Analyse de données non stationnaires : représentations, théorie, algorithmes et applications.

Barbara Pascal

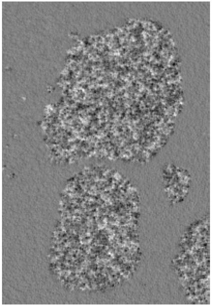
March 7th 2022

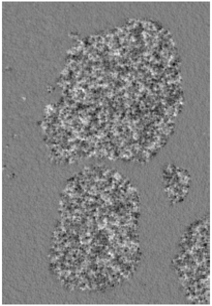
Laboratoire Mathématiques Appliquées À Paris 5 (MAP5)

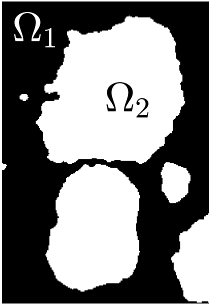
Groupe de travail Images

Part I: Texture segmentation based on fractal attributes

Textured image segmentation

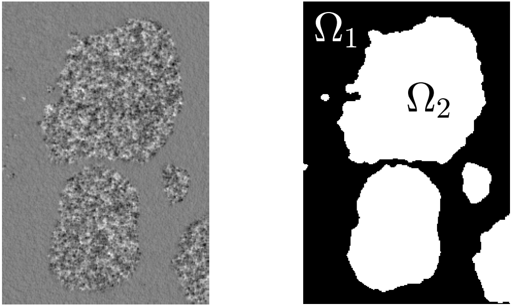






Goal: obtain a partition of the image into K homogeneous textures

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_K$$



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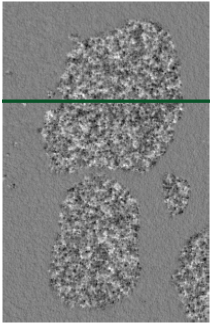
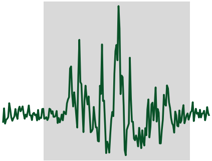
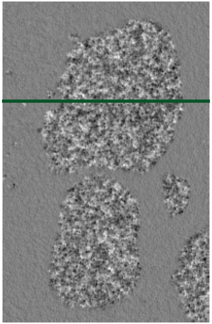
Piecewise monofractal model

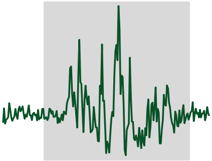




Fractals attributes

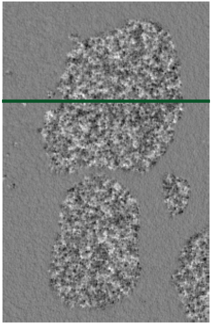
- variance σ^2 *amplitude of variations*

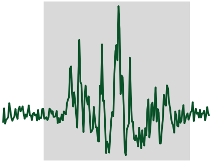




Fractals attributes

- variance σ^2 *amplitude of variations*
- local regularity h *scale invariance*

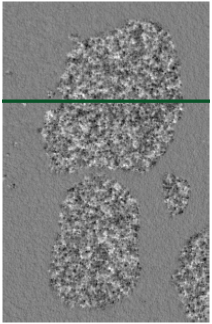


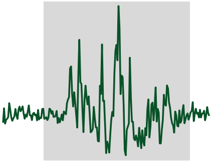


Fractals attributes

- variance σ^2 — *amplitude of variations*
- local regularity h — *scale invariance*

$$|f(x) - f(y)| \leq \sigma(x)|x - y|^{h(x)}$$





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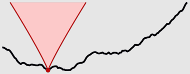
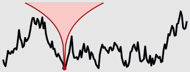
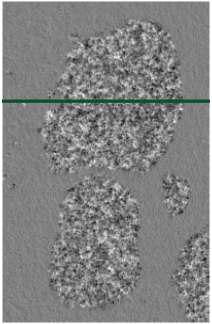
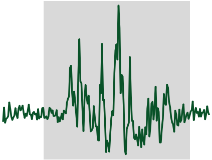


$h(x) \equiv h_1 = 0.9$



$h(x) \equiv h_2 = 0.3$





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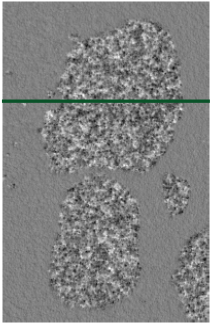
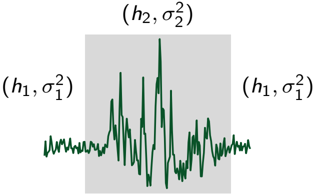


Segmentation

- σ^2 and h piecewise constant



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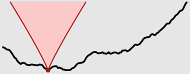

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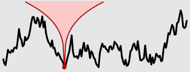
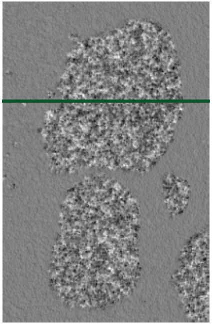
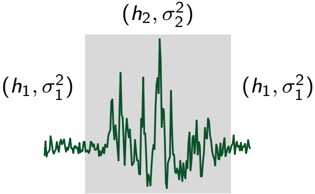


$h(x) \equiv h_2 = 0.3$

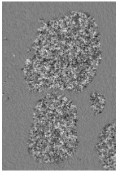
Segmentation

- σ^2 and h piecewise constant
- region Ω_k characterized by (σ_k^2, h_k)



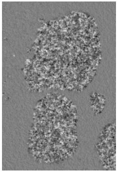


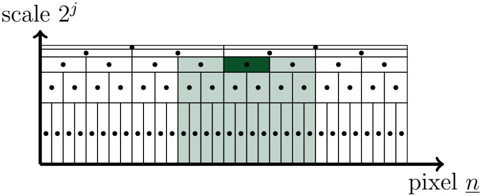
Textured image



Textured image

Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$





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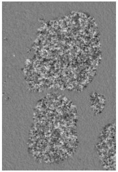
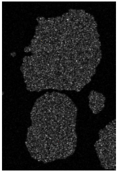


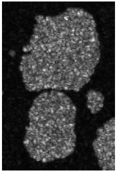
Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$

Scale

$a = 2^1$

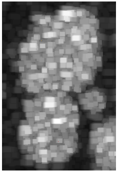


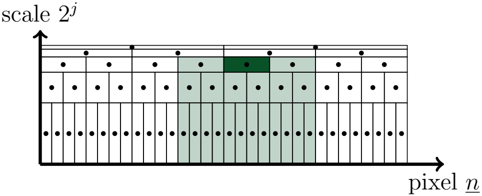
$a = 2^2$



...

$a = 2^5$





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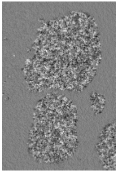

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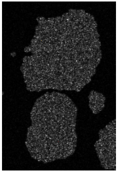

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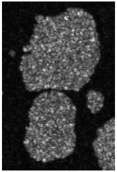

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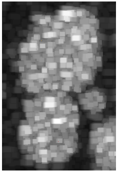

Proposition (Jaffard, 2004), (Wendt, 2008)

$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}$$

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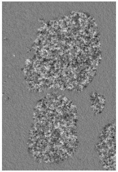

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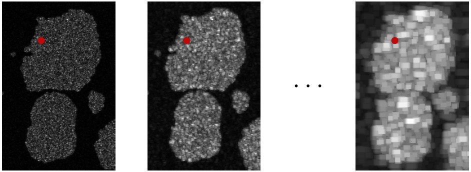

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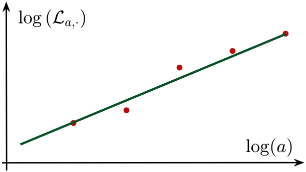

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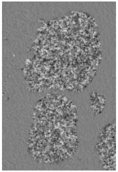

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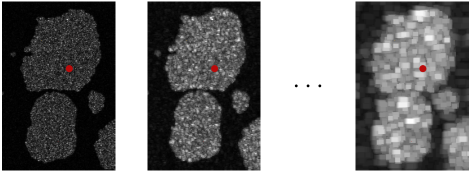

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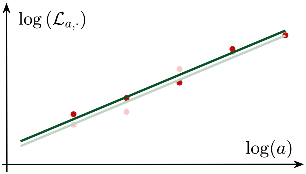
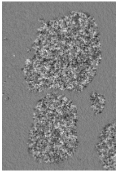


Multiscale analysis

Textured image



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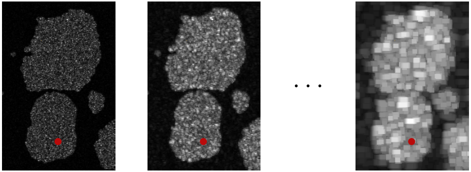

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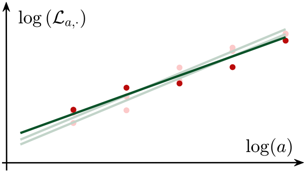

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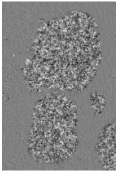

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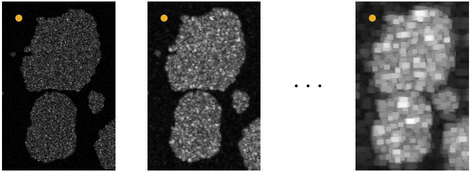

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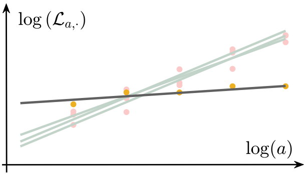

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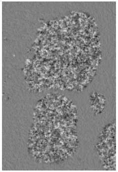

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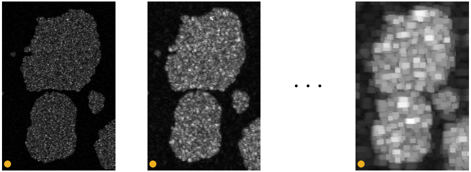

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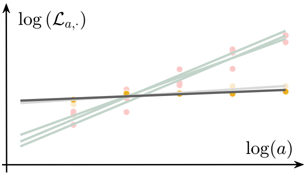

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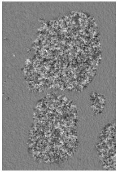

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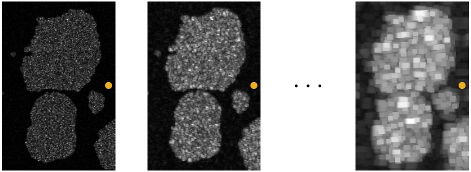

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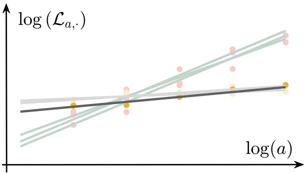
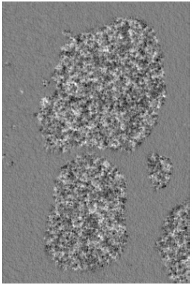


Linear regression $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

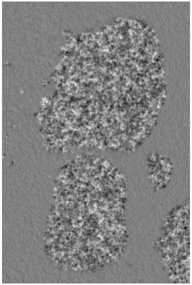
Textured image



Linear regression $\log(\mathcal{L}_{a,\cdot}) \simeq \log(a) \underset{\text{regularity}}{\boldsymbol{h}} + \underset{\propto \log(\boldsymbol{\sigma}^2)}{\boldsymbol{v}}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\| \log(\mathcal{L}_{a,\cdot}) - \log(a)\boldsymbol{h} - \boldsymbol{v} \right\|^2$$

Textured image



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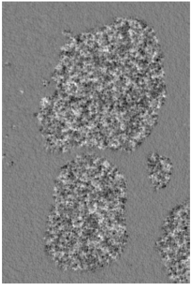
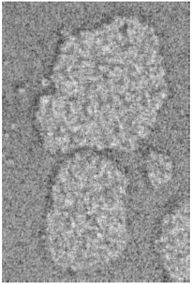


Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

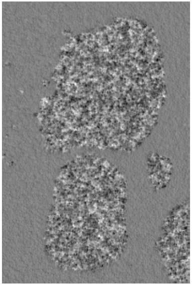
Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$



Linear regression $\underbrace{\mathbb{E}\log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{expected value}} = \log(a)\underbrace{\bar{\boldsymbol{h}}}_{\text{regularity}} + \underbrace{\bar{\boldsymbol{v}}}_{\propto\log(\boldsymbol{\sigma}^2)}$

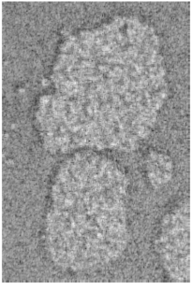
$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Textured image



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$



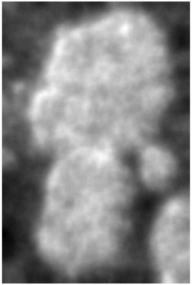
$\longrightarrow$ large estimation variance

Filter smoothing (linear)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Lissage

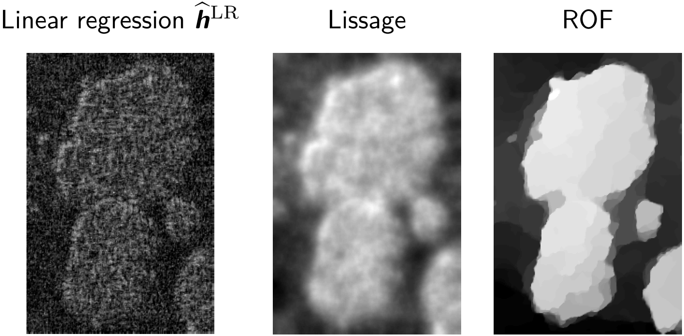


Filter smoothing (linear)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

ROF denoising (nonlinear)

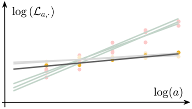
$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \ \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$



$\longrightarrow$ cumulative estimation variance and regularization bias

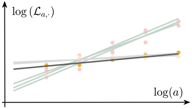
$$\sum_a \underbrace{\|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}}$$

$\rightarrow$ fidelity to the log-linear model

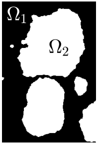


$$\sum_a \underbrace{\|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to the log-linear model

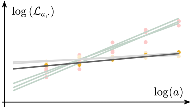


$\rightarrow$ favors piecewise constancy

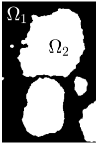


$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model



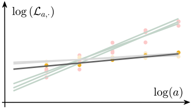
→ favors piecewise constancy

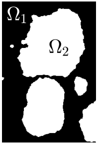


$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

→ favors piecewise constancy

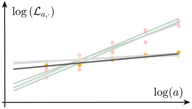




Finite differences $\mathbf{D}_1\boldsymbol{x}$ (horizontal), $\mathbf{D}_2\boldsymbol{x}$ (vertical) in each pixel

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to the log-linear model



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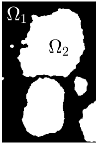


Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

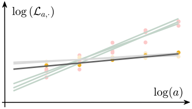
Free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

$$\mathcal{Q}_{\mathsf{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

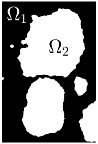
Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to the log-linear model



$\rightarrow$ favors piecewise constancy



Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

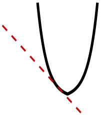
Free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

$$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

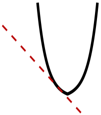
Co-localized: $\boldsymbol{h}$, $\boldsymbol{v}$ are **concomitantly** piecewise constant

$$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \|[\alpha\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}]\|_{2,1}$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



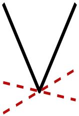
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \ + \ \lambda \ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth

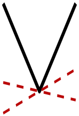


- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$
- implicit subgradient descent: proximal point algorithm

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \boldsymbol{u}^n, \ \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \ \Leftrightarrow \ \boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



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- splitting proximal algorithm

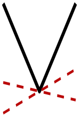
$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau \mathbf{D}^\top \boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



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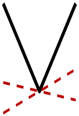
nonsmooth

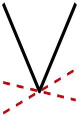


Primal-dual algorithm (Chambolle, 2011)

δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



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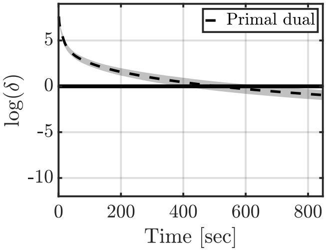
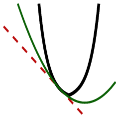
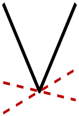


$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underset{\textbf{Least-Squares}}{\underline{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}} \; + \; \lambda \, \underset{\textbf{Total Variation}}{\underline{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}}$$



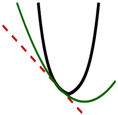
$\boldsymbol{\mu}$-strongly convex

nonsmooth



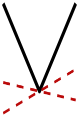
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \ + \ \lambda \ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



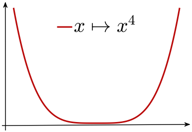
$\boldsymbol{\mu}$-strongly convex

nonsmooth

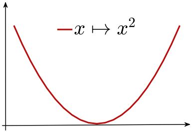


Strong-convexity

- φ $\boldsymbol{\mu}$-strongly convex iff $\varphi - \dfrac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convex



✓ strictly convex
✗ non strongly convex



✓ strictly convex
✓ 1-strongly convex

Convexity properties

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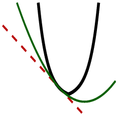
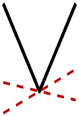


$\boldsymbol{\mu}$-strongly convex

nonsmooth



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- φ $\mathcal{C}^2$ with Hessian matrix $\boldsymbol{H}\varphi \succeq 0 \implies \boldsymbol{\mu} = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

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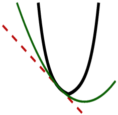

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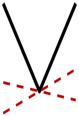
nonsmooth



Strong-convexity

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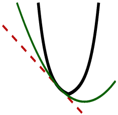
Proposition (Pascal, 2019)

$\sum_a \|\log \boldsymbol{\mathcal{L}} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2$ is $\boldsymbol{\mu}$-strongly convex.

$a_{\min} = 2^1$, $a_{\max}$	2^2	2^3	2^4	2^5	2^6
$\boldsymbol{\mu} = \min \mathrm{Sp}\left(2\boldsymbol{\Phi}^\top \boldsymbol{\Phi}\right)$	0.29	**0.72**	1.20	1.69	2.20

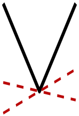
Accelerated algorithm based on strong-convexity

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \;\; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;+\; \underbrace{\lambda\, \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



μ-strongly convex

nonsmooth



Accelerated Primal-dual algorithm *(Chambolle, 2011)*

for $n = 0, 1, \ldots$ $\qquad \boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})$

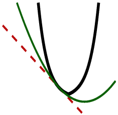
$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma_n(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma_n \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau_n\|\boldsymbol{\mathcal{L}} - \Phi\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau_n \mathbf{D}^\top \boldsymbol{y}^{n+1}\right)$$

$$\theta_n = \sqrt{1 + 2\mu\tau_n}, \quad \tau_{n+1} = \tau_n/\theta_n, \quad \sigma_{n+1} = \theta_n\sigma_n$$

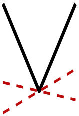
$$\bar{\boldsymbol{x}}^{n+1} = \boldsymbol{x}^{n+1} + \theta_n^{-1}\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



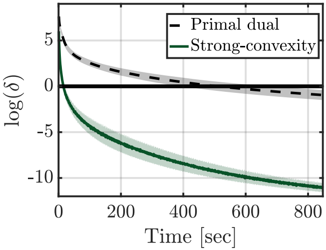
$\boldsymbol{\mu}$-strongly convex

nonsmooth



Accelerated Primal-dual algorithm *(Chambolle, 2011)*

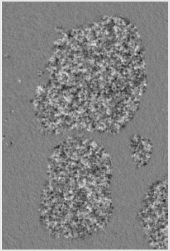
δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$



Segmentation *via* iterated thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image

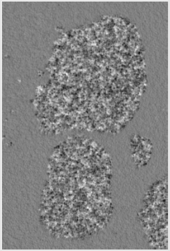


Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Segmentation *via* iterated thresholding

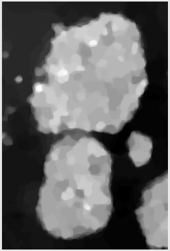
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \;\; \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;+\; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image

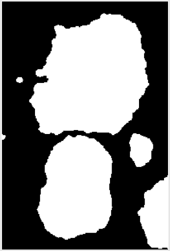


Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours $\widehat{\boldsymbol{h}}^{\mathrm{C}}$



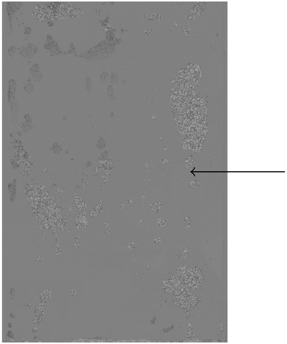
Threshold estimate† $T\widehat{\boldsymbol{h}}^{\mathrm{C}}$



†(Cai, 2013)

Multiphase flow through porous media

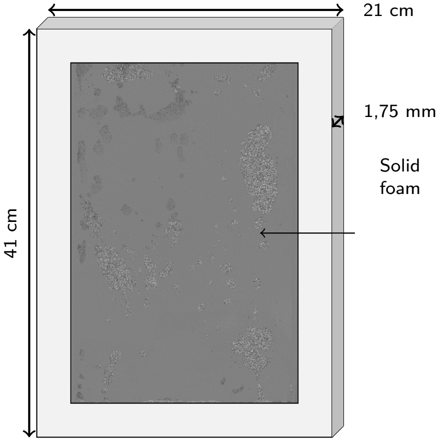
Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



Solid
foam

Multiphase flow through porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



Multiphase flow through porous media

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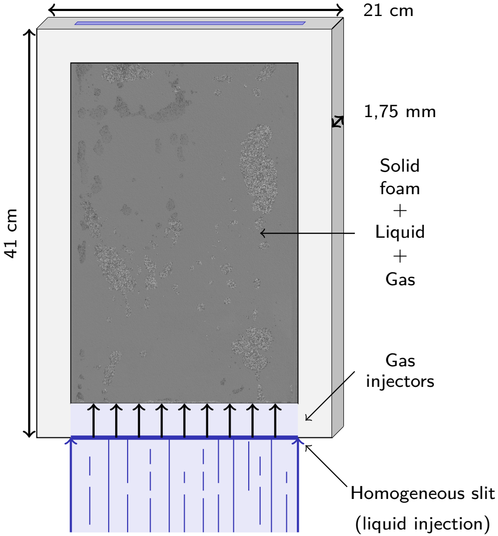
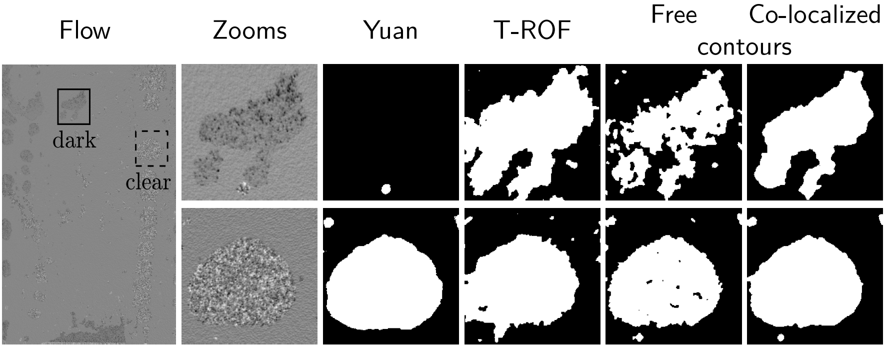
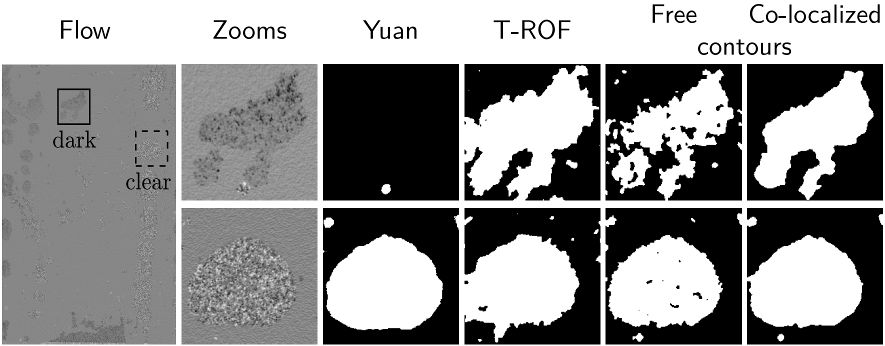


→ 1600 × 1100 pixels
→ video: ∼ 1000 images
→ phase diagram: ∼ 10 flow rates

Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$

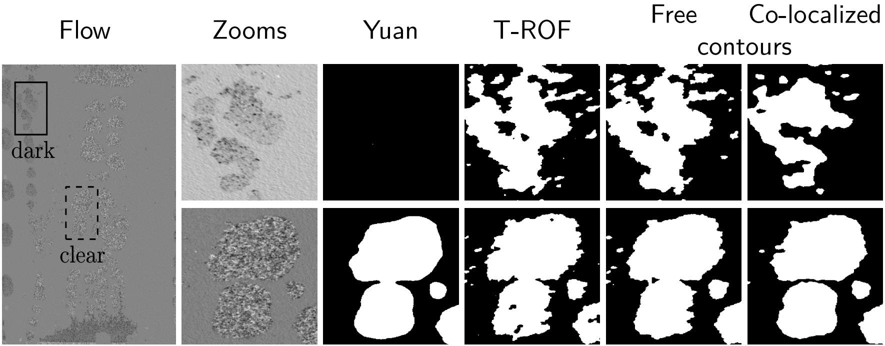


Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$



Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

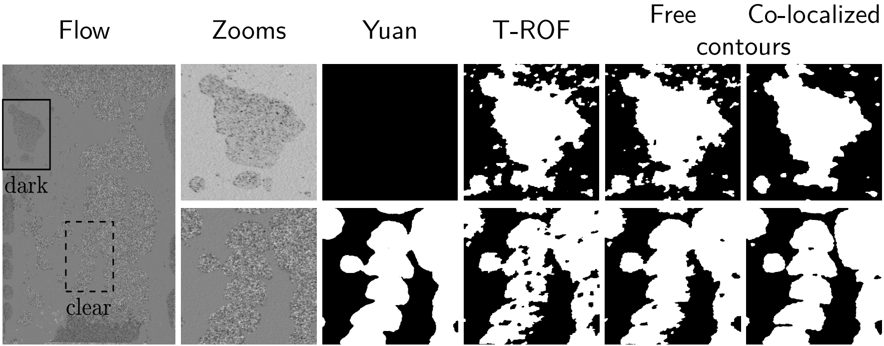
Gas: $h_{\mathrm{G}} = 0.9$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$ (dark bubbles)
$\sigma^2_{\mathrm{clear}} = 10^{-1}$ (clear bubbles).



Liquid: $h_{\mathrm{L}} = 0.4$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\quad \left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

High activity: $Q_{\mathrm{G}} = 1200$mL/min - $Q_{\mathrm{L}} = 300$mL/min

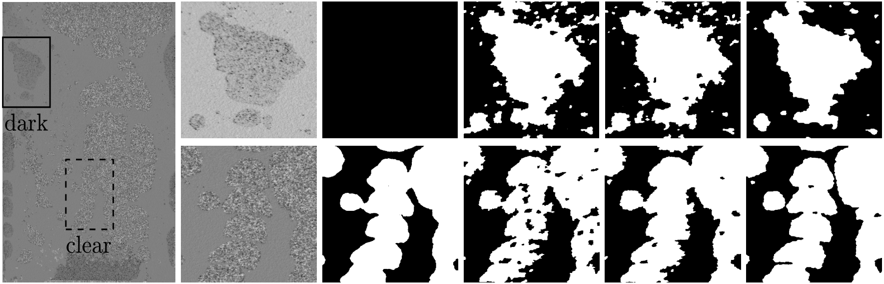


Liquid: $h_{\mathrm{L}} = 0.4 \qquad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9 \quad \left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

High activity: $Q_{\mathrm{G}} = 1200\text{mL/min}$ - $Q_{\mathrm{L}} = 300\text{mL/min}$

Flow | Zooms | Yuan | T-ROF | Free contours | Co-localized contours



Computational load	Yuan	T-ROF	Free contours	Co-localized contours
	1s	12s	700s	**2100s**

Liquid: $h_{\mathrm{L}} = 0.4 \qquad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9 \quad \left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

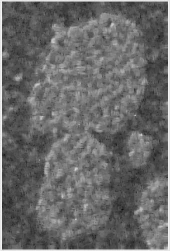
$(\lambda, \alpha) = (0, 0)$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\mathcal{L}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

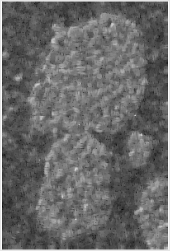
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

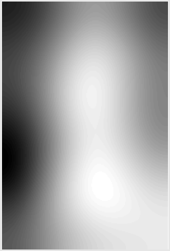
$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

$(\lambda, \alpha) = (500, 500)$



too large

Regularization parameters selection

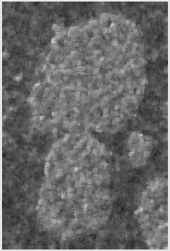
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

$(\lambda, \alpha) = (0, 0)$

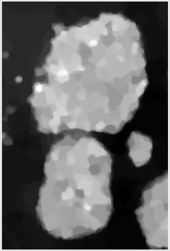
Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0.5, 0.5)$



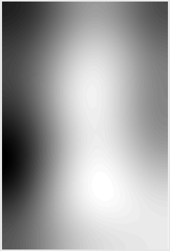
too small

$(\lambda^\dagger, \alpha^\dagger) = (11.5, 0.8)$



optimal

$(\lambda, \alpha) = (500, 500)$



too large

What *optimal* means? How to determine $\lambda^\dagger$ and $\alpha^\dagger$?

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\mathrm{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

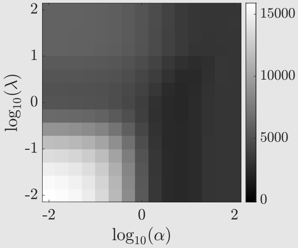
$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\mathcal{L}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$

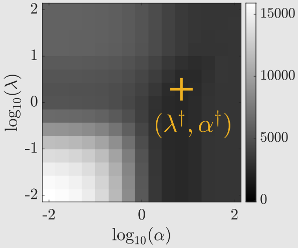


$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



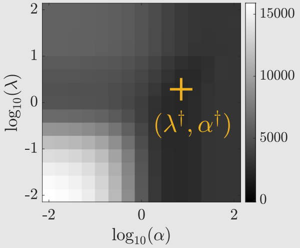
Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

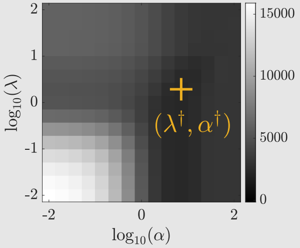
?

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \| \log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v} \|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

?

Stein Unbiased Risk Estimate
(SURE)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

$$\textbf{Ex.} \quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Quadratic error $R(\lambda) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \stackrel{?}{=} \mathbb{E}_{\boldsymbol{\zeta}} \widehat{R}(\boldsymbol{y}; \lambda)$ $\quad$ $\bar{\boldsymbol{x}}$ unknown

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

$$\textbf{Ex.} \quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \, \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Quadratic error $R(\lambda) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \overset{?}{=} \mathbb{E}_{\boldsymbol{\zeta}} \widehat{R}(\boldsymbol{y}; \lambda)$ $\qquad$ $\bar{\boldsymbol{x}}$ unknown

Theorem (Stein, 1981)

Let $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\boldsymbol{\zeta} \mapsto \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \boldsymbol{\zeta} \rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$.

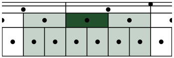
$$\widehat{R}(\boldsymbol{y}; \lambda) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\rho^2 \operatorname{tr}\left(\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)\right) - \rho^2 P$$
$$\Longrightarrow R(\lambda) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{y}; \lambda)].$$

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\quad$ $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$ $\quad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

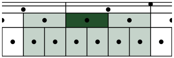
Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Observations $\boldsymbol{y} = \Phi \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\Phi : \mathbb{R}^{P \times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \Phi(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$ $\quad\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$ $\quad\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\Pi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\Pi}(\Lambda) \triangleq \mathbb{E}_\zeta \|\Pi \widehat{\boldsymbol{x}}(\boldsymbol{y}; \Lambda) - \Pi \bar{\boldsymbol{x}}\|^2$

Theorem (Pascal, 2020)

Let $(\boldsymbol{y}; \Lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \Lambda)$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\zeta \mapsto \langle \Pi \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \zeta; \lambda), \mathbf{A}\zeta \rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$.

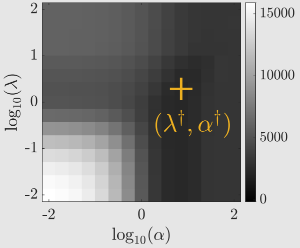
$$\widehat{R}(\Lambda) \triangleq \|\mathbf{A}(\Phi \widehat{\boldsymbol{x}}(\boldsymbol{y}; \Lambda) - \boldsymbol{y})\|^2 + 2\mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \Pi \partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \Lambda)\right) - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

$$\Longrightarrow R_{\Pi}(\Lambda) = \mathbb{E}_\zeta[\widehat{R}(\Lambda)].$$

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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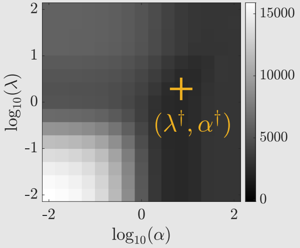

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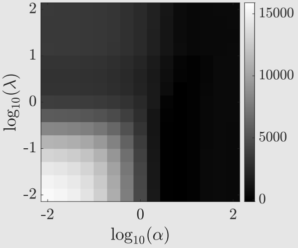

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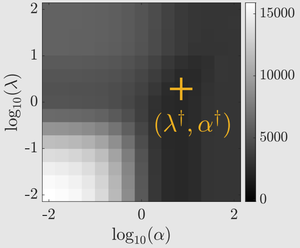

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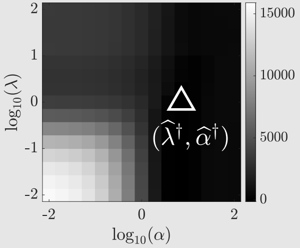

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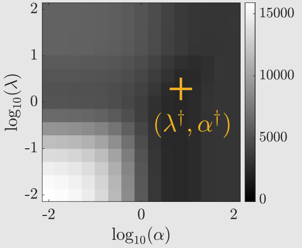

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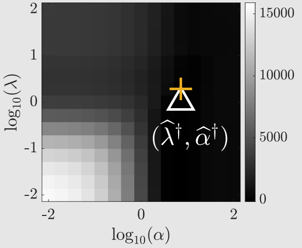
$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$



$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha)=\underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}}\sum_{a}\|\log\boldsymbol{\mathcal{L}}_{a,\cdot}-\log(a)\boldsymbol{h}-\boldsymbol{v}\|^2+\lambda\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

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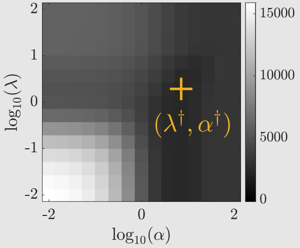

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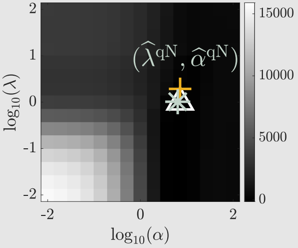
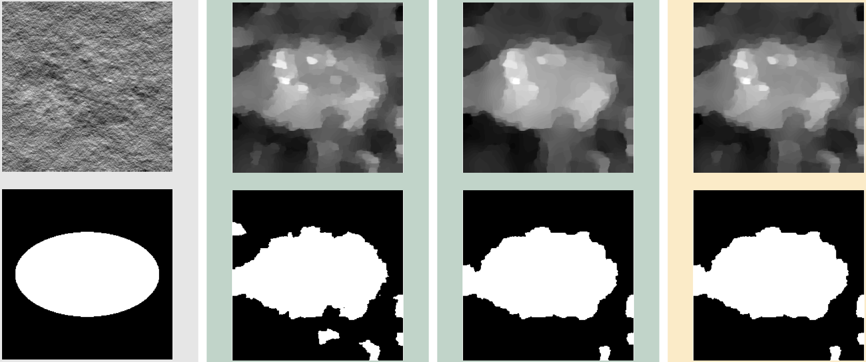


Automated selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \| \log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v} \|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Example	$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}}; \lambda^{\dagger}, \alpha^{\dagger})$ (grid)	$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\dagger}, \widehat{\alpha}^{\dagger})$ (grid)	$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\mathrm{qN}}, \widehat{\alpha}^{\mathrm{qN}})$ (quasi-Newton)



225 calls of the estimator over the grid v.s. **40** for quasi-Newton

Take home messages

- Fractal texture model based on local *regularity* and *variance*
  - appropriate for real-world texture characterization
  - complementary attributes able to finely discriminate

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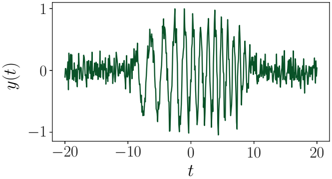
Take home messages

- Fractal texture model based on local *regularity* and *variance*
  - ∗ appropriate for real-world texture characterization
  - ∗ complementary attributes able to finely discriminate
- Simultaneous estimation and regularization
  - ∗ significant decrease of the estimation error
  - ∗ accurate and regular contours thanks to co-localized penalization
- Fast algorithms for automated tuning of hyperparameters
  - ∗ possibility to manage huge amount of data
  - ∗ amenable to process data corrupted by *correlated* Gaussian noise
  - ∗ ensured objectivity and reproducibility

Part II: Point processes in time-frequency analysis

$y : \mathbb{R} \to \mathbb{C}$ function of time.



- electrical cardiac activity,
- audio recording,
- seismic activity,
- light intensity on a photosensor
- ...

Information of interest:

– time events e.g., an earthquake and its replica

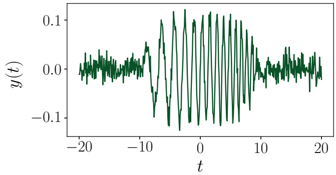
– frequency content e.g., monitoring of the heart beating rate

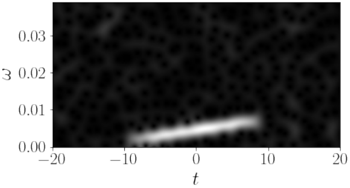
time
ever-changing world
marker of events and evolutions

frequency
waves, oscillations, rhythms
intrinsic mechanisms

Short-Time Fourier Transform with window h:

$$V_h y(t,\omega) \triangleq \int_{-\infty}^{\infty} \overline{y(u)} h(u-t) \exp(-\mathrm{i}\omega u) \, \mathrm{d}u$$





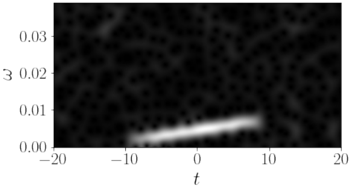
Energy density interpretation

$$\int \int_{-\infty}^{+\infty} |V_h y(t,\omega)|^2 \, \mathrm{d}t \frac{\mathrm{d}\omega}{2\pi} = \int_{-\infty}^{+\infty} |x(t)|^2 \, \mathrm{d}t \quad \text{if} \quad \|h\|_2^2 = 1$$

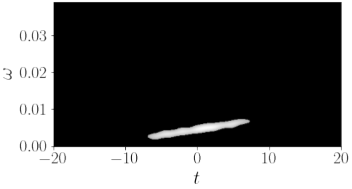
Signal, i.e., information of interest: regions of maximal energy.

Inversion formula $$y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u,\omega)} h(t-u) \exp(\mathrm{i}\omega u) \,\mathrm{d}u \frac{\mathrm{d}\omega}{2\pi}$$

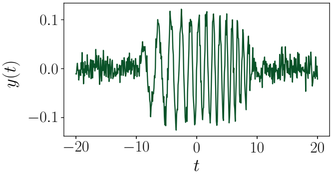
spectrogram



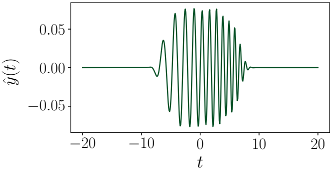
only maxima



noisy observation



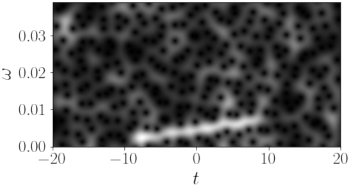
estimate



Denoising in the time-frequency plane   snr $= 0.5$

Inversion formula $\quad y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u,\omega)} h(t-u) \exp(\mathrm{i}\omega u) \, \mathrm{d}u \frac{\mathrm{d}\omega}{2\pi}$

spectrogram



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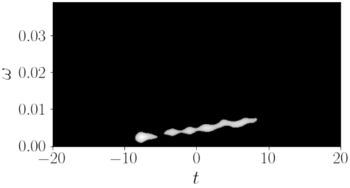

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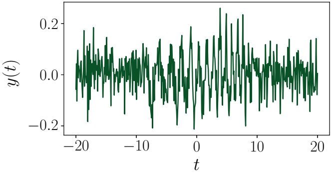

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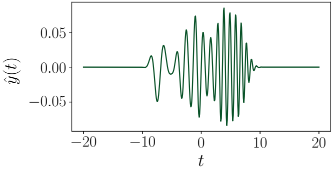
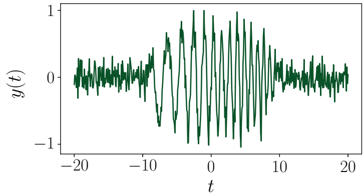
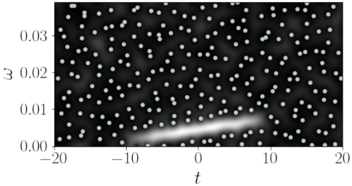


Maxima detection: *reassignment, synchrosqueezing, ridge extraction*

Restriction to the *circular Gaussian window*: $g(t) = \pi^{-1/4} e^{-t^2/2}$

Look for the zeros, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.





Observations: (Flandrin, 2015)

- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.
- Zeros are repelled by the signal.

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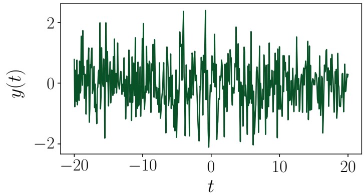

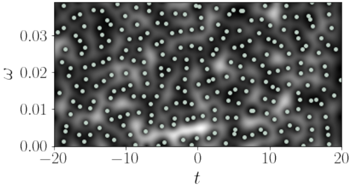

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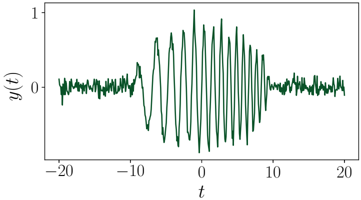

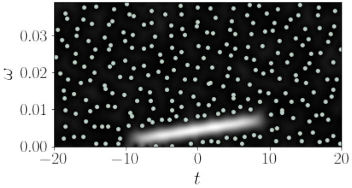

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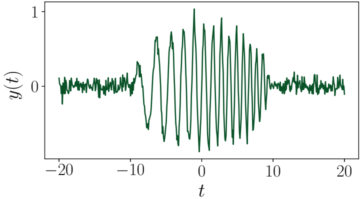

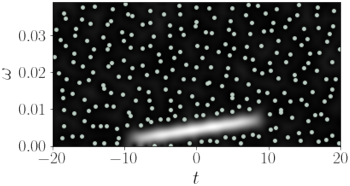

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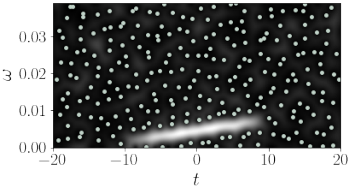
What can be said theoretically about the zeros of the spectrogram?

Idea assimilate the time-frequency plane with $\mathbb{C}$ through $z = (\omega + \mathrm{i}t)/\sqrt{2}$



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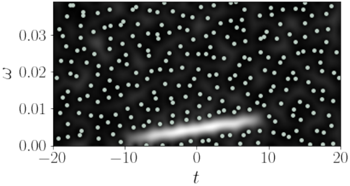


Bargmann factorization

$$V_g y(t, \omega) = \mathrm{e}^{-|z|^2/2} \mathrm{e}^{-\mathrm{i}\omega t/2} By(z)$$

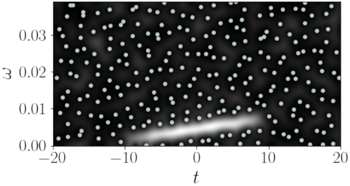
Bargmann transform of the signal y

$$By(z) \triangleq \pi^{-1/4} \mathrm{e}^{-z^2/2} \int_{\mathbb{R}} \overline{y(u)} \exp\left(\sqrt{2}uz - u^2/2\right) \,\mathrm{d}u,$$

By is an **entire** function, almost characterized by its infinitely many zeros:

$$By(z) = z^m \mathrm{e}^{C_0 + C_1 z + C_2 z^2} \prod_{n \in \mathbb{N}} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2\right).$$

Idea assimilate the time-frequency plane with $\mathbb{C}$ through $z = (\omega + \mathrm{i}t)/\sqrt{2}$



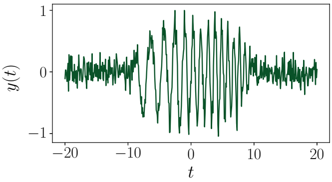
Bargmann factorization

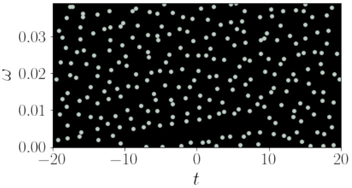
$$V_g y(t,\omega) = \mathrm{e}^{-|z|^2/2}\mathrm{e}^{-\mathrm{i}\omega t/2} By(z)$$

Theorem The zeros of the Gaussian spectrogram $V_g y(t,\omega)$

- coincide with the zeros of the **entire** function By,
- hence are **isolated** and constitute a **Point Process**,
- which almost completely **characterizes** the spectrogram.

(Flandrin, 2015)





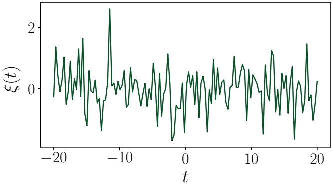
Advantages of working with the zeros

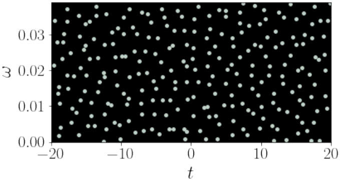
- Easy to find compared to relative maxima.
- Form a robust pattern in the time-frequency plane.
- Require little memory space for storage.
- Efficient tools were recently developed in **stochastic geometry**.

Need for a rigorous characterization of the distribution of the zeros.

Continuous complex white Gaussian noise

$$\xi(t) = \sum_{n=0}^{\infty} \xi[n] h_n(t),\ \xi[n] \sim \mathcal{N}_{\mathbb{C}}(0,1), \quad \{h_n, k = 0, 1, \ldots\} \text{ Hermite functions}$$





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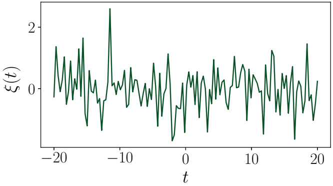
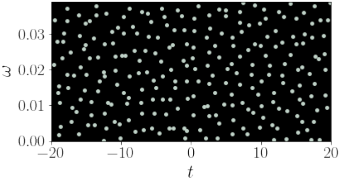
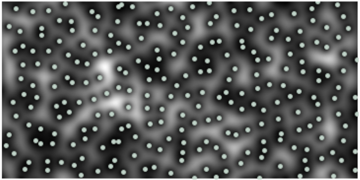




Theorem $V_g\xi(t,\omega) = \mathrm{e}^{-|z|^2/4}\mathrm{e}^{-\mathrm{i}\omega t/2}\,\mathrm{GAF}_{\mathbb{C}}(z)$ (Bardenet & Hardy, 2021)

$\mathrm{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$ is the *planar Gaussian Analytic Function*.



$$V_g\xi(t,\omega) \overset{\text{non-vanishing}}{\propto} \mathrm{GAF}_{\mathbb{C}}(z)$$

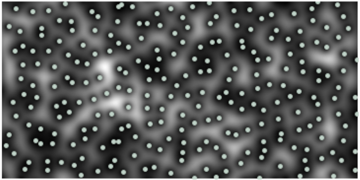
$$z = (\omega + \mathrm{i}t)/\sqrt{2}$$

Zeros of $\mathrm{GAF}_{\mathbb{C}}$: random set of points, i.e., a **Point Process** characterized by a probability distribution on point configurations

Properties of the Point Process of the zeros of $\mathrm{GAF}_{\mathbb{C}}$:

- invariant under the isometries of $\mathbb{C}$, i.e., **stationary**,
- has a uniform density $\rho^{(1)}(z) = \rho^{(1)} = 1/\pi$,
- explicit pair correlation function $\rho^{(2)}(z, z') = g_0(|z - z'|)$,
- scaling of the *hole probability*: $r^{-4} \log p_r \to -3\mathrm{e}^2/4$, as $r \to \infty$

$$p_r = \mathbb{P}\,(\text{no point in the disk of center 0 and radius } r)$$



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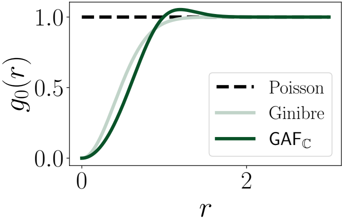
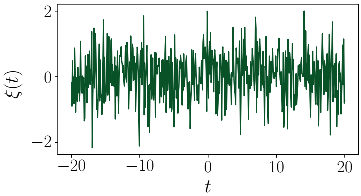


Pair correlation $\rho^{(2)}(z, z')\mathrm{d}z\mathrm{d}z' =$

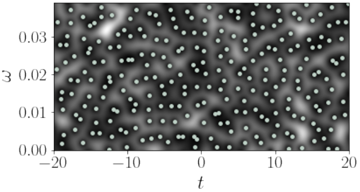
$$\mathbb{P}\left(1 \text{ point in } B(z, \mathrm{d}z) \text{ and } 1 \text{ in } B(z', \mathrm{d}z')\right)$$

The point process of the zeros of the spectrogram is not **determinantal**.

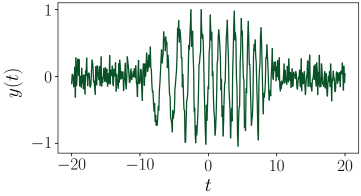
- **$\mathbf{H}_0$** white noisy only, i.e., $y(t) = \xi(t)$
- **$\mathbf{H}_1$** presence of a signal i.e., $y(t) = \mathrm{snr} \times x(t) + \xi(t), \quad \mathrm{snr} > 0$

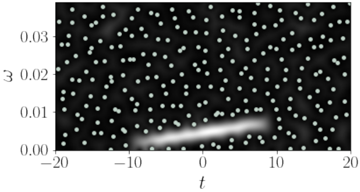


null hypothesis



alternative hypothesis





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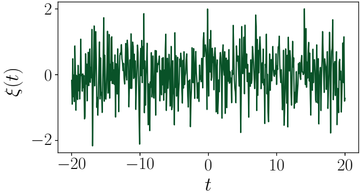
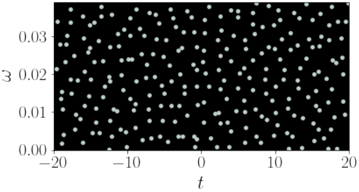
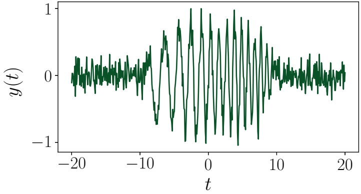
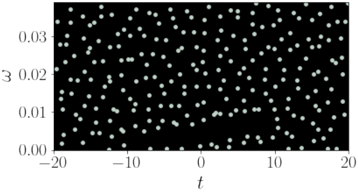




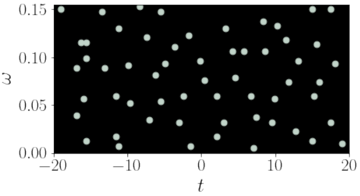
alternative hypothesis

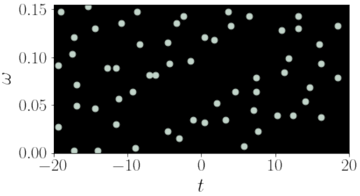




A functional statistic: the F-function

$$F(r) = \mathbb{P}\left(\inf_{z_i \in \mathcal{Z}} \mathrm{d}(z_0, z_i) < r\right) : \textbf{ empty space function}$$





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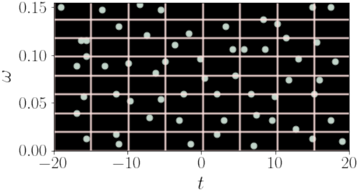
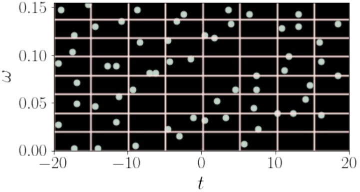
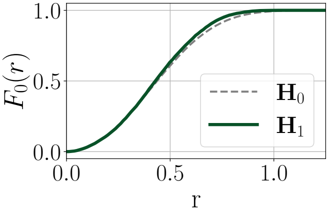




$$\widehat{F}(r) = \frac{1}{N_\#} \sum_{j=1}^{N_\#} \mathbf{1}\left(\inf_{z \in \textbf{Zeros}} \mathrm{d}\left(z_j, z\right) < r\right)$$



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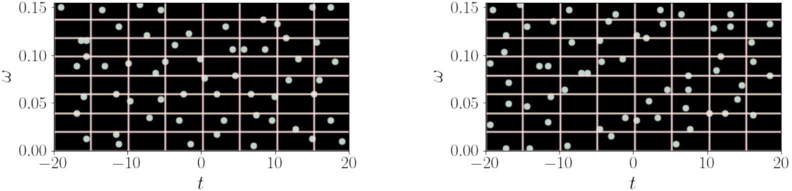

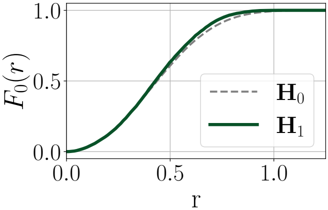

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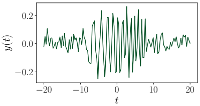
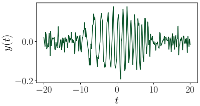
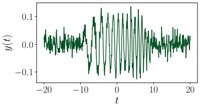
- Monte Carlo envelope test based on the discrepancy between $\widehat{F}$ and F_0



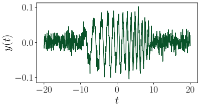
$N = 128$



$N = 256$

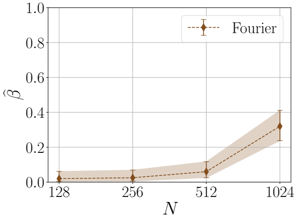


$N = 512$



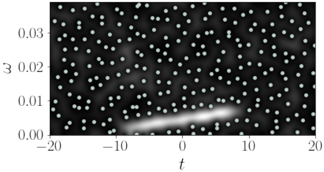
$N = 1024$

Performance: power of the test computed over 200 samples



✗ low detection power

✗ requires large number of samples

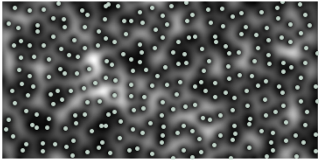


Limitations:

- necessary discretization of the STFT: arbitrary resolution
- observe only a bounded window: edge correction to compute $\widehat{F}(r)$

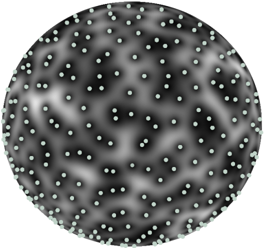
Short-Time Fourier Transform

$$V_g\xi(t,\omega) \propto \mathrm{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$$



New transform?

$$? \quad \propto \mathrm{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^{N} \boldsymbol{\xi}[n] \sqrt{\binom{N}{n}} z^n$$



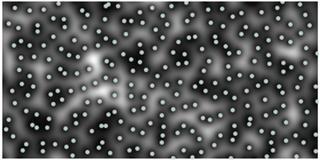
Time and frequency shifts $\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$

$$V_h[\boldsymbol{W}_{(t,\omega)} y](t',\omega') \overset{\text{covariance}}{=} \mathrm{e}^{-\mathrm{i}(\omega'-\omega)t} V_h y(t'-t, \omega'-\omega),$$

Coherent state interpretation

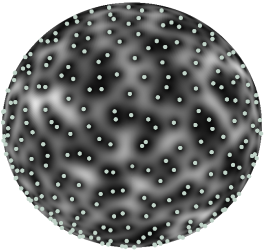
$$V_h y(t,\omega) = \langle y, \boldsymbol{W}_{(t,\omega)} h \rangle$$

$\{\boldsymbol{W}_{(t,\omega)} h,\ t,\omega \in \mathbb{R}\}$ covariant family



Weyl-Heisenberg group $\{\mathrm{e}^{\mathrm{i}\gamma} \boldsymbol{W}_{(t,\omega)}, (\gamma, t, \omega) \in [0, 2\pi] \times \mathbb{R}^2\}$

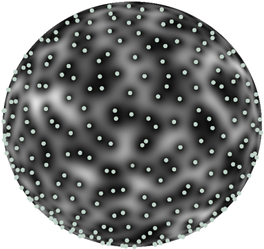
$$\boldsymbol{W}_{(t',\omega')} \boldsymbol{W}_{(t,\omega)} = \mathrm{e}^{\mathrm{i}\omega t'} \boldsymbol{W}_{(t+t',\omega+\omega')}.$$



Coherent state interpretation $\quad \boldsymbol{y} \in \mathbb{C}^{N+1}$

$$T\boldsymbol{y}(\vartheta, \varphi) = \langle \boldsymbol{y}, \boldsymbol{\Psi}_{(\vartheta,\varphi)} \rangle$$

$\vartheta \in [0, \pi], \varphi \in [0, 2\pi]$



The Kravchuk transform: covariance under $\mathrm{SO}(3)$

Coherent state interpretation $\boldsymbol{y} \in \mathbb{C}^{N+1}$

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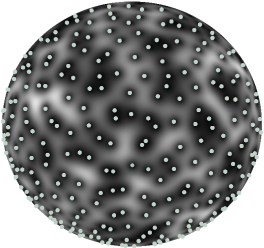
$\vartheta \in [0,\pi], \varphi \in [0,2\pi]$



$\mathrm{SO}(3)$ **coherent states** (Gazeau, 2009)

$$\boldsymbol{\Psi}_{\vartheta,\varphi} = \sum_{n=0}^{N} \sqrt{\binom{N}{n}} \left(\cos\frac{\vartheta}{2}\right)^{n} \left(\sin\frac{\vartheta}{2}\right)^{N-n} \mathrm{e}^{\mathrm{i}n\varphi} \boldsymbol{q}_n = \boldsymbol{R}_{\boldsymbol{u}(\vartheta,\varphi)} \boldsymbol{\Psi}_{(0,0)},$$

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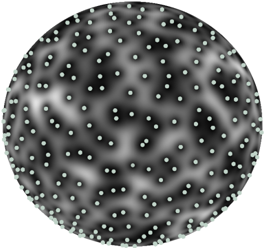

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Kravchuk transform $\{\boldsymbol{q}_n, n = 0, 1, ..., N\}$ the *Kravchuk functions*

$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^{N} \langle \boldsymbol{y}, \boldsymbol{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$$

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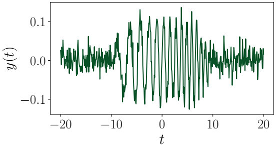
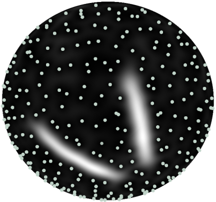
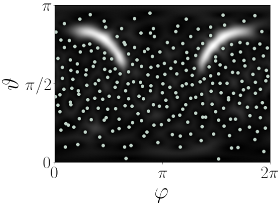
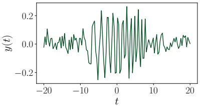






Contributions

- rigorous link: $T\boldsymbol{\xi}(z) \stackrel{\text{(law)}}{=} \sqrt{(1+|z|^2)}^{-N} \mathrm{GAF}_{\mathbb{S}}(z)$
- design of a robust implementation avoiding to compute $\langle \boldsymbol{y}, \boldsymbol{q}_n \rangle$
- spatial statistics on the sphere using the chordal distance: $\widehat{F}(r)$



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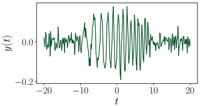

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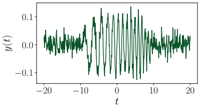
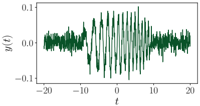
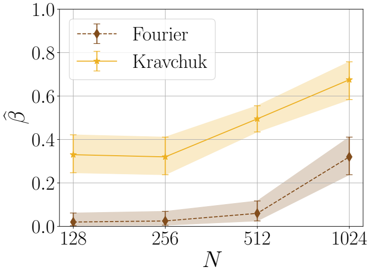


$N = 512$



$N = 1024$

Performance: power of the test computed over 200 samples



✓ higher detection power

✓ robust to small number of samples

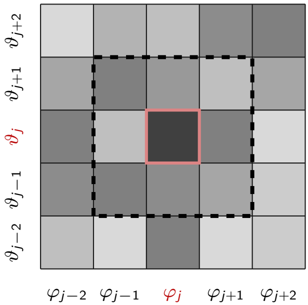
- intrinsically encoded resolution: no need for prior knowledge
- compact phase space: no edge correction

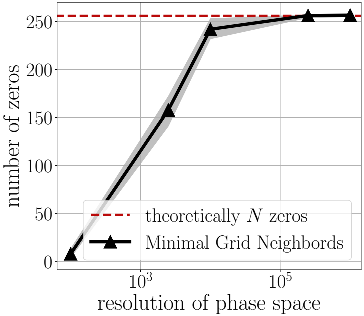
Take home messages

- a novel covariant discrete transform
  - interpreted as a coherent state decomposition
  - zeros of the Kravchuk spectrogram of white noise fully characterized
- signal processing based on spectrogram zeros
  - preliminary work using the zeros of the Fourier spectrogram
  - significant improvement using the Kravchuk spectrogram

Work in progress and perspectives

- convergence of the Kravchuk spectrogram toward the Fourier spectrogram
- interpretation of the action of $\mathrm{SO}(3)$ on $\mathbb{C}^{N+1}$
- design of a FFT counterpart to compute the Kravchuk transform





Minimal Grid Neighbors

Purpose: summary statistic s, such that $\mathbb{E}[s(y)|\mathbf{H}_0] = 0$, $\mathbb{E}[s(y)|\mathbf{H}_1] > 0$

Test settings

- Level of significance α
- Number of samples under the null hypothesis m
- Index k, chosen so that $\alpha = k/(m+1)$

Monte Carlo strategy

(i) generate m independent samples of complex white Gaussian noise;

(ii) compute their summary statistics $s_1 \geq s_2 \geq \ldots \geq s_m$;

(iii) compute the summary statistics of the observations $\boldsymbol{y}$ under concern;

(iv) if $s(\boldsymbol{y}) \geq s_k$, then reject the null hypothesis with confidence $1 - \alpha$.