Texture segmentation based on fractal attributes

Convex functional minimization and generalized Stein formalism for automated regularization parameter selection

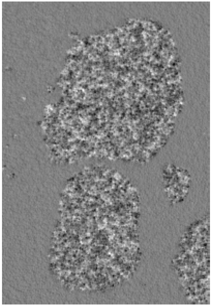
Barbara Pascal

April 7th 2023

Journées ANR Mistic

Joint work with **Patrice Abry**, **Nelly Pustelnik**, **Valérie Vidal** and **Samuel Vaiter**

Image segmentation



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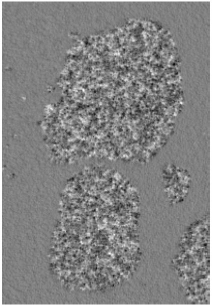
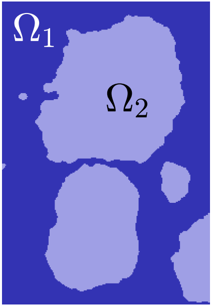
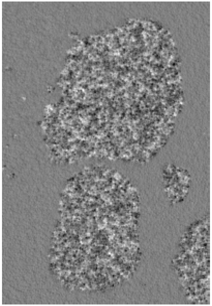


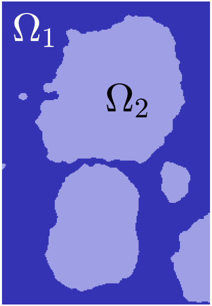


Goal : partitioning the image into K homogeneous regions

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_K$$

Image segmentation





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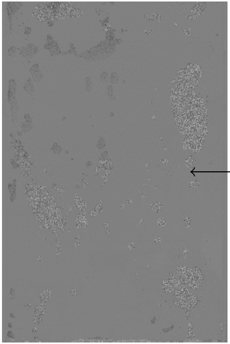
Multiphase flows through porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



Solid foam

Multiphase flows through porous media

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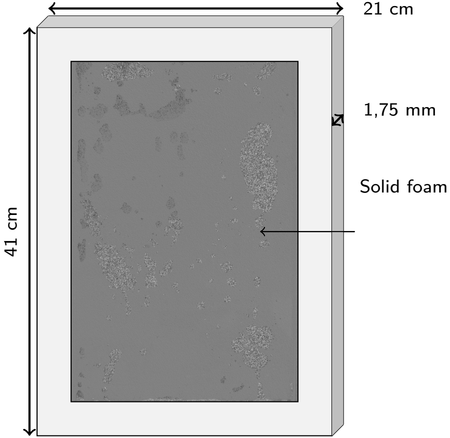

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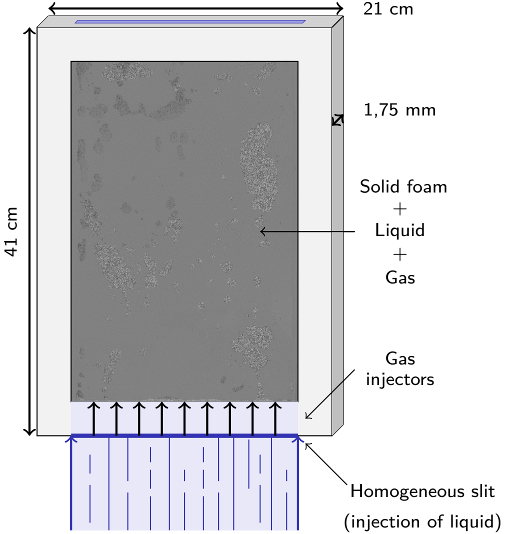


→ 1600 × 1100 pixels
→ video : ∼ 1000 images
→ phase space : ∼ 10 flow rates

Outline of the presentation

1. Texture characterization and synthesis

→ fractal attributes
- local variance σ^2
- local regularity h

2. Functional design for variational approaches

→ Penalized least squares
- free contours
- co-localized contours

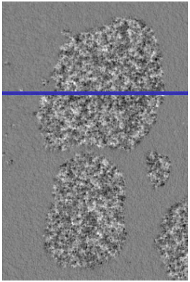
3. Accelerated minimization algorithms

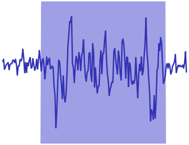
→ proximal splitting algorithms
- computation of proximity operators
- strong convexity acceleration

4. Hyperparameter tuning

→ SURE under correlated Gaussian noise
- projected estimation error

Piecewise monofractal model

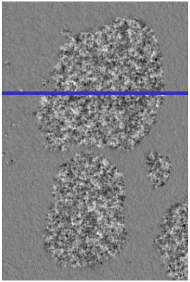


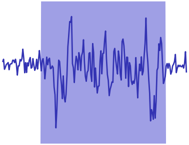


Piecewise monofractal model

Fractals attributes

- variance σ^2 — *amplitude of variations*

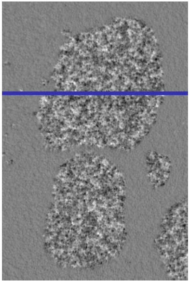


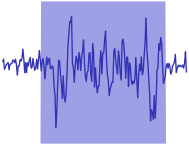


Piecewise monofractal model

Fractals attributes

- variance σ^2 — *amplitude of variations*
- local regularity h — *scale invariance*



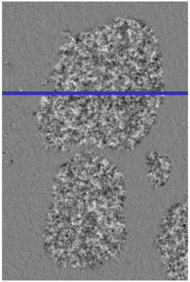


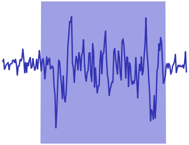
Piecewise monofractal model

Fractals attributes

- variance σ^2 *amplitude of variations*
- local regularity h *scale invariance*

$$|f(x) - f(y)| \leq \sigma(x)|x - y|^{h(x)}$$





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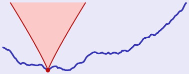
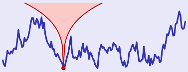
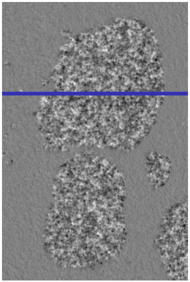
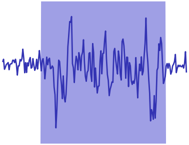


$h(x) \equiv h_1 = 0.9$



$h(x) \equiv h_2 = 0.3$





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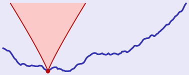

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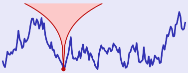

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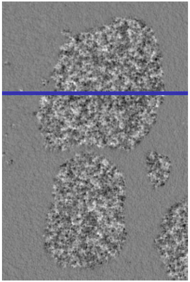
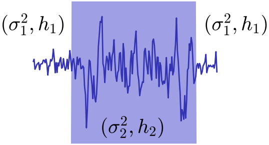
Segmentation

- h and σ^2 piecewise constant





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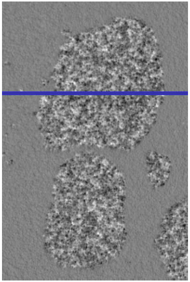
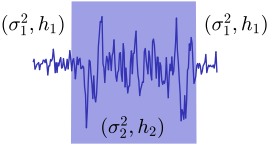
$h(x) \equiv h_1 = 0.9$ $\qquad$ $h(x) \equiv h_2 = 0.3$

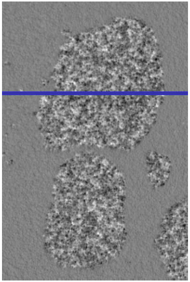
Segmentation

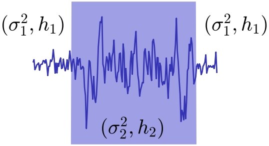
- h and σ^2 piecewise constant
- region Ω_k characterized by (h_k, σ_k^2)





Synthesis of piecewise fractal textures





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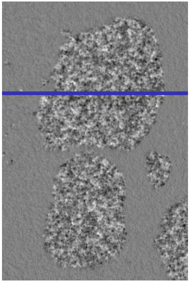
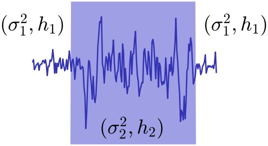
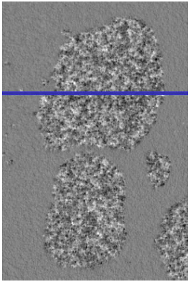
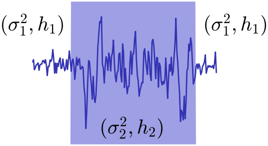




Which synthetic textures model

Synthesis of piecewise fractal textures





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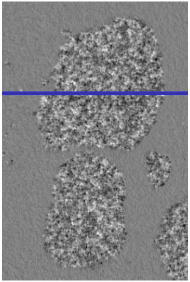
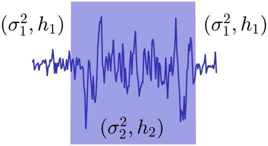
Synthesis of piecewise fractal textures





Which synthetic textures model

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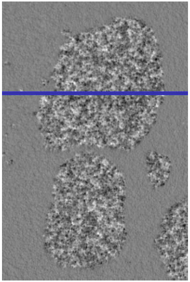
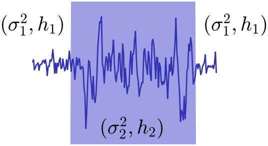
Synthesis of piecewise fractal textures





Which synthetic textures model

- resembling real-world textures,
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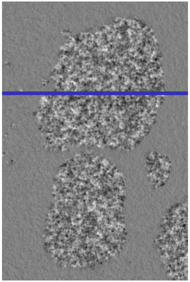
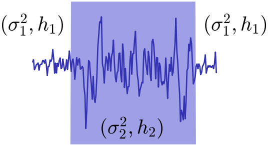
- easy to "patch"?

Synthesis of piecewise fractal textures





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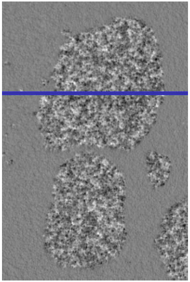
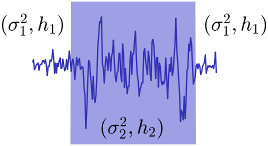
Proposition of a Gaussian random field being

Synthesis of piecewise fractal textures





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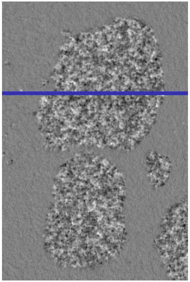
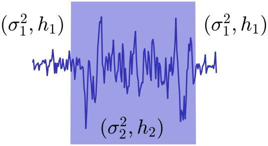
Proposition of a Gaussian random field being

- isotropic,

Synthesis of piecewise fractal textures





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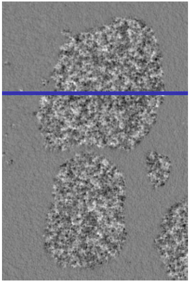
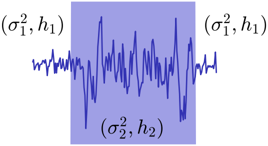
Proposition of a Gaussian random field being

- isotropic,
- self-similar, with local regularity h,

Synthesis of piecewise fractal textures





Which synthetic textures model

- resembling real-world textures,
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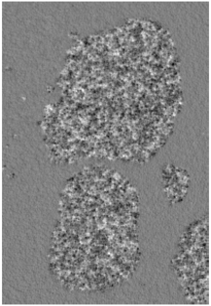
Proposition of a Gaussian random field being

- isotropic,
- self-similar, with local regularity h,
- stationary, with variance σ^2.

Synthesis of piecewise fractal textures

Real-world texture



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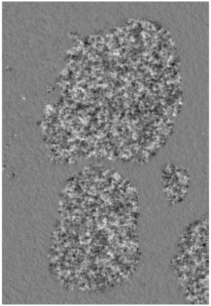
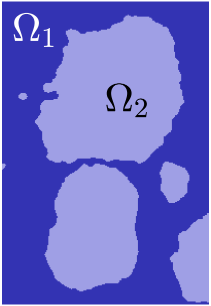


Mask



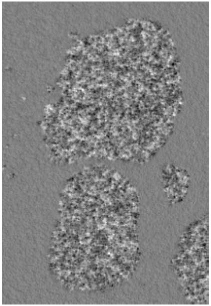
$\Omega_1 : (\sigma_1^2, h_1) \quad \Omega_2 : (\sigma_2^2, h_2)$

Synthesis of piecewise fractal textures

Fractional Brownian field

$$b_h(\underline{x}) = \sigma \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-\mathrm{i}\langle \underline{x}, \underline{k} \rangle} - 1}{C_h^{1/2} \|\underline{k}\|^{H+1}} \, \widehat{w}(\mathrm{d}\underline{k})$$

Real-world texture



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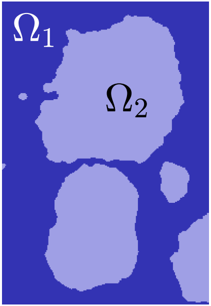

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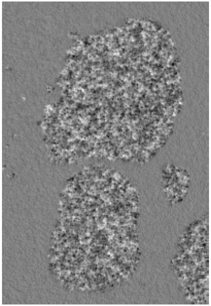

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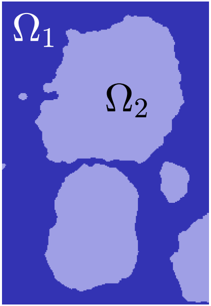
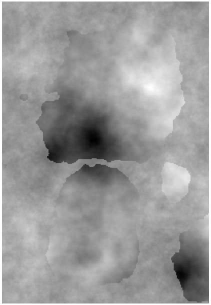


Synthetic texture



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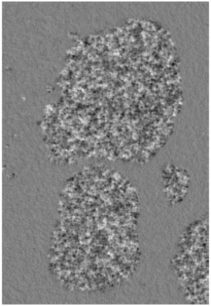
Synthesis of piecewise fractal textures

Fractional Gaussian field *stationary*

$$g_h(\underline{x}) = \frac{1}{2}\underbrace{\left(b_h(\underline{x}+\underline{e}_1) - b_h(\underline{x})\right)}_{\text{horizontal increment}} + \frac{1}{2}\underbrace{\left(b_h(\underline{x}+\underline{e}_2) - b_h(\underline{x})\right)}_{\text{vertical increment}}$$

Real-world texture



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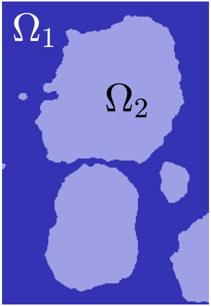

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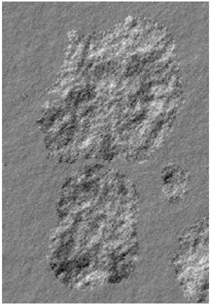

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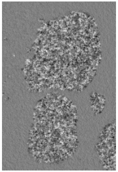
Multiscale analysis

Textured image



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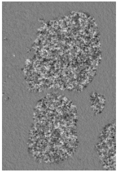
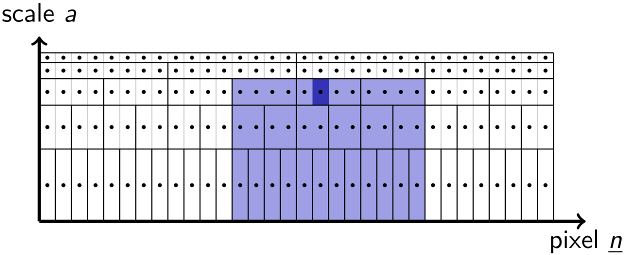


Wavelet coefficient local maximum: $\mathcal{L}_{a,\cdot}$



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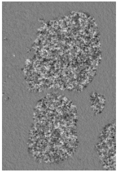
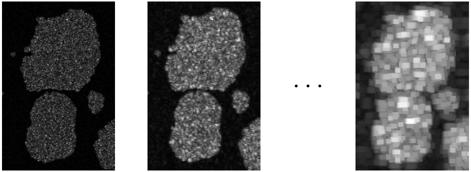
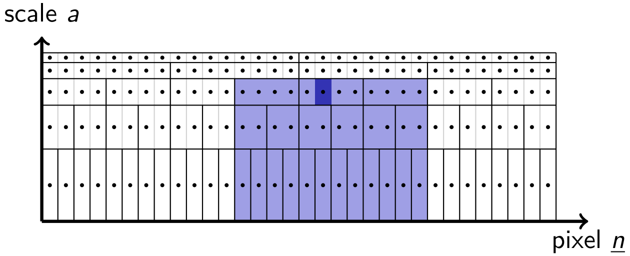


Wavelet coefficient local maximum: $\mathcal{L}_{a,\cdot}$

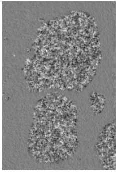
Scale $a = 2^1$ $a = 2^2$ … $a = 2^5$





Multiscale analysis

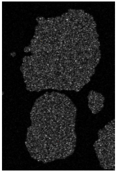
Textured image



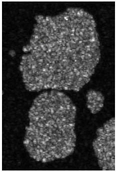
Wavelet coefficient local maximum: $\mathcal{L}_{a,\cdot}$

Scale

$a = 2^1$

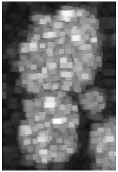


$a = 2^2$



$\ldots$

$a = 2^5$

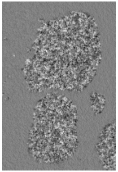


Proposition *(Jaffard, 2004), (Wendt, 2008)*

$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underset{\text{regularity}}{\boldsymbol{h}} + \underset{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}{\boldsymbol{v}}$$

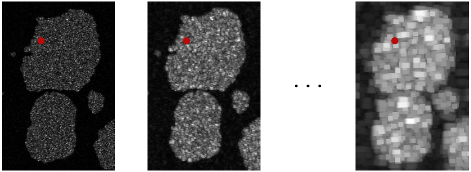
Multiscale analysis

Textured image



Wavelet coefficient local maximum: $\mathcal{L}_{a,\cdot}$

Scale $a = 2^1$ $a = 2^2$ $a = 2^5$



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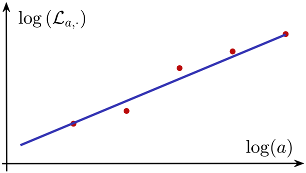
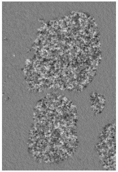


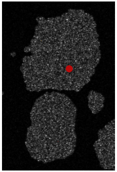
Multiscale analysis

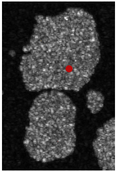
Textured image



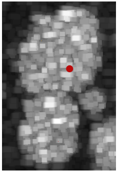
Wavelet coefficient local maximum: $\boldsymbol{\mathcal{L}}_{a,\cdot}$

Scale $a = 2^1$ $a = 2^2$ $a = 2^5$





$\cdots$



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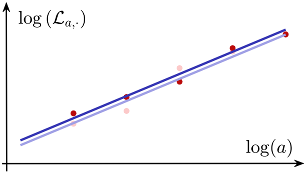
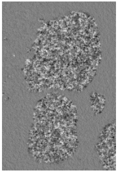


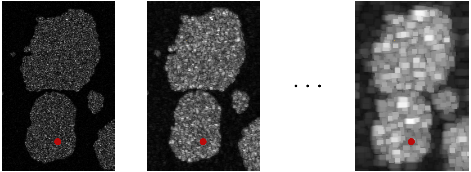
Multiscale analysis

Textured image



Wavelet coefficient local maximum: $\mathcal{L}_{a,\cdot}$

Scale $a = 2^1$ $a = 2^2$ $\cdots$ $a = 2^5$



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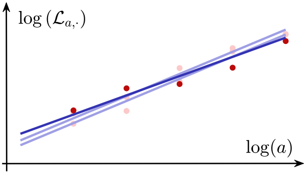
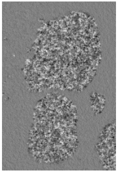


Multiscale analysis

Textured image



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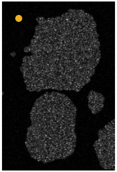
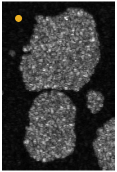
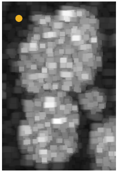




$\cdots$



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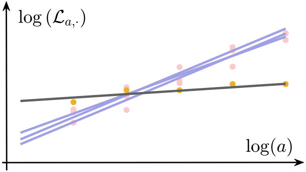
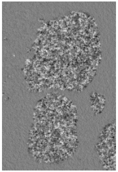


Multiscale analysis

Textured image



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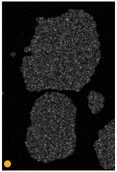
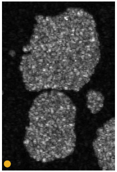
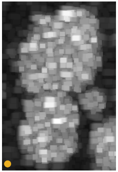




$\cdots$



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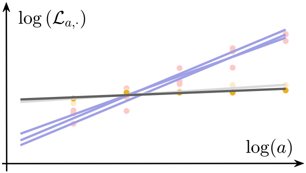
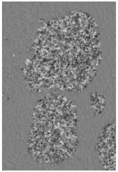


Multiscale analysis

Textured image



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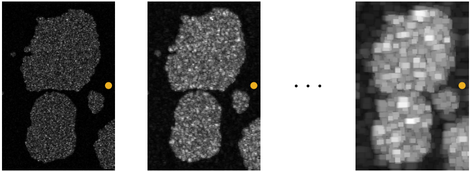

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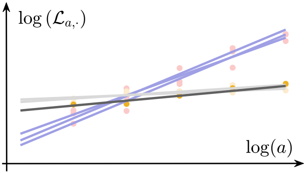
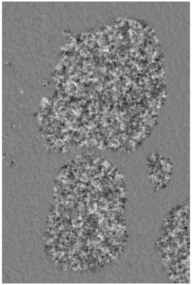


Direct pixelwise estimation

Linear regression $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

Textured image

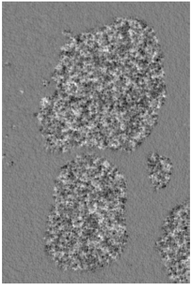


Direct pixelwise estimation

Linear regression $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\| \log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v} \right\|^2$$

Textured image

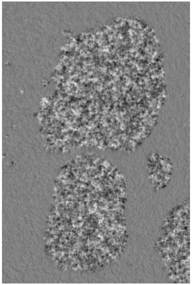


Direct pixelwise estimation

Linear regression $\log(\mathcal{L}_{a,\cdot}) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

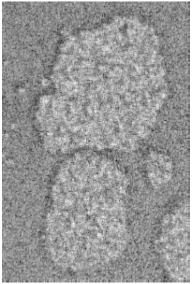
$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \|\log(\mathcal{L}_{a,\cdot}) - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2$$

Textured image



Local regularity locale $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$

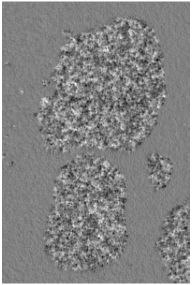


Direct pixelwise estimation

Linear regression $\underbrace{\mathbb{E}\log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{expectation}} = \log(a)\underbrace{\bar{\boldsymbol{h}}}_{\text{regularity}} + \underbrace{\bar{\boldsymbol{v}}}_{\propto\log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Textured image



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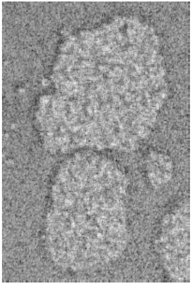


$\longrightarrow$ high estimation variance

A posteriori regularization

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

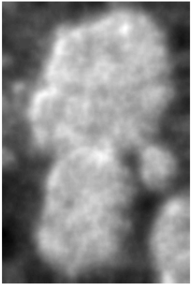
A posteriori regularization

Smoothing via filtering (linear)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Smoothing



A posteriori regularization

Smoothing via filtering (linear)

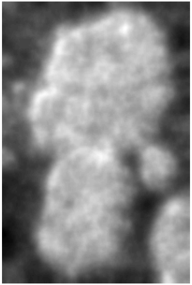
$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

ROF denoising (nonlinear)

$$\operatorname*{argmin}_{\boldsymbol{h}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Smoothing



ROF

A posteriori regularization

Smoothing via filtering (linear)

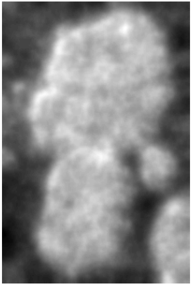
$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

ROF denoising (nonlinear)

$$\operatorname*{argmin}_{\boldsymbol{h}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Smoothing



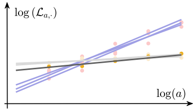
ROF

$\longrightarrow$ accumulation of estimation variance and regularization bias

Free contour and co-localized contour functionals

$$\sum_a \underbrace{\|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}}$$

$\rightarrow$ fidelity to log-linear mdoel

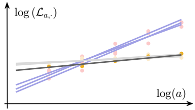


Free contour and co-localized contour functionals

$$\sum_a \underbrace{\|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

$\rightarrow$ fidelity to log-linear mdoel

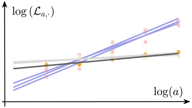
$\rightarrow$ favors piecewise constancy



Free contour and co-localized contour functionals

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \quad + \quad \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

$\rightarrow$ fidelity to log-linear mdoel



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Free contour and co-localized contour functionals

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} + \underbrace{\lambda \ \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

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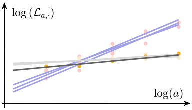


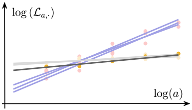
Finite differences $\mathbf{D}_1\boldsymbol{x}$ (horizontal), $\mathbf{D}_2\boldsymbol{x}$ (vertical) at each pixel

Free contour and co-localized contour functionals

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

$\rightarrow$ fidelity to log-linear mdoel

$\rightarrow$ favors piecewise constancy



Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

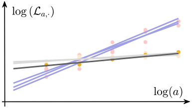
$$\mathcal{Q}_{\mathrm{L}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

Free contour and co-localized contour functionals

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

→ fidelity to log-linear mdoel

→ favors piecewise constancy



Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

$$\mathcal{Q}_{\mathrm{L}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

co-localized: $\boldsymbol{h}$, $\boldsymbol{v}$ are **concomitantly** piecewise constant

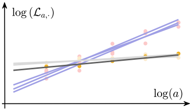
$$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \|[\alpha\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}]\|_{2,1}$$

Free contour and co-localized contour functionals

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

→ fidelity to log-linear model

→ favors piecewise constancy



Disjoint contours

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

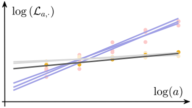
Common contours

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

Free contour and co-localized contour functionals

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

Least squares
→ fidelity to log-linear model



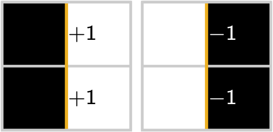
Total variation
→ favors piecewise constancy

Disjoint contours

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad \boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

Common contours



$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad \boldsymbol{v} \in \mathbb{R}^{2\times 2}$

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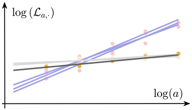
Free contour and co-localized contour functionals

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

→ fidelity to log-linear model

→ favors piecewise constancy



Disjoint contours

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

$\mathcal{Q}_C(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 2 + \sqrt{2} \simeq 3,4$

Common contours

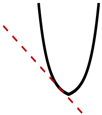
$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

$\mathcal{Q}_C(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 2\sqrt{2} \simeq 2,8$

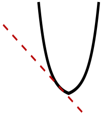
Functional minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \;\; \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \;\; + \;\; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



Functional minimization

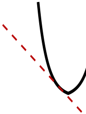
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



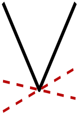
- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$

Functional minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \ + \ \lambda \ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



nonsmooth

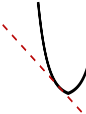


- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$
- implicit subgradient descent: proximal point algorithm

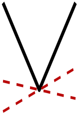
$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \boldsymbol{u}^n, \ \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \ \Leftrightarrow \ \boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

Functional minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



nonsmooth



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- proximal splitting algorithm

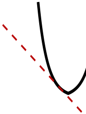
$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau\|\mathcal{L}-\Phi\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau\mathbf{D}^\top \boldsymbol{y}^{n+1}\right), \qquad \Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

Functional minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \quad + \quad \lambda \; \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



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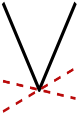

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$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \boldsymbol{u}^n, \;\; \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \;\; \Leftrightarrow \;\; \boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

- proximal splitting algorithm $\qquad \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}) = \underset{\boldsymbol{u}}{\mathrm{argmin}} \, \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{u}\|^2 + \tau\varphi(\boldsymbol{u})$

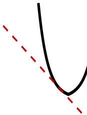
$$\boldsymbol{y}^{n+1} = \mathrm{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

$$\boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau \mathbf{D}^\top \boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

Computation of proximity operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \ + \ \lambda \ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



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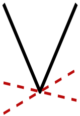
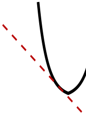


Least squares : $\|\log \boldsymbol{\mathcal{L}} - \Phi(\boldsymbol{h}, \boldsymbol{v})\|^2, \quad \Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Computation of proximity operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



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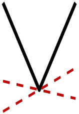


Least squares : $\|\log \boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}(\boldsymbol{h}, \boldsymbol{v})\|^2, \quad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

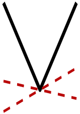
Proposition *(Pascal, 2019)*

$$(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \operatorname{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|^2}(\boldsymbol{h}, \boldsymbol{v}) \iff (\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \left(\mathbf{I} + \tau\boldsymbol{\Phi}^\top\boldsymbol{\Phi}\right)^{-1}\left((\boldsymbol{h}, \boldsymbol{v}) + \tau\boldsymbol{\Phi}^\top \log \boldsymbol{\mathcal{L}}\right)$$

Computation of proximity operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

nonsmooth



Least squares : $\|\log \boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}(\boldsymbol{h}, \boldsymbol{v})\|^2$, $\quad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Proposition *(Pascal, 2019)*

Let $S_m = \sum_a \log^m(a)$, $\mathcal{D} = (1 + \tau S_2)(1 + \tau S_0) - \tau^2 S_1^2$,
$\boldsymbol{\mathcal{T}} = \sum_a \log \boldsymbol{\mathcal{L}}_a$ and $\boldsymbol{\mathcal{G}} = \sum_a \log(a) \log \boldsymbol{\mathcal{L}}_a$, then

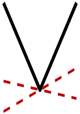
$$(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|^2}(\boldsymbol{h}, \boldsymbol{v}) \iff (\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \left(\mathbf{I} + \tau \boldsymbol{\Phi}^\top \boldsymbol{\Phi}\right)^{-1} \left((\boldsymbol{h}, \boldsymbol{v}) + \tau \boldsymbol{\Phi}^\top \log \boldsymbol{\mathcal{L}}\right)$$

$$\iff \begin{cases} \widetilde{\boldsymbol{h}} = \mathcal{D}^{-1}\left((1 + \tau S_0)(\tau \boldsymbol{\mathcal{G}} + \boldsymbol{h}) - \tau S_1(\tau \boldsymbol{\mathcal{T}} + \boldsymbol{v})\right) \\ \widetilde{\boldsymbol{v}} = \mathcal{D}^{-1}\left((1 + \tau S_2)(\tau \boldsymbol{\mathcal{T}} + \boldsymbol{v}) - \tau S_1(\tau \boldsymbol{\mathcal{G}} + \boldsymbol{h})\right) \end{cases}$$

Strong convexity accelerated algorithm

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{minimize}} \;\; \sum_a \underset{\textbf{Least squares}}{\underline{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}} \;+\; \lambda\, \underset{\textbf{Total variation}}{\underline{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}}$$

nonsmooth

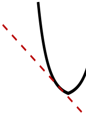


primal-dual algorithm *(Chambolle, 2011)*

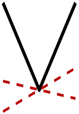
δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to+\infty}{\longrightarrow} 0$

Strong convexity accelerated algorithm

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



nonsmooth



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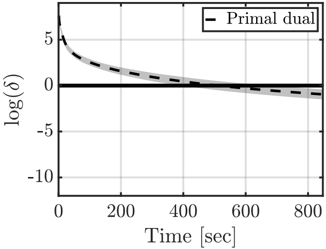
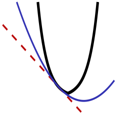
$$\delta\text{: duality gap, } \delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to+\infty}{\longrightarrow} 0$$



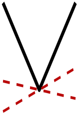
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underset{\textbf{Least squares}}{\underline{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}} \; + \; \lambda \; \underset{\textbf{Total variation}}{\underline{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}}$$



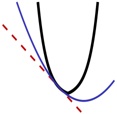
$\boldsymbol{\mu}$-strongly convex

nonsmooth



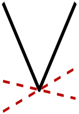
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



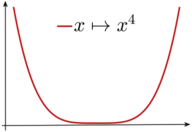
$\boldsymbol{\mu}$-strongly convex

nonsmooth

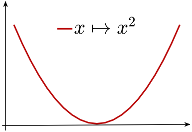


Forte convexité

- φ $\boldsymbol{\mu}$-strongly convex ssi $\varphi - \frac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convex



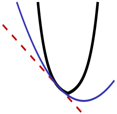
✓ strictly convex
✗ not strongly convex



✓ strictly convex
✓ 1-strongly convex

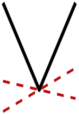
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



$\boldsymbol{\mu}$-strongly convex

nonsmooth

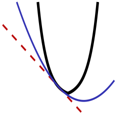


Forte convexité

- φ $\boldsymbol{\mu}$-strongly convex ssi $\varphi - \dfrac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convex
- φ $\mathcal{C}^2$ de hessienne $\boldsymbol{H}\varphi \succeq 0 \implies \boldsymbol{\mu} = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

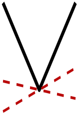
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



μ-strongly convex

nonsmooth



Forte convexité

- φ μ-strongly convex ssi $\varphi - \frac{\mu}{2}\|\cdot\|^2$ convex
- φ $\mathcal{C}^2$ de hessienne $\boldsymbol{H}\varphi \succeq 0 \implies \mu = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

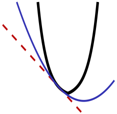
Proposition *(Pascal, 2019)*

$\sum_a \|\log \boldsymbol{\mathcal{L}} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2$ is μ-strongly convex.

$a_{\min} = 2^1$, $a_{\max}$	2^2	2^3	2^4	2^5	2^6
$\mu = \min \mathrm{Sp}\left(2\boldsymbol{\Phi}^\top \boldsymbol{\Phi}\right)$	0.29	**0.72**	1.20	1.69	2.20

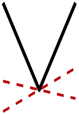
Strong convexity accelerated algorithm

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



μ-strongly convex

nonsmooth



Accelerated primal-dual algorithm *(Chambolle, 2011)*

for $n = 0, 1, \ldots$ $\qquad \boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})$

$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma_n(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma_n \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

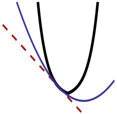
$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau_n\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau_n \mathbf{D}^\top \boldsymbol{y}^{n+1}\right)$$

$$\theta_n = \sqrt{1 + 2\mu\tau_n}, \quad \tau_{n+1} = \tau_n/\theta_n, \quad \sigma_{n+1} = \theta_n\sigma_n$$

$$\bar{\boldsymbol{x}}^{n+1} = \boldsymbol{x}^{n+1} + \theta_n^{-1}\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

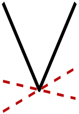
Strong convexity accelerated algorithm

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



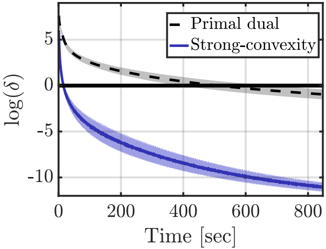
μ-strongly convex

nonsmooth



Accelerated primal-dual algorithm *(Chambolle, 2011)*

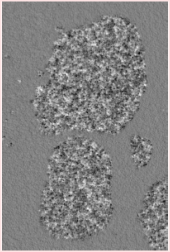
δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to+\infty}{\longrightarrow} 0$



Segmentation by iterative thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

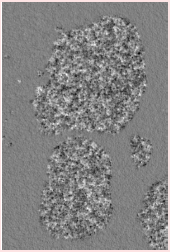
Textured image



Segmentation by iterative thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

Textured image

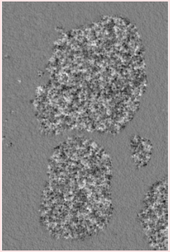


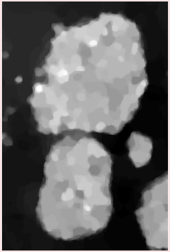
Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Segmentation by iterative thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

Textured image | Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Co-localized contours $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

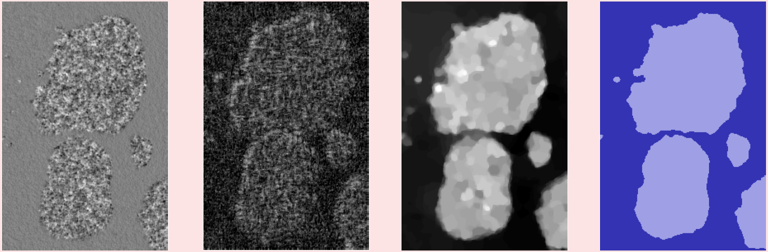




Segmentation by iterative thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$

Textured image | Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Co-localized contours $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ | Thresholded estimate† $S\widehat{\boldsymbol{h}}^{\mathrm{C}}$



† *(Cai, 2013)*

State-of-the-art methods for texture segmentation

Threshold–ROF on $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

(Nafornita, 2014), (Pustelnik, 2016)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Based only on local regularity $\boldsymbol{h}$.

State-of-the-art methods for texture segmentation

Threshold–ROF on $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

(Nafornita, 2014), (Pustelnik, 2016)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \; \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Lin. reg.

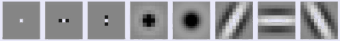
ROF

Threshold

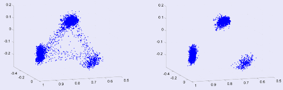
Based only on local regularity $\boldsymbol{h}$.

Matrix factorization based segmentation† *(Yuan, 2015)*

(i) local histograms



(ii) matrix factorization



(a) (b)

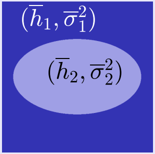
Fig. 2. Scatterplot of features in subspace. (a) Scatterplot of features projected onto the 3-d subspace. (b) Scatterplot after removing features with high edgeness.

†https://sites.google.com/site/factorizationsegmentation/

Compared performances on synthetic textures

Piecewise monofractal texture synthesis *(Pascal, 2019)*

- mask : $\Omega = \Omega_1 \sqcup \Omega_2$,
- attributes : $(\bar{h}_k, \bar{\sigma}^2_k)_{k=1,2}$

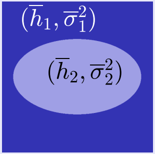


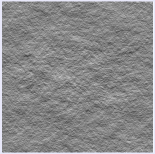
Compared performances on synthetic textures

Piecewise monofractal texture synthesis *(Pascal, 2019)*

- mask : $\Omega = \Omega_1 \sqcup \Omega_2$,
- attributes : $(\bar{h}_k, \bar{\sigma}_k^2)_{k=1,2}$

E.g., $\bar{h}_1 = 0.5$, $\bar{\sigma}_1^2 = 0.6$
$\bar{h}_2 = 0.6$, $\bar{\sigma}_2^2 = 0.7$



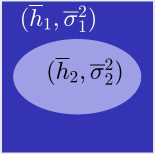


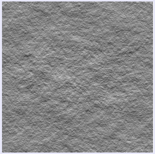
Compared performances on synthetic textures

Piecewise monofractal texture synthesis *(Pascal, 2019)*

- mask : $\Omega = \Omega_1 \sqcup \Omega_2$,
- attributes : $(\overline{h}_k, \overline{\sigma}^2_k)_{k=1,2}$

E.g., $\overline{h}_1 = 0.5,\ \overline{\sigma}^2_1 = 0.6$
$\overline{h}_2 = 0.6,\ \overline{\sigma}^2_2 = 0.7$



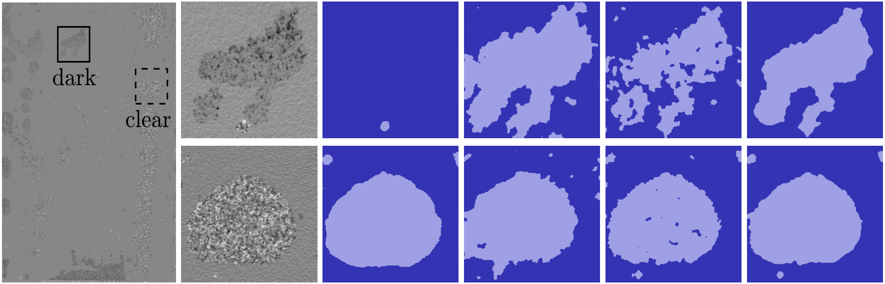


Averaged segmentation performance over 5 realizations

Yuan	T-ROF	Free	Co-localized
		contours	
			
$71.1 \pm 1.3\%$	$78.5 \pm 1.1\%$	$90.2 \pm 1.9\%$	$91.1 \pm 1.5\%$

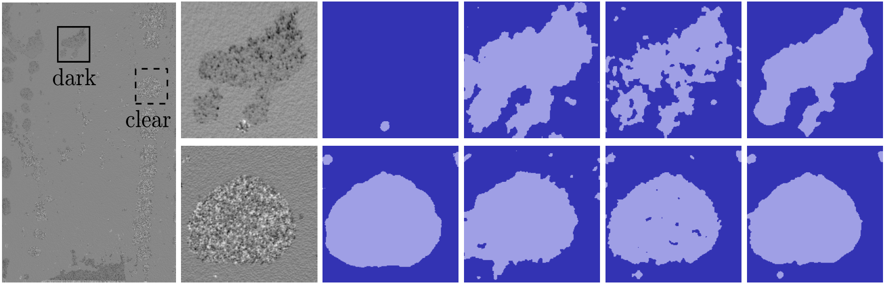
Low activity : $Q_\mathrm{G} = 300\mathrm{mL}/\mathrm{min}$ - $Q_\mathrm{L} = 300\mathrm{mL}/\mathrm{min}$

Flow | Zooms | Yuan | T-ROF | free contours | co-localized contours



Low activity : $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$

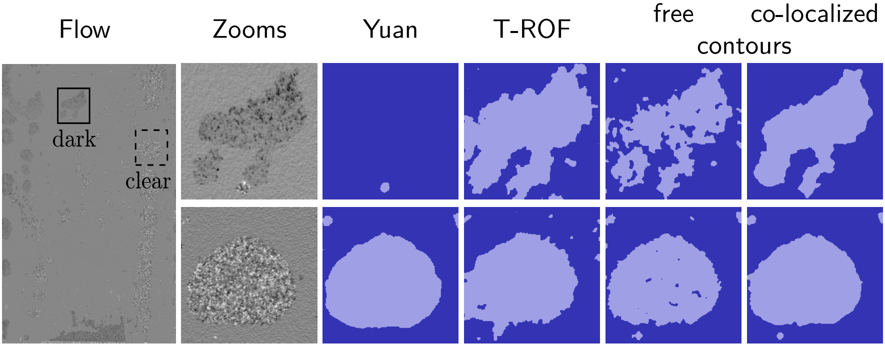
Flow Zooms Yuan T-ROF free contours co-localized



Liquid: $h_{\mathrm{L}} = 0.4$

Gas: $h_{\mathrm{G}} = 0.9$

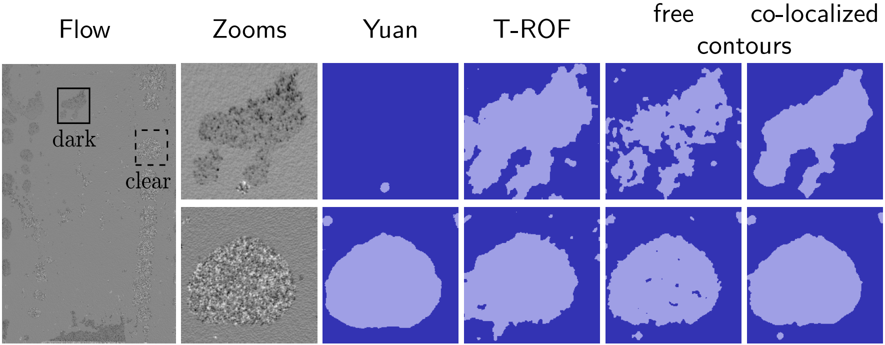
Low activity : $Q_{\mathrm{G}} = 300\text{mL}/\text{min}$ - $Q_{\mathrm{L}} = 300\text{mL}/\text{min}$



Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$

Low activity : $Q_\mathrm{G} = 300\mathrm{mL}/\mathrm{min}$ - $Q_\mathrm{L} = 300\mathrm{mL}/\mathrm{min}$



Liquid: $h_\mathrm{L} = 0.4$ $\sigma^2_\mathrm{dark} = 10^{-2}$

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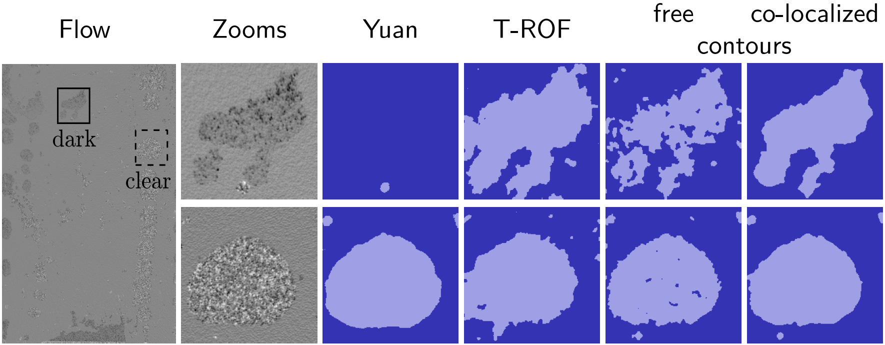
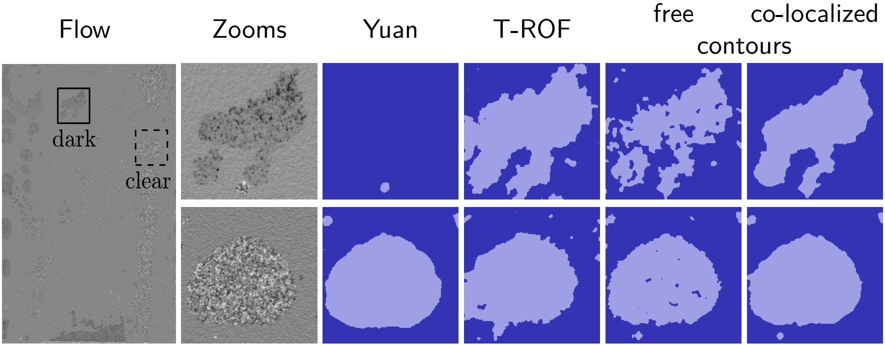
Low activity : $Q_{\mathrm{G}} = 300\text{mL}/\text{min}$ - $Q_{\mathrm{L}} = 300\text{mL}/\text{min}$



Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\text{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ | $\sigma^2_{\text{dark}} = 10^{-2}$ (dark bubbles)

Low activity : $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$

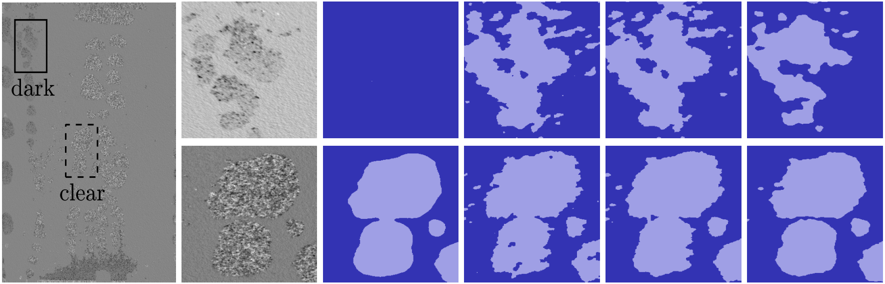


Liquid: $h_{\mathrm{L}} = 0.4$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$ (dark bubbles)
$\quad \sigma^2_{\mathrm{clear}} = 10^{-1}$ (clear bubbles)

Transition : $Q_{\mathrm{G}} = 400\mathrm{mL}/\min$ - $Q_{\mathrm{L}} = 700\mathrm{mL}/\min$

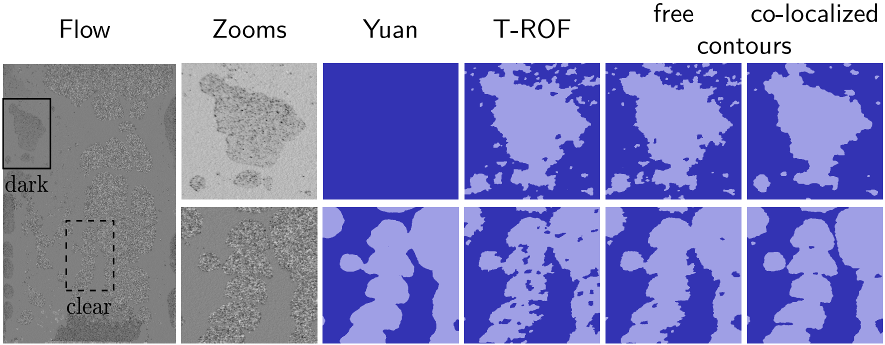
Flow | Zooms | Yuan | T-ROF | free contours | co-localized contours



Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$ (dark bubbles)
$\sigma^2_{\mathrm{clear}} = 10^{-1}$ (clear bubbles).

High activity : $Q_{\mathrm{G}} = 1200\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$

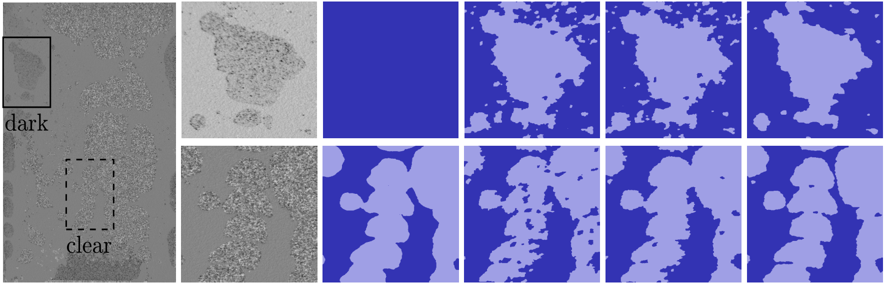


Liquid: $h_{\mathrm{L}} = 0.4$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\quad \Big| \quad \sigma^2_{\mathrm{dark}} = 10^{-2}$ (dark bubbles)
$\sigma^2_{\mathrm{clear}} = 10^{-1}$ (clear bubbles).

High activity : $Q_{\mathrm{G}} = 1200\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$

Flow	Zooms	Yuan	T-ROF	free contours	co-localized contours



Computational time		1s	12s	700s	**2100s**

Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ | $\sigma^2_{\mathrm{dark}} = 10^{-2}$ (dark bubbles)
$\sigma^2_{\mathrm{clear}} = 10^{-1}$ (clear bubbles).

Selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\mathcal{L}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Selection of regularization parameters

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Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

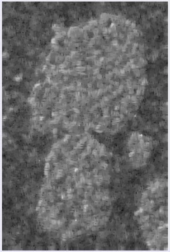
$(\lambda; \alpha) = (0; 0)$

Selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ Co-localized contour estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda; \alpha) = (0; 0)$ $(\lambda, \alpha) = (0.5; 0.5)$



too small

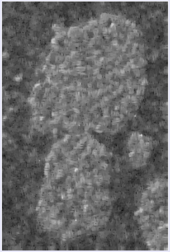
Selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Co-localized contour estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda; \alpha) = (0; 0)$

$(\lambda, \alpha) = (0.5; 0.5)$



too small

$(\lambda; \alpha) = (500; 500)$

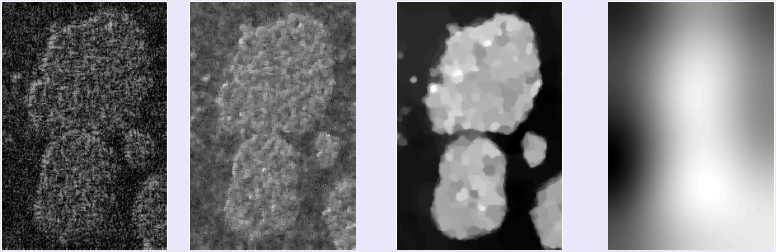
too large

Selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Co-localized contour estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda; \alpha) = (0; 0)$ | $(\lambda, \alpha) = (0.5; 0.5)$ | $(\lambda^{\dagger}, \alpha^{\dagger}) = (11.5; 0.8)$ | $(\lambda; \alpha) = (500; 500)$



too small | optimal | too large

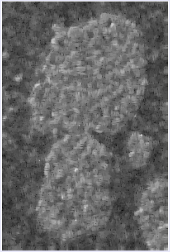
Selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ Co-localized contour estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

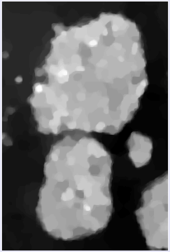
$(\lambda; \alpha) = (0; 0)$

$(\lambda, \alpha) = (0.5; 0.5)$



too small

$(\lambda^\dagger, \alpha^\dagger) = (11.5; 0.8)$



optimal

$(\lambda; \alpha) = (500; 500)$

too large

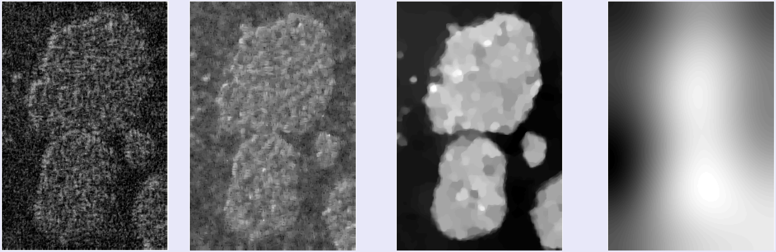
What *optimal* means?

Selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ Co-localized contour estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda;\alpha) = (0;0)$ $(\lambda,\alpha) = (0.5;0.5)$ $(\lambda^\dagger,\alpha^\dagger) = (11.5;0.8)$ $(\lambda;\alpha) = (500;500)$



too small optimal too large

What *optimal* means? How to find $\lambda^\dagger$ and $\alpha^\dagger$?

Parameter tuning (systematic search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

Parameter tuning (systematic search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$

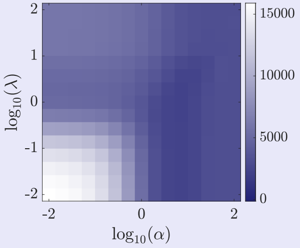
Parameter tuning (systematic search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \| \log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v} \|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



Parameter tuning (systematic search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

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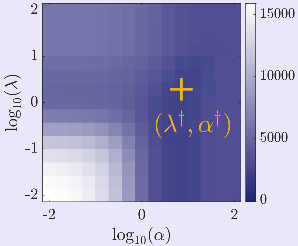
$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



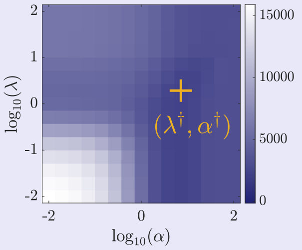
Parameter tuning (systematic search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

?

Parameter tuning (systematic search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \operatorname*{argmin}_{\boldsymbol{h},\boldsymbol{v}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

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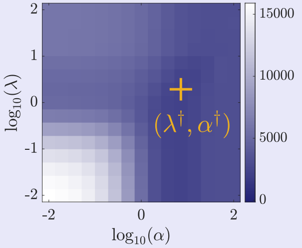

$\bar{\boldsymbol{h}}$: unknown!

?

Stein Unbiased Risk Estimate
(SURE)

Stein Unbiased Risk Estimate (Principle)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: ground truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

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Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

E.g., $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

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Quadratic error $R(\lambda) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \stackrel{?}{=} \mathbb{E}_{\boldsymbol{\zeta}} \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ unknown

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Theorem *(Stein, 1981)*

Let $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$ be an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable with respect to $\boldsymbol{y}$,
- such that $\boldsymbol{\zeta} \mapsto \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \boldsymbol{\zeta} \rangle$ is integrable w. r. t. $\mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$.

$$\widehat{R}(\boldsymbol{y}; \lambda) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\rho^2 \operatorname{tr}\left(\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)\right) - \rho^2 P$$
$$\Longrightarrow R(\lambda) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{y}; \lambda)].$$

Stein Unbiased Risk Estimate generalized

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

Stein Unbiased Risk Estimate generalized

Observations $\boldsymbol{y} = \Phi \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \Phi : \mathbb{R}^{P \times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

E.g., free and co-localized contour estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$

$\log \mathcal{L} = \Phi(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$

$\Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Stein Unbiased Risk Estimate generalized

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

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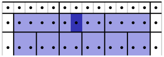

Stein Unbiased Risk Estimate generalized

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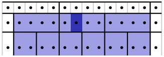

$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \boldsymbol{0})$

Stein Unbiased Risk Estimate generalized

Observations $\boldsymbol{y} = \Phi\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\Phi : \mathbb{R}^{P\times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

E.g., free and co-localized contour estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$

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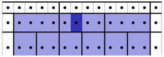

$\Pi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\Pi}(\Lambda) \triangleq \mathbb{E}_\zeta \|\Pi\widehat{\boldsymbol{x}}(\boldsymbol{y}; \Lambda) - \Pi\bar{\boldsymbol{x}}\|^2$

Stein Unbiased Risk Estimate generalized

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

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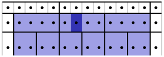

$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Theorem *(Pascal, 2020)*

Let $\mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top$ and $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

Stein Unbiased Risk Estimate generalized

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

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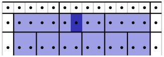

$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

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- weakly differentiable with respect to $\boldsymbol{y}$,
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Stein Unbiased Risk Estimate generalized

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

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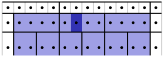

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$$\widehat{R}(\boldsymbol{\Lambda}) \triangleq \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right) - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

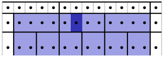
Stein Unbiased Risk Estimate generalized

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

E.g., free and co-localized contour estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$ $\quad$ $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$ $\quad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Theorem *(Pascal, 2020)*

Let $\mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top$ and $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable with respect to $\boldsymbol{y}$,
- such that $\zeta \mapsto \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \zeta; \lambda), \mathbf{A}\zeta \rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \mathcal{S})$.

$$\widehat{R}(\boldsymbol{\Lambda}) \triangleq \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathrm{tr}\left(\mathcal{S}\mathbf{A}^\top\boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right) - \mathrm{tr}\left(\mathbf{A}\mathcal{S}\mathbf{A}^\top\right)$$

$$\Longrightarrow R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \mathbb{E}_\zeta[\widehat{R}(\boldsymbol{\Lambda})].$$

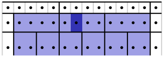
Stein Unbiased Risk Estimate generalized

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N,\ \boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

E.g., free and co-localized contour estimators $\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha)$

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\qquad \boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$ $\qquad \mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \boldsymbol{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Theorem *(Pascal, 2020)*

Let $\mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top \boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top$ and $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable with respect to $\boldsymbol{y}$,
- such that $\boldsymbol{\zeta} \mapsto \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \mathbf{A}\boldsymbol{\zeta} \rangle$ is integrable w.r.t. $\mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$.

$$\widehat{R}(\boldsymbol{\Lambda}) \triangleq \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\operatorname{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right) - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

$$\Longrightarrow R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{\Lambda})].$$

Stein Unbiased Risk Estimate (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

Erreur d'estimation projetée $R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \| \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}} \|^2$

Generalized Finite Differences Monte Carlo SURE

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) \triangleq \left\| \mathbf{A} \left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y} \right) \right\|^2 + \frac{2}{\nu} \left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi} \left(\widehat{\boldsymbol{x}}(\boldsymbol{y} + \nu\boldsymbol{\varepsilon}; \boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) \right), \boldsymbol{\varepsilon} \right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top \right)$$

Stein Unbiased Risk Estimate (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

Erreur d'estimation projetée $R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Generalized Finite Differences Monte Carlo SURE

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) \triangleq \|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\|^2 +$$
$$\frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

Theorem *(Pascal, 2020)*

Let $(\boldsymbol{y};\boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})$ un estimateur de $\bar{\boldsymbol{x}}$

- uniformly Lipschitz w.r.t. $\boldsymbol{y}$,
- such that $\forall \boldsymbol{\Lambda} \in \mathbb{R}^L$, $\widehat{\boldsymbol{x}}(\mathbf{0}_P;\boldsymbol{\Lambda}) = \mathbf{0}_N$. Then

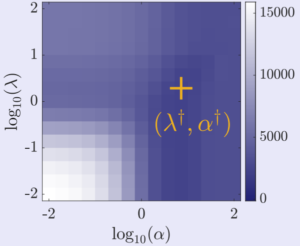
$$R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \lim_{\nu\to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}}\left[\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}})\right]$$

Parameter tuning (Systematic search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

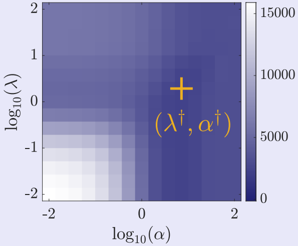
$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

Parameter tuning (Systematic search)

$$\left(\widehat{\boldsymbol{h}}\ ,\widehat{\boldsymbol{v}}\ \right)(\boldsymbol{\mathcal{L}};\lambda,\alpha)=\underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}}\sum_a \|\log\boldsymbol{\mathcal{L}}_{a,\cdot}-\log(a)\boldsymbol{h}-\boldsymbol{v}\|^2+\lambda\mathcal{Q}\ (\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

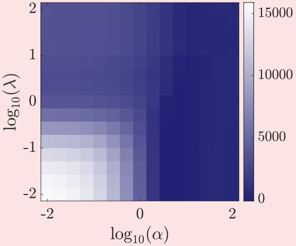
$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda,\alpha)=\left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}};\lambda,\alpha)-\bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}};\lambda,\alpha|\,\boldsymbol{\mathcal{S}})$$

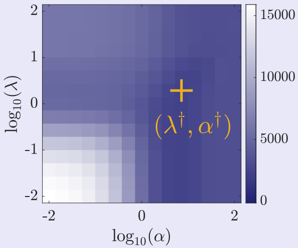


Parameter tuning (Systematic search)

$$\left(\widehat{\boldsymbol{h}}\ , \widehat{\boldsymbol{v}}\ \right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}\ (\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v};\alpha)$$

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda,\alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}};\lambda,\alpha) - \bar{\boldsymbol{h}}\right\|^2$$



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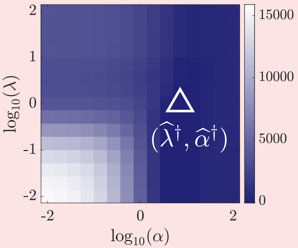

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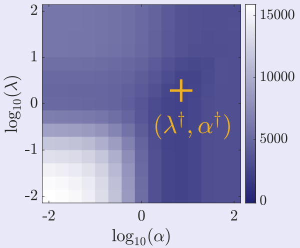

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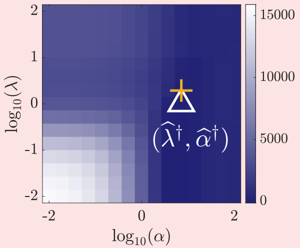
$$\widehat{R}_{\nu,\varepsilon}(\mathcal{L}; \lambda, \alpha | \mathcal{S})$$



Systematic search of regularization parameters

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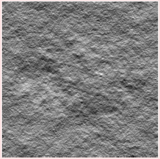
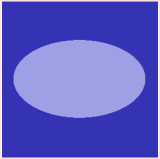
Example





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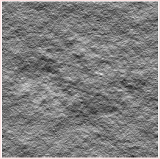
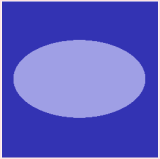
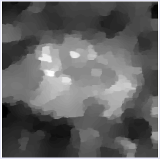




$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}}; \lambda^\dagger, \alpha^\dagger)$
(grid)



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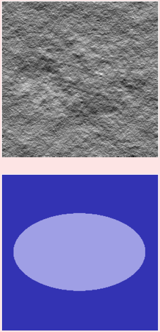
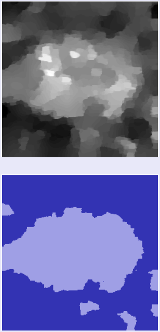
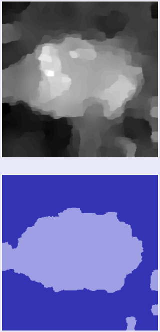


$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}}; \lambda^\dagger, \alpha^\dagger)$ (grid)



$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^\dagger, \widehat{\alpha}^\dagger)$ (grid)



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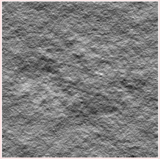	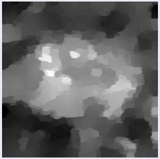	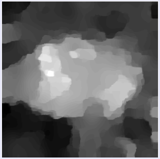
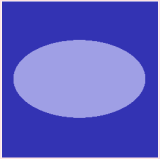		

$15 \times 15 = \mathbf{225}$ parameters $\Rightarrow$ grid search is very costly!

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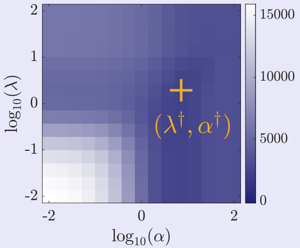
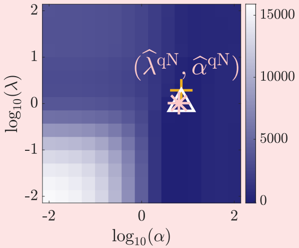


$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$



Automated selection of regularization parameters

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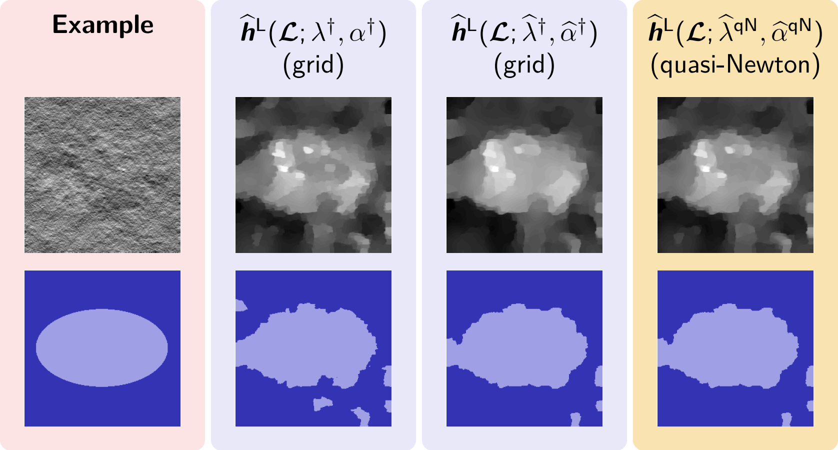

40 calls of the estimator v.s. **225** over a grid

Conclusion

- Local regularity and local variance
  - able to characterize real-world textures
  - complementary attributes $\rightarrow$ ability to discriminate finely

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Conclusion

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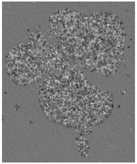
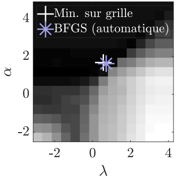
  - able to characterize real-world textures
  - complementary attributes → ability to discriminate finely
- Simultaneous estimation and regularization
  - significant reduction of the estimation error
  - accurate and regular contours thanks to the co-localized penalization
- Fast and automated algorithms
  - capacity to handle large amount of data
  - objectivity and reproducibility



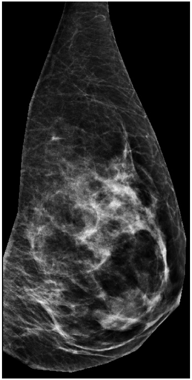


Perspectives: application to anticipate breast cancer

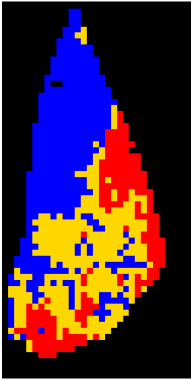
Observation: local regularity reflects health of breast tissues

(Marin et al., *Medical Physics*, 2017)

Mammogram



Local regularity (WTMM)



Microenvironment analysis via density fluctuations

Healthy $\Leftrightarrow$ structured

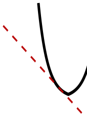
- $h < 0.5$: negative correlations
- $h > 0.5$: positive correlations

Prone to cancer $\Leftrightarrow$ unstructured

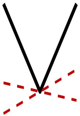
- $h = 0.5$: absence of correlations

Computation of proximity operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least squares}} \quad + \quad \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total variation}}$$



nonsmooth



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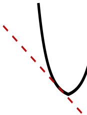

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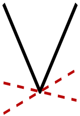

E.g., Mixed norm: for $\boldsymbol{z} = [\boldsymbol{z}_1; \ldots, ; \boldsymbol{z}_I]$

$$\mathcal{Q}(\boldsymbol{z}) = \|\boldsymbol{z}\|_{2,1} = \sum_{\underline{n}\in\Omega} \sqrt{\sum_{i=1}^{I} z_i^2(\underline{n})} = \sum_{\underline{n}\in\Omega} \|\boldsymbol{z}(\underline{n})\|_2$$

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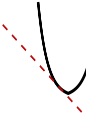

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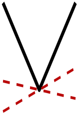

E.g., **Mixed norm:** for $\boldsymbol{z} = [\boldsymbol{z}_1; \ldots; \boldsymbol{z}_I]$

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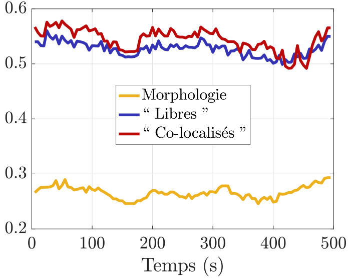
$$\boldsymbol{p} = \mathrm{prox}_{\lambda\|\cdot\|_{2,1}}(\boldsymbol{z}) \quad \Leftrightarrow \quad p_i(\underline{n}) = \max\left(0, 1 - \frac{\lambda}{\|\boldsymbol{z}(\underline{n})\|_2}\right) z_i(\underline{n})$$

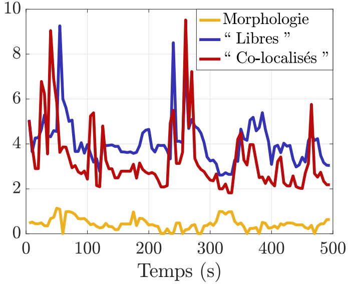
Écoulement multiphasiques en milieu poreux

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)

Fraction de gaz dans la cellule



Périmètre d'interface



Computation of degree of freedom

Degrees of freedom $\qquad \mathrm{dof} \triangleq \mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)$

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$$\mathrm{tr}(\mathbf{M}) = \mathbb{E}_{\boldsymbol{\varepsilon}} \langle \mathbf{M}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}\rangle, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_P)$$

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Proposition *(Pascal, 2020)*

Let $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

- uniformly Lipschitz with respect to $\boldsymbol{y}$,
- such that $\forall \boldsymbol{\Lambda} \in \mathbb{R}^L$, $\widehat{\boldsymbol{x}}(\mathbf{0}_P; \boldsymbol{\Lambda}) = \mathbf{0}_N$. Then

$$\mathbb{E}_{\boldsymbol{\zeta}} [\mathrm{dof}] = \lim_{\nu\to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}} \left[\frac{1}{\nu} \left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi} \left(\widehat{\boldsymbol{x}}(\boldsymbol{y} + \nu\boldsymbol{\varepsilon}; \boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon} \right\rangle \right]$$

Automated minimization of SURE

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

Generalized FDMC SURE $\lim_{\nu \to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) = R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda})$

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Goal : $\underset{\boldsymbol{\Lambda}}{\text{minimize}}\ \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \mid \boldsymbol{\mathcal{S}})$ for given $\boldsymbol{y}$, $\boldsymbol{\mathcal{S}}$

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Quasi-Newton of Broyden-Fletcher-Goldfarb-Shanno *(Nocedal, 2006)*

for $t = 0, 1, \dots$

$\boldsymbol{d}^{[t]} = -\boldsymbol{H}^{[t]} \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *descent direction*

$\alpha^{[t]} \in \underset{\alpha\in\mathbb{R}}{\mathrm{Argmin}}\ \widehat{R}(\boldsymbol{\Lambda}^{[t]} + \alpha \boldsymbol{d}^{[t]})$ *line search*

$\boldsymbol{\Lambda}^{[t+1]} = \boldsymbol{\Lambda}^{[t]} + \alpha^{[t]} \boldsymbol{d}^{[t]}$

$\boldsymbol{u}^{[t]} = \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t+1]}) - \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *gradient increment*

$\boldsymbol{H}^{[t+1]} = \mathrm{BFGS}(\boldsymbol{H}^{[t]}, \boldsymbol{d}^{[t]}, \boldsymbol{u}^{[t]})$ *inverse Hessian update*

Automated minimization of SURE

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

Generalized FDMC SURE $\lim_{\nu \to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) = R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda})$

Goal : $\underset{\boldsymbol{\Lambda}}{\mathrm{minimize}}\ \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) \equiv \widehat{R}(\boldsymbol{\Lambda})$ for given $\boldsymbol{y}$, $\boldsymbol{\mathcal{S}}$

Quasi-Newton of Broyden-Fletcher-Goldfarb-Shanno *(Nocedal, 2006)*

for $t = 0, 1, \ldots$

$\boldsymbol{d}^{[t]} = -\boldsymbol{H}^{[t]} \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *descent direction*

$\alpha^{[t]} \in \underset{\alpha \in \mathbb{R}}{\mathrm{Argmin}}\ \widehat{R}(\boldsymbol{\Lambda}^{[t]} + \alpha \boldsymbol{d}^{[t]})$ *line search*

$\boldsymbol{\Lambda}^{[t+1]} = \boldsymbol{\Lambda}^{[t]} + \alpha^{[t]} \boldsymbol{d}^{[t]}$

$\boldsymbol{u}^{[t]} = \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t+1]}) - \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *gradient increment*

$\boldsymbol{H}^{[t+1]} = \mathrm{BFGS}(\boldsymbol{H}^{[t]}, \boldsymbol{d}^{[t]}, \boldsymbol{u}^{[t]})$ *inverse Hessian update*

Stein Unbiased GrAdient Risk estimate

Generalized FDMC SURE

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = \left\|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\right\|^2 +$$
$$\frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

Stein Unbiased GrAdient Risk estimate

Generalized FDMC SURE

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = \left\|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\right\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

Generalized Finite Difference Monte Carlo SUGAR

$$\begin{aligned}\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) &= 2\left(\mathbf{A}\boldsymbol{\Phi}\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)^\top\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right) \\ &\quad + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle\end{aligned}$$

Stein Unbiased GrAdient Risk estimate

Generalized FDMC SURE

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = \|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

Generalized Finite Difference Monte Carlo SUGAR

$$\begin{aligned}\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) &= 2\left(\mathbf{A}\boldsymbol{\Phi}\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)^\top\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\\ &\quad + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle\end{aligned}$$

Theorem *(Pascal, 2020)*

Let $(\boldsymbol{y};\boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})$ be an estimator of $\bar{\boldsymbol{x}}$

- uniformly Lipschitz with respect to $\boldsymbol{y}$
- such that $\forall \boldsymbol{\Lambda} \in \mathbb{R}^L$, $\widehat{\boldsymbol{x}}(\mathbf{0}_P;\boldsymbol{\Lambda}) = \mathbf{0}_N$,
- uniformly L-Lipschitz with respect to $\boldsymbol{\Lambda}$, L indep. of $\boldsymbol{y}$. Then

$$\partial_{\boldsymbol{\Lambda}} R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \lim_{\nu\to 0}\mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}}\left[\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}})\right]$$