process (Møller, 2007)

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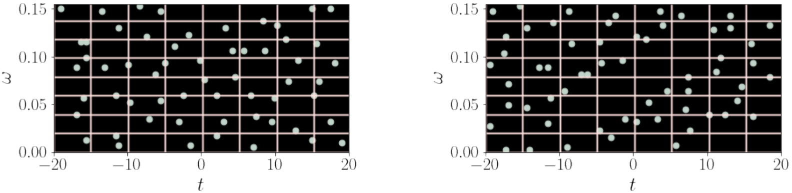

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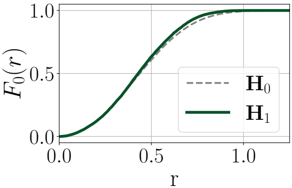
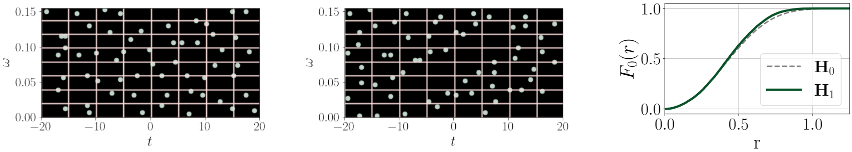


- Monte Carlo envelope test based on the discrepancy between $\widehat{F}$ and F_0

Monte Carlo envelope test



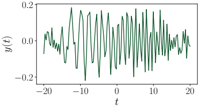
$$s(\boldsymbol{y}) = \sqrt{\int_0^{r_{\max}} \left|\widehat{F}_{\boldsymbol{y}}(r) - F_0(r)\right|^2 \mathrm{d}r},$$

<u>Test settings:</u> α level of significance, m number of samples under $\mathbf{H}_0$

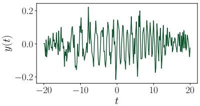
Index k, chosen so that $\alpha = k/(m+1)$

(i) generate m independent samples of complex white Gaussian noise;
(ii) compute their summary statistics $s_1 \geq s_2 \geq \ldots \geq s_m$;
(iii) compute the summary statistic of the observation $\boldsymbol{y}$ under concern;
(iv) if $s(\boldsymbol{y}) \geq s_k$, then reject the null hypothesis with confidence $1 - \alpha$.

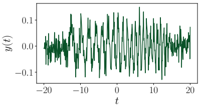
Detection of a noisy chirp of duration $2\nu = 30$ s



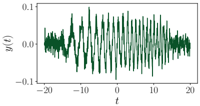
$N = 128$



$N = 256$



$N = 512$



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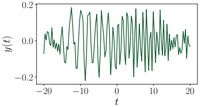

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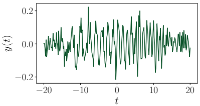

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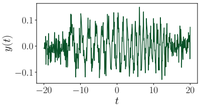

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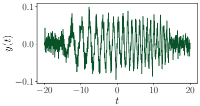

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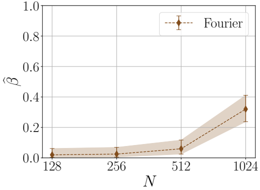
Performance: power of the test computed over 200 samples



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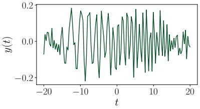

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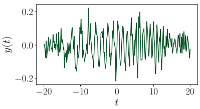

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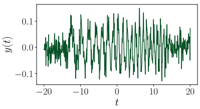

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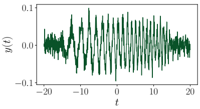

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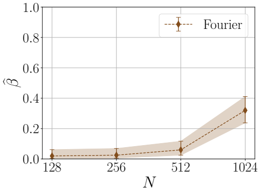
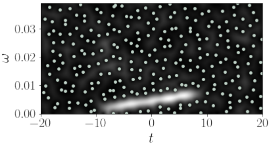


- ✓ Fast Fourier Transform ;
- ✗ Low detection power ;
- ✗ Requires large number of samples



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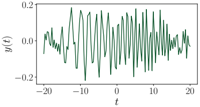

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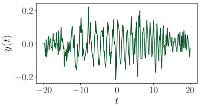

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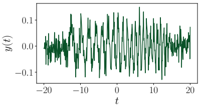

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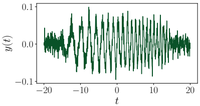

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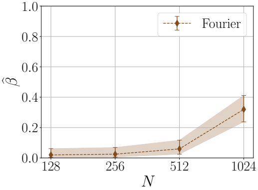

✓ Fast Fourier Transform ;

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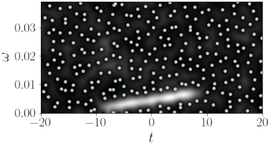
✗ Requires large number of samples

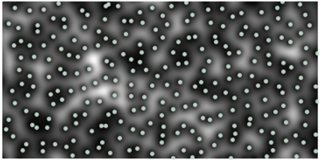


Limitations:

- Necessary discretization of the STFT: arbitrary resolution ;
- Observe only a bounded window: edge corrections to compute $\widehat{F}(r)$.

Unbounded phase space $\mathbb{C}$

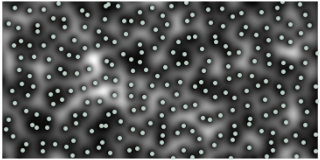
Short-Time Fourier Transform



$$V_g\xi(t,\omega) \propto \mathrm{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$$

$\rightarrow$ edge corrections

Unbounded phase space $\mathbb{C}$

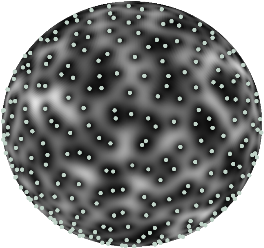


Short-Time Fourier Transform

$$V_g\xi(t,\omega) \propto \mathrm{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$$

$\rightarrow$ edge corrections

Compact phase space S^2?



New transform?

$$? \quad \propto \mathrm{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^{N} \boldsymbol{\xi}[n] \sqrt{\binom{N}{n}} z^n$$

stereographic projection $z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$

$\rightarrow$ spherical coordinates $(\vartheta, \varphi) \in S^2$

$\rightarrow$ no border!

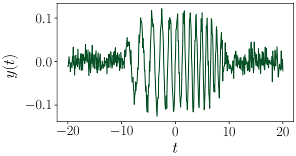
- Signal detection: the role of representations

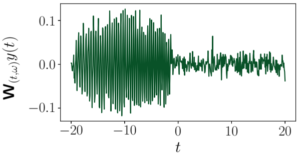
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- **Covariance principle and stationary point processes**

- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

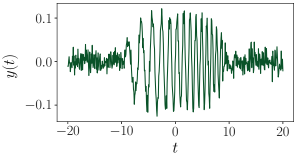
Time and frequency shifts

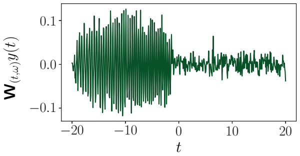
$$\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$$



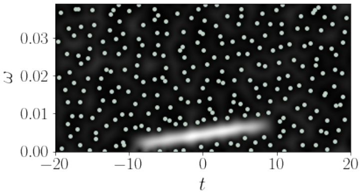


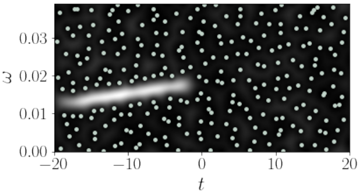
Time and frequency shifts $\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$





$$V_h[\boldsymbol{W}_{(t,\omega)} y](t',\omega') \overset{\text{(covariance)}}{=} \mathrm{e}^{-\mathrm{i}(\omega'-\omega)t} V_h y(t'-t, \omega'-\omega),$$





Time and frequency shifts $\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$

$$\left| V_h[\boldsymbol{W}_{(t,\omega)} y](t', \omega') \right|^2 \overset{\text{(covariance)}}{=} \left| V_h y(t'-t, \omega'-\omega) \right|^2 ,$$

Time and frequency shifts $\boldsymbol{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$

$$\left|V_h[\boldsymbol{W}_{(t,\omega)}y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Complex white Gaussian noise $\widetilde{\xi} = \boldsymbol{W}_{(t,\omega)}\xi$

- $\mathbb{E}[\widetilde{\xi}(u)] \quad = e^{-\mathrm{i}\omega u}\mathbb{E}[\xi(u-t)] \quad = 0$

Time and frequency shifts $\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$

$$\left| V_h[\boldsymbol{W}_{(t,\omega)} y](t',\omega') \right|^2 \overset{\text{(covariance)}}{=} \left| V_h y(t'-t,\omega'-\omega) \right|^2,$$

Complex white Gaussian noise $\widetilde{\xi} = \boldsymbol{W}_{(t,\omega)} \xi$

- $\mathbb{E}[\widetilde{\xi}(u)] = e^{-\mathrm{i}\omega u} \mathbb{E}[\xi(u-t)] = 0$
- $\mathbb{E}[\overline{\widetilde{\xi}(u)} \widetilde{\xi}(u')] = e^{\mathrm{i}\omega(u-u')} \mathbb{E}[\overline{\xi(u)} \xi(u')] = \delta(u-u')$

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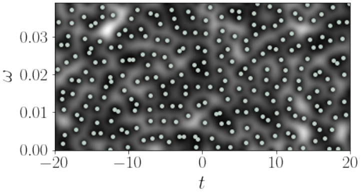
Invariance under time-frequency shifts:

$$\widetilde{\xi} = \boldsymbol{W}_{(t,\omega)}\xi \overset{\text{(law)}}{=} \xi$$



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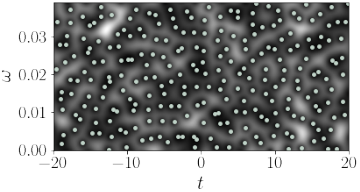

Covariance is the key to get **stationarity**: how to get covariant transforms?

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$$\left|V_h[\boldsymbol{W}_{(t,\omega)}y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Algebraic interpretation: covariance under a symmetry group

Time and frequency shifts $\boldsymbol{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$

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Weyl-Heisenberg group $\{\mathrm{e}^{\mathrm{i}\gamma}\boldsymbol{W}_{(t,\omega)},\ (\gamma,t,\omega)\in[0,2\pi]\times\mathbb{R}^2\}$

$$\boldsymbol{W}_{(t',\omega')}\boldsymbol{W}_{(t,\omega)} = \mathrm{e}^{\mathrm{i}\omega t'}\boldsymbol{W}_{(t+t',\omega+\omega')}.$$

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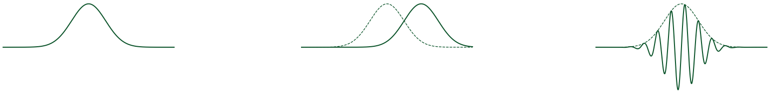

$g(t) = \pi^{-1/4}\exp\left(-t^2/2\right)$ $\quad$ $\mathbf{T}_u g(t) = g(t-u)$ $\quad$ $\mathbf{M}_\omega g(t) = g(t)\exp\left(-\mathrm{i}\omega t\right)$

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Coherent state interpretation $\quad \{\boldsymbol{W}_{(t,\omega)} h,\ t,\omega \in \mathbb{R}\}$ covariant family

$$V_h y(t,\omega) = \int_{-\infty}^{\infty} \overline{y(u)} h(u-t) \exp(-\mathrm{i}\omega u)\, \mathrm{d}u = \langle y, \boldsymbol{W}_{(t,\omega)} h \rangle$$

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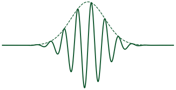

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- Signal detection: the role of representations

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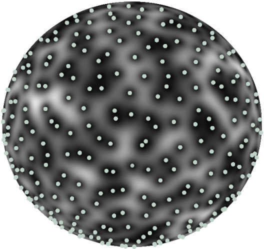
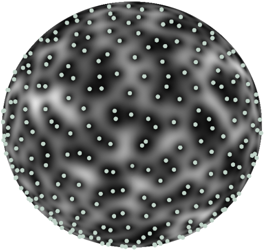
The Kravchuk transform: covariance under $\mathrm{SO}(3)$



Coherent state interpretation $\boldsymbol{y} \in \mathbb{C}^{N+1}$



$$T\boldsymbol{y}(\vartheta, \varphi) = \langle \boldsymbol{y}, \boldsymbol{\Psi}_{(\vartheta,\varphi)} \rangle$$

$\vartheta \in [0, \pi], \varphi \in [0, 2\pi]$

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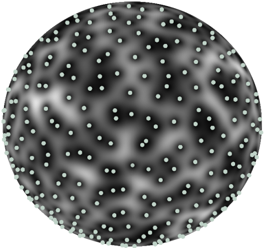
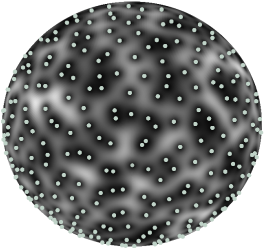


$\mathrm{SO}(3)$ **coherent states** (Gazeau, 2009)

$$\boldsymbol{\Psi}_{\vartheta,\varphi} = \sum_{n=0}^{N} \sqrt{\binom{N}{n}} \left(\cos\frac{\vartheta}{2}\right)^{n} \left(\sin\frac{\vartheta}{2}\right)^{N-n} \mathrm{e}^{\mathrm{i}n\varphi} \boldsymbol{q}_n = \boldsymbol{R}_{\boldsymbol{u}(\vartheta,\varphi)} \boldsymbol{\Psi}_{(0,0)},$$



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Kravchuk transform $\quad \{\boldsymbol{q}_n, n = 0, 1, ..., N\}$ the *Kravchuk functions*

$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^{N} \langle \boldsymbol{y}, \boldsymbol{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$$

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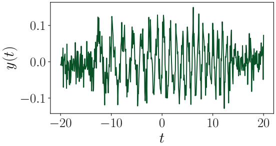

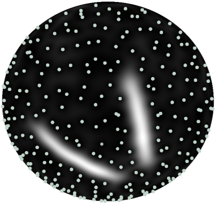

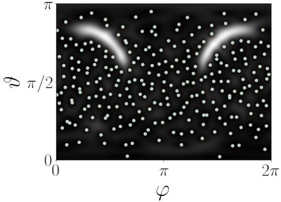

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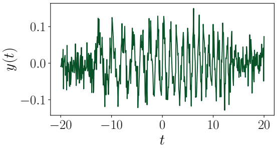

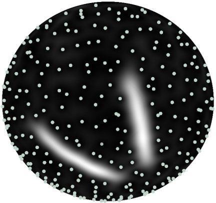

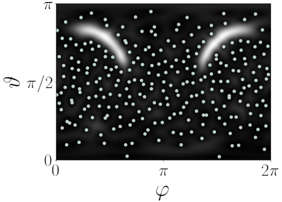

Theorem $$T\boldsymbol{\xi}(\vartheta, \varphi) = \sqrt{(1+|z|^2)}^{-N} \mathrm{GAF}_{\mathbb{S}}(z), \qquad z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$$

$\mathrm{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^{N} \xi[n] \sqrt{\binom{N}{n}} z^n$ the *spherical Gaussian Analytic Function*

(Pascal & Bardenet, 2022)

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$\rightarrow$ first: basis change, i.e., computation of $\langle \boldsymbol{y}, \boldsymbol{q}_n \rangle = \sum_{\ell=0}^{N} \overline{\boldsymbol{y}[\ell]} q_n(\ell; N)$

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Evaluation of Kravchuk functions $q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$

$$(N-n)Q_{n+1}(t; N) = (N-2t)Q_n(t; N) - nQ_{n-1}(t; N),$$

$\{Q_n(t; N), n = 0, 1, \ldots, N\}$ orthogonal family of Kravchuk polynomials

$$\sum_{\ell=0}^{N} \binom{N}{\ell} Q_n(\ell; N) Q_{n'}(\ell; N) = 2^N \binom{N}{n}^{-1} \delta_{n,n'}$$

Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

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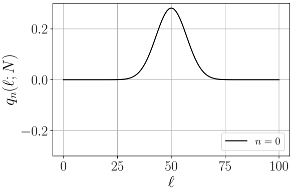

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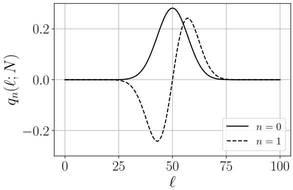

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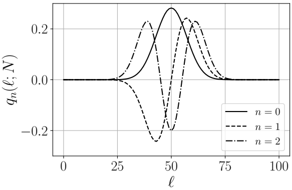

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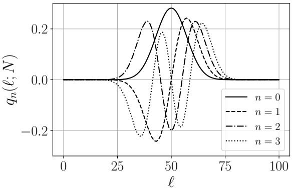

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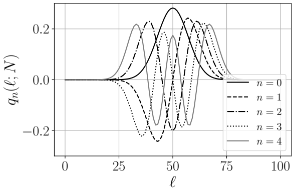

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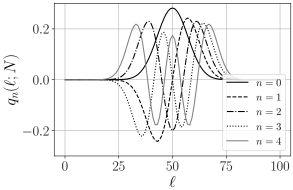

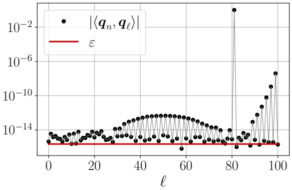

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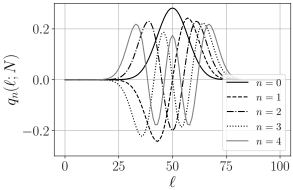
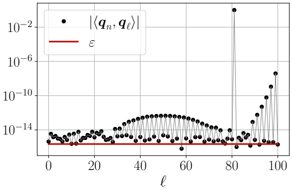




$\rightarrow$ estimated basis is **not orthogonal**! Not possible to compute $\langle \boldsymbol{y}, \boldsymbol{q}_n\rangle$.

Kravchuk transform $\{\boldsymbol{q}_n, n = 0, 1, \ldots, N\}$ the Kravchuk basis

$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^{N} \left(\sum_{\ell=0}^{N} \overline{\boldsymbol{y}[\ell]} q_n(\ell; N) \right) \sqrt{\binom{N}{n}} z^n \rightarrow \text{intractable}$$

A generative function for Kravchuk polynomials

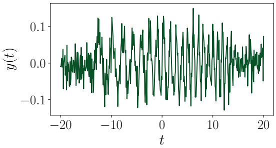
$$\sum_{n=0}^{N} \binom{N}{n} Q_n(\ell; N) z^n = (1-z)^{\ell}(1+z)^{N-\ell}$$

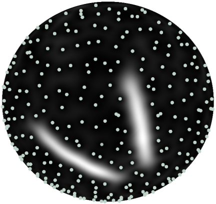
$$\implies \sum_{n=0}^{N} \sqrt{\binom{N}{n}} q_n(\ell; N) z^n = \sqrt{\binom{N}{\ell}} \frac{(1-z)^{\ell}(1+z)^{N-\ell}}{\sqrt{2^N}}$$

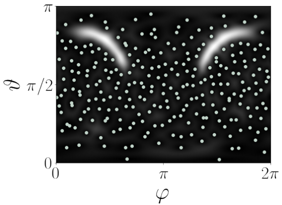
$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z^2|)^N}} \sum_{\ell=0}^{N} \sqrt{\binom{N}{\ell}} \overline{\boldsymbol{y}[\ell]} \frac{(1-z)^{\ell}(1+z)^{N-\ell}}{\sqrt{2^N}}$$

✗ no more Fast Fourier Transform algorithm using $z^n = \cot(\vartheta/2)^n \mathrm{e}^{\mathrm{i}n\varphi}$

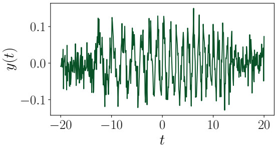
Detection of the zeros of the Kravchuk spectrogram $|T\boldsymbol{y}(z_i)|^2 = 0$

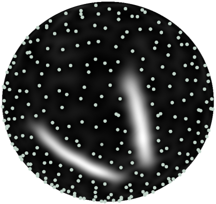


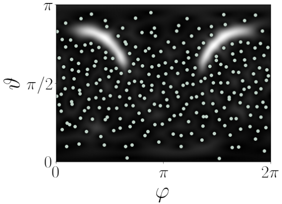




Advantage compared to Fourier: can tune the resolution of phase space.







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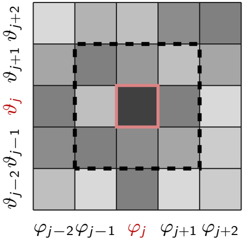

Minimal Grid Neighbors

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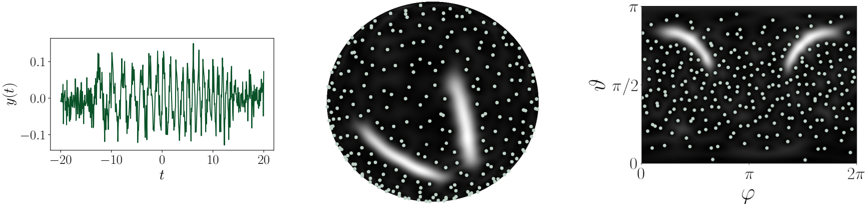
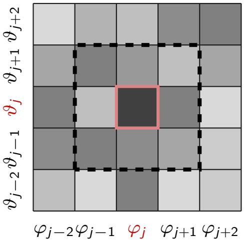
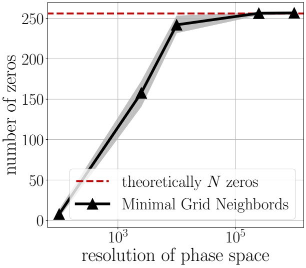


Advantage compared to Fourier: can tune the resolution of phase space.



Minimal Grid Neighbors



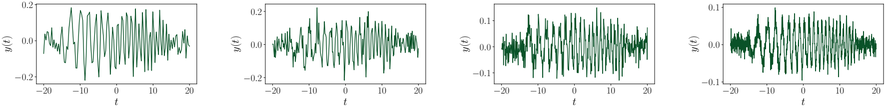
Proposition: all local minima of $|T\boldsymbol{y}(z)|^2$ are zeros.

- Signal detection: the role of representations

- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- Covariance principle and stationary point processes

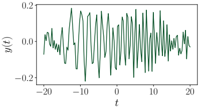
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- **Signal detection based on the spectrogram zeros - II**

Detection of a noisy chirp of duration $2\nu = 30$ s

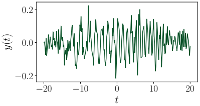


$N = 128$ $N = 256$ $N = 512$ $N = 1024$

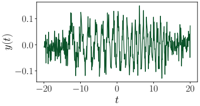
Detection of a noisy chirp of duration $2\nu = 30$ s



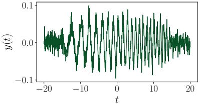
$N = 128$



$N = 256$

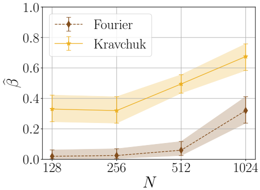


$N = 512$

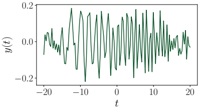


$N = 1024$

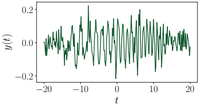
Performance: power of the test computed over 200 samples



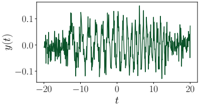
Detection of a noisy chirp of duration $2\nu = 30$ s



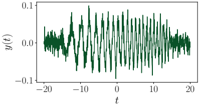
$N = 128$



$N = 256$

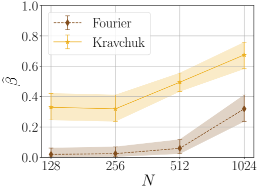


$N = 512$



$N = 1024$

Performance: power of the test computed over 200 samples



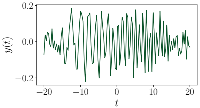
✓ higher detection power

✓ more robust to small N

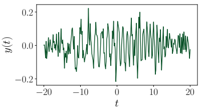
✗ no fast algorithm yet

Signal detection based on the spectrogram zeros - II snr $= 1.5$

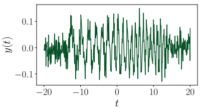
Detection of a noisy chirp of duration $2\nu = 30$ s



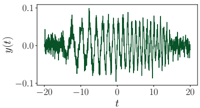
$N = 128$



$N = 256$

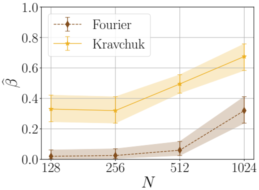


$N = 512$



$N = 1024$

Performance: power of the test computed over 200 samples



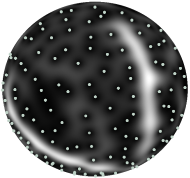
✓ higher detection power

✓ more robust to small N

✗ no fast algorithm yet

Advantages of using Kravchuk vs. Fourier spectrogram

- intrinsically encoded resolution: no need for prior knowledge
- compact phase space: no edge correction



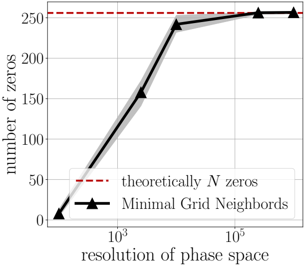
Take home messages

- Novel covariant discrete Kravchuk transform $T\boldsymbol{y}(\vartheta, \varphi)$
  - ∗ Interpreted as a coherent state decomposition,
  - ∗ Representation on a compact phase space,
  - ∗ Zeros of the Kravchuk spectrogram of white noise fully characterized.
- Signal detection based on spectrogram zeros
  - ∗ Preliminary work using the zeros of the Fourier spectrogram,
  - ∗ Significant improvement using the Kravchuk spectrogram.

Pascal & Bardenet, 2022: `arxiv:2202.03835`
`GitHub: bpascal-fr/kravchuk-transform-and-its-zeros`

Work in progress and perspectives

- Interpretation of the action of $\mathrm{SO}(3)$ on $\mathbb{C}^{N+1}$;
- Implementation of the inversion formula: denoising based on zeros ;
- Design of a Kravchuk FFT counterpart ;
- Convergence of Kravchuk toward the Fourier spectrogram as $N \to \infty$.



Spherical Gaussian Analytic Function

$$\mathrm{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^{N} \boldsymbol{\xi}[n] \sqrt{\binom{N}{n}} z^n$$

with $\boldsymbol{\xi}[n] \sim \mathcal{N}_{\mathbb{C}}(0,1)$ i.i.d.

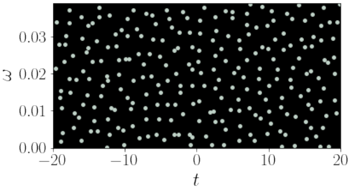
$\rightarrow$ only **simple** zeros

General case $T\boldsymbol{y}(z) = \sqrt{(1+|z|^2)}^{-N} \sum_{n=0}^{N} \sqrt{\binom{N}{n}} \left(\mathbf{Q}\boldsymbol{y}\right)[n] z^n$

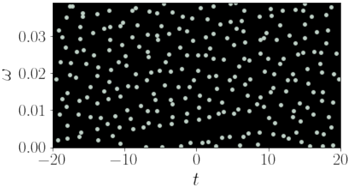
If $\boldsymbol{y}$ deterministic, such that $\left(\mathbf{Q}\boldsymbol{y}\right)[n] = \sqrt{\binom{N}{n}} a^{N-n} b^n, a \in \mathbb{C}, b \in \mathbb{C}^*$,

$$\sqrt{(1+|z|^2)}^{-N} \sum_{n=0}^{N} \sqrt{\binom{N}{n}} \left(\mathbf{Q}\boldsymbol{y}\right)[n] z^n = (a+bz)^N$$

$\rightarrow -a/b$ multiple root of order of degeneracy N



snr $= 0$

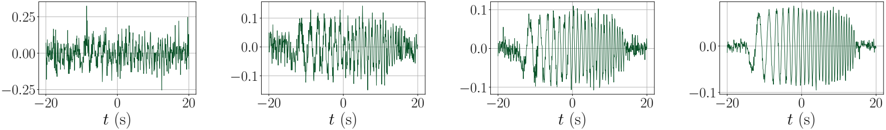


snr > 0

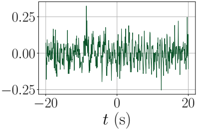
The signal creates **holes** in the zeros pattern: **sedond order** statistics.

Functional statistics:

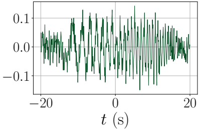
- the empty space function
  $F(r) = \mathbb{P}\left(\inf_{z_i \in Z} \mathrm{d}(z_0, z_i) < r\right)$: **probability to find a zero at less than** r
- Ripley's K-function
  $K(r) = 2\pi \int_0^r s g_0(s)\,\mathrm{d}s$: **expected # of pairs at distance less than** r



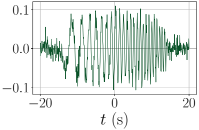
snr $= 0.5$ snr $= 1$ snr $= 2$ snr $= 5$



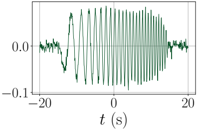
snr = 0.5



snr = 1

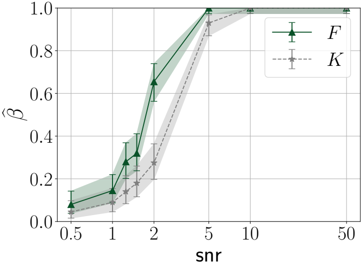


snr = 2

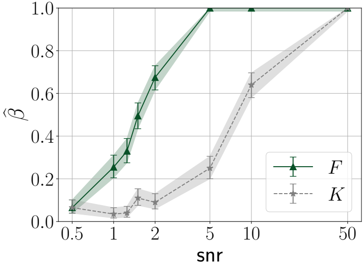


snr = 5

Ripley's K functional vs. empty space functional F



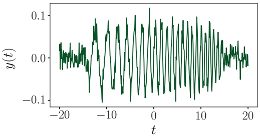
$N+1 = 257$ points



$N+1 = 513$ points

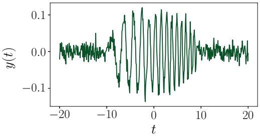
Fixed observation window of 40 **s**

long time event



duration $2\nu = 30$ s

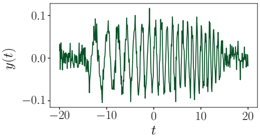
short time event



duration $2\nu = 20$ s

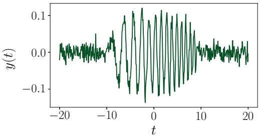
Fixed observation window of 40 s

long time event



duration $2\nu = 30$ s

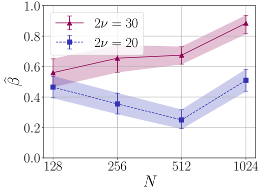
short time event



duration $2\nu = 20$ s

Robustness to small number of samples and short duration.

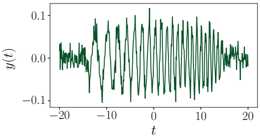
medium noise level



snr $= 2$

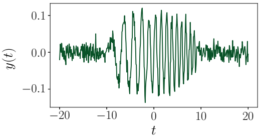
Fixed observation window of 40 s

long time event



duration $2\nu = 30$ s

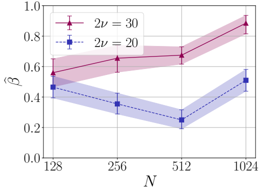
short time event



duration $2\nu = 20$ s

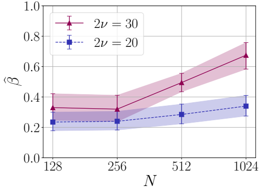
Robustness to small number of samples and short duration.

medium noise level



snr $= 2$

high noise level



snr $= 1.5$