Processing nonstationary data: representations, theory, algorithms and applications.

Barbara Pascal

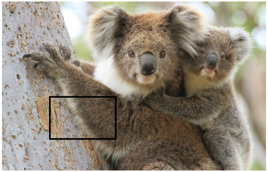
December 10$^{\text{th}}$ 2021

Laboratoire des Sciences du Numérique de Nantes (LS2N)

Team Signal, IMage et Son (SIMS)

Part I: Texture segmentation based on fractal attributes

Textures



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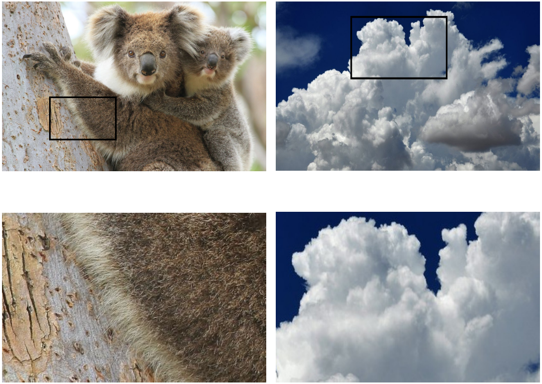

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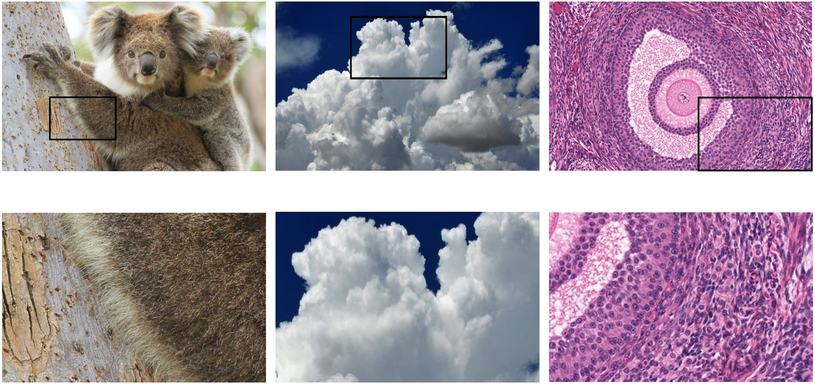

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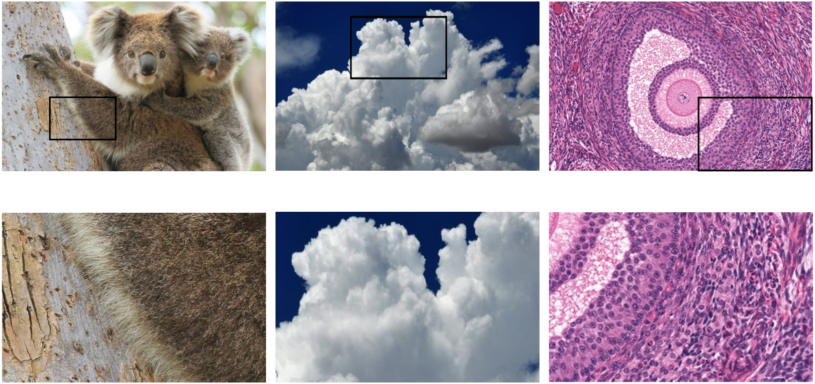
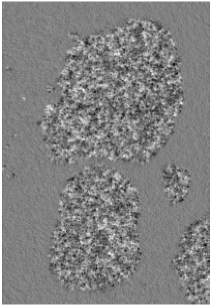


Crucial to describe real-world images

Textured image segmentation



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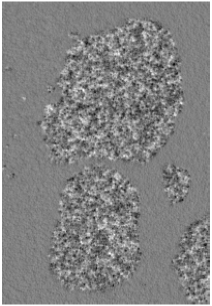

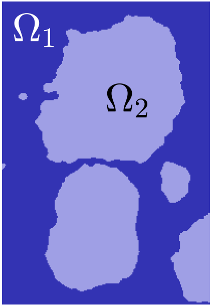

Goal: obtain a partition of the image into K homogeneous textures

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_K$$

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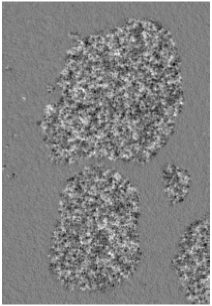

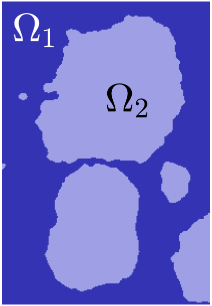

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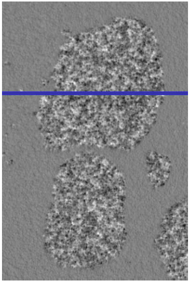
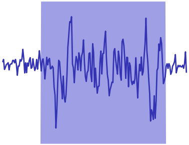
Piecewise monofractal model

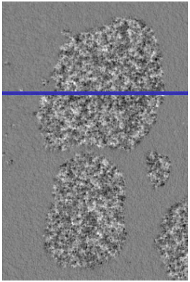


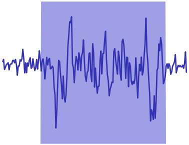


Piecewise monofractal model

Fractals attributes

- variance σ^2 *amplitude of variations*

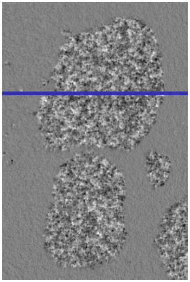


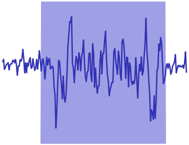


Piecewise monofractal model

Fractals attributes

- variance σ^2 — *amplitude of variations*
- local regularity h — *scale invariance*





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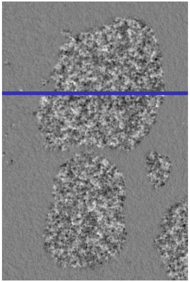

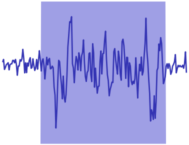

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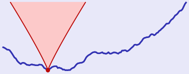
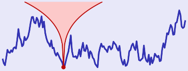


$h(x) \equiv h_1 = 0.9$



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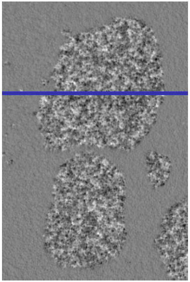

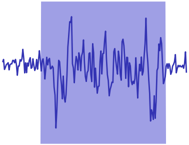

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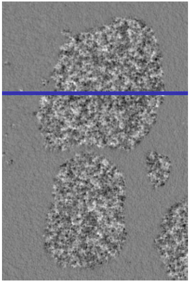
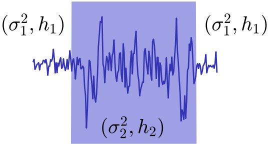
Segmentation

- σ^2 and h piecewise constant





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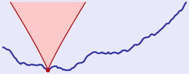

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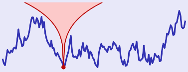

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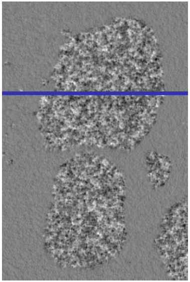
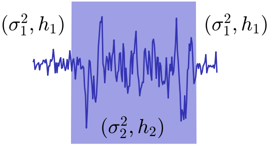


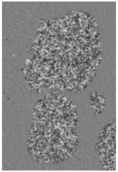


Segmentation

- σ^2 and h piecewise constant
- region Ω_k characterized by (σ_k^2, h_k)

Multiscale analysis

Textured image



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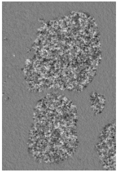

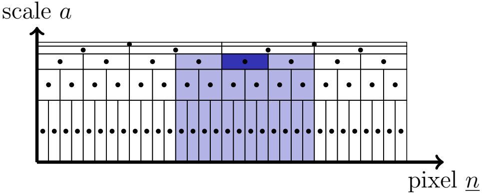

Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$

Multiscale analysis

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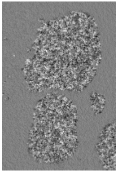

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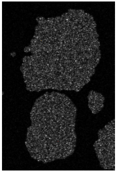
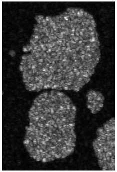
Scale

$a = 2^1$

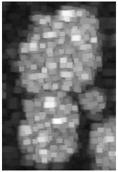


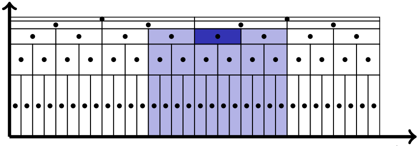
$a = 2^2$



...

$a = 2^5$





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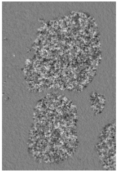

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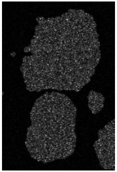

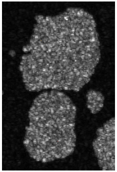

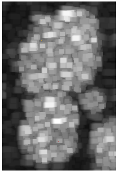

Proposition *(Jaffard, 2004), (Wendt, 2008)*

$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underset{\text{regularity}}{\boldsymbol{h}} + \underset{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}{\boldsymbol{v}}$$

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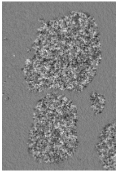

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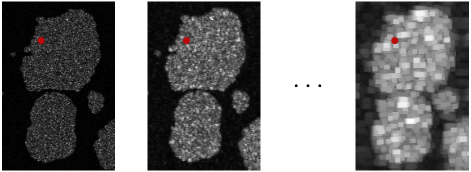

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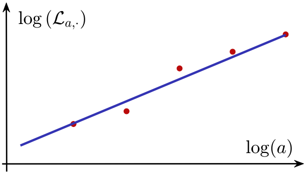

Multiscale analysis

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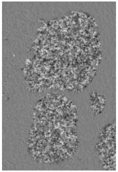

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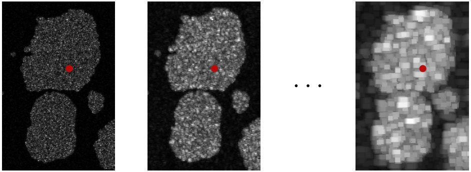

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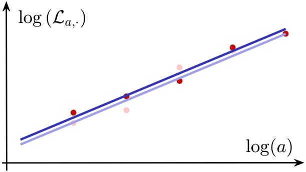

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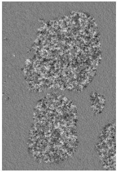

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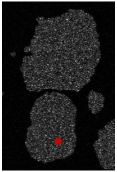

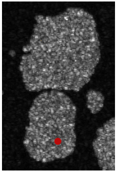

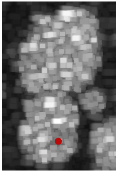

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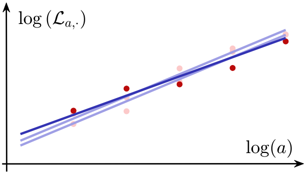

Multiscale analysis

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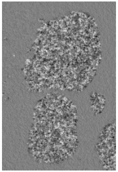

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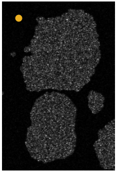

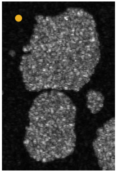

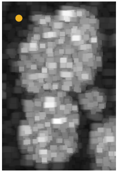

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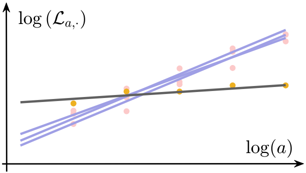

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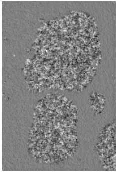

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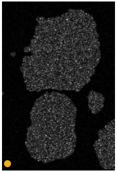

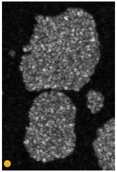

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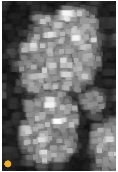

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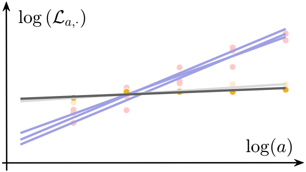

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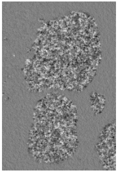

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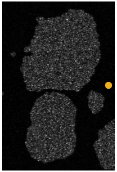
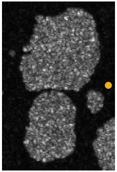
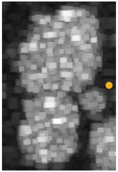
Scale $a = 2^1$ $a = 2^2$ $\cdots$ $a = 2^5$

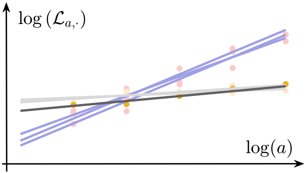






Proposition *(Jaffard, 2004), (Wendt, 2008)*

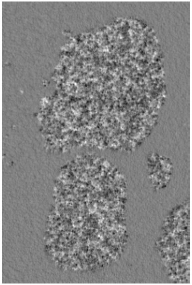
$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underset{\text{regularity}}{\boldsymbol{h}} + \underset{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}{\boldsymbol{v}}$$



Direct punctual estimation

Linear regression $\log(\mathcal{L}_{a,\cdot}) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

Textured image

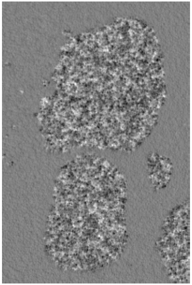


Direct punctual estimation

Linear regression $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\| \log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v} \right\|^2$$

Textured image



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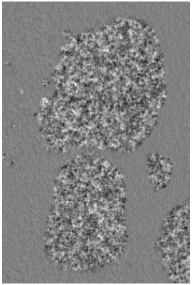
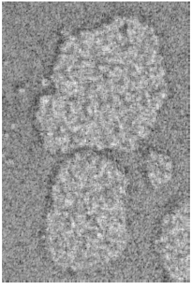
Textured image



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$

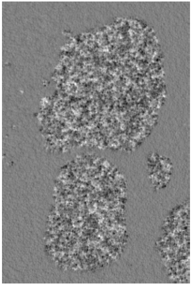


Direct punctual estimation

Linear regression $\underbrace{\mathbb{E}\log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{expected value}} = \log(a)\underbrace{\bar{\boldsymbol{h}}}_{\text{regularity}} + \underbrace{\bar{\boldsymbol{v}}}_{\propto\log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Textured image



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

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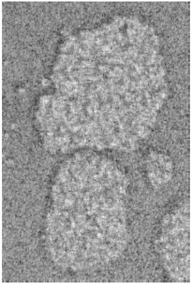


$\longrightarrow$ large estimation variance

A posteriori regularization

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

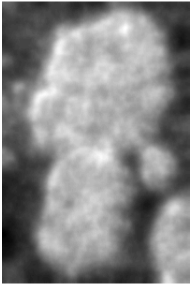
A posteriori regularization

Filter smoothing (linear)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Lissage



A posteriori regularization

Filter smoothing (linear)

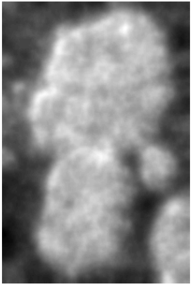
$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

ROF denoising (nonlinear)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \ \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Lissage



ROF

A posteriori regularization

Filter smoothing (linear)

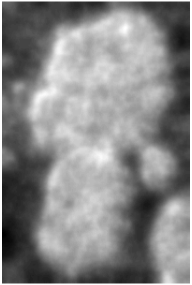
$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

ROF denoising (nonlinear)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \ \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Lissage



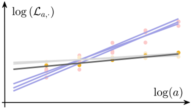
ROF

$\longrightarrow$ cumulative estimation variance and regularization bias

Functionals with either free or co-localized contours

$$\sum_{a} \underbrace{\|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}}$$

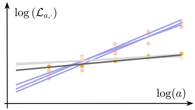
→ fidelity to the log-linear model



Functionals with either free or co-localized contours

$$\sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model



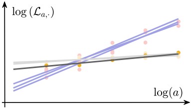
→ favors piecewise constancy

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

→ favors piecewise constancy



Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

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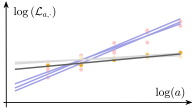


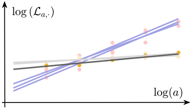
Finite differences $\mathbf{D}_1\boldsymbol{x}$ (horizontal), $\mathbf{D}_2\boldsymbol{x}$ (vertical) in each pixel

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to the log-linear model

$\rightarrow$ favors piecewise constancy



Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

<u>Free:</u> $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

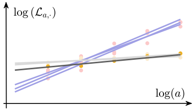
$$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to the log-linear model

$\rightarrow$ favors piecewise constancy



Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

Free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

$$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

Co-localized: $\boldsymbol{h}$, $\boldsymbol{v}$ are **concomitantly** piecewise constant

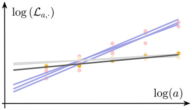
$$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \|[\alpha\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}]\|_{2,1}$$

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

$\rightarrow$ fidelity to the log-linear model

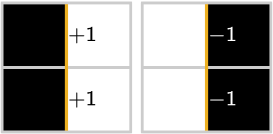
$\rightarrow$ favors piecewise constancy



Disjoint contours

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

Common contours



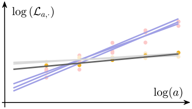
$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

→ favors piecewise constancy

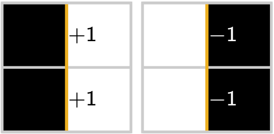


Disjoint contours

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_{\mathsf{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

Common contours



$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_{\mathsf{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

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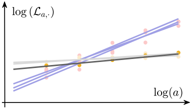


Disjoint contours

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 2 + \sqrt{2} \simeq 3.4$

Common contours

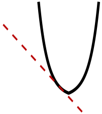
$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 2\sqrt{2} \simeq 2.8$

Functionals minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



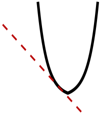
Functionals minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_{a} \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

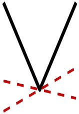
- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$

Functionals minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



nonsmooth



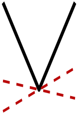
- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$
- implicit subgradient descent: proximal point algorithm

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \boldsymbol{u}^n, \; \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \; \Leftrightarrow \; \boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

Functionals minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$
- implicit subgradient descent: proximal point algorithm

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \boldsymbol{u}^n, \ \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \ \Leftrightarrow \ \boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

- splitting proximal algorithm

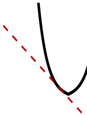
$$\boldsymbol{y}^{n+1} = \mathrm{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

$$\boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}}-\boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau \mathbf{D}^\top \boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

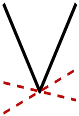
$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

Functionals minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



nonsmooth



- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$
- implicit subgradient descent: proximal point algorithm

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \boldsymbol{u}^n, \ \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \ \Leftrightarrow \ \boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

- splitting proximal algorithm $\qquad \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}) = \underset{\boldsymbol{u}}{\mathrm{argmin}} \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{u}\|^2 + \tau\varphi(\boldsymbol{u})$

$$\boldsymbol{y}^{n+1} = \mathrm{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

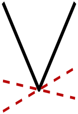
$$\boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \Phi\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau \mathbf{D}^\top \boldsymbol{y}^{n+1}\right), \qquad \Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

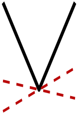
nonsmooth



Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



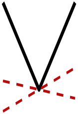
Ex. **Mixed norm:** for $\boldsymbol{z} = [\boldsymbol{z}_1; \dots, ; \boldsymbol{z}_I]$

$$\mathcal{Q}(\boldsymbol{z}) = \|\boldsymbol{z}\|_{2,1} = \sum_{\underline{n}\in\Omega} \sqrt{\sum_{i=1}^{I} z_i^2(\underline{n})} = \sum_{\underline{n}\in\Omega} \|\boldsymbol{z}(\underline{n})\|_2$$

Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



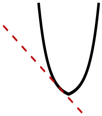
Ex. Mixed norm: for $\boldsymbol{z} = [\boldsymbol{z}_1; \ldots; \boldsymbol{z}_I]$

$$\mathcal{Q}(\boldsymbol{z}) = \|\boldsymbol{z}\|_{2,1} = \sum_{\underline{n}\in\Omega} \sqrt{\sum_{i=1}^{I} z_i^2(\underline{n})} = \sum_{\underline{n}\in\Omega} \|\boldsymbol{z}(\underline{n})\|_2$$

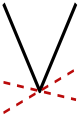
$$\boldsymbol{p} = \mathrm{prox}_{\lambda\|\cdot\|_{2,1}}(\boldsymbol{z}) \quad \Leftrightarrow \quad p_i(\underline{n}) = \max\left(0, 1 - \frac{\lambda}{\|\boldsymbol{z}(\underline{n})\|_2}\right) z_i(\underline{n})$$

Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;+\; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



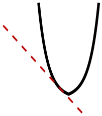
nonsmooth



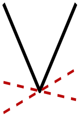
Least-Squares: $\|\log \boldsymbol{\mathcal{L}} - \Phi(\boldsymbol{h}, \boldsymbol{v})\|^2, \quad \Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



nonsmooth



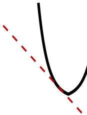
Least-Squares: $\|\log \boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}(\boldsymbol{h}, \boldsymbol{v})\|^2, \quad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Proposition *(Pascal, 2019)*

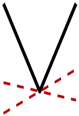
$$(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \operatorname{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|^2}(\boldsymbol{h}, \boldsymbol{v}) \Longleftrightarrow (\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \left(\mathbf{I} + \tau\boldsymbol{\Phi}^\top\boldsymbol{\Phi}\right)^{-1}\left((\boldsymbol{h}, \boldsymbol{v}) + \tau\boldsymbol{\Phi}^\top \log \boldsymbol{\mathcal{L}}\right)$$

Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



nonsmooth



Least-Squares: $\|\log \boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}(\boldsymbol{h}, \boldsymbol{v})\|^2, \quad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Proposition *(Pascal, 2019)*

Let $S_m = \sum_a \log^m(a)$, $\mathcal{D} = (1 + \tau S_2)(1 + \tau S_0) - \tau^2 S_1^2$,
$\boldsymbol{\mathcal{T}} = \sum_a \log \boldsymbol{\mathcal{L}}_a$ and $\boldsymbol{\mathcal{G}} = \sum_a \log(a) \log \boldsymbol{\mathcal{L}}_a$, alors

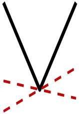
$$(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|^2}(\boldsymbol{h}, \boldsymbol{v}) \iff (\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \left(\mathbf{I} + \tau \boldsymbol{\Phi}^\top \boldsymbol{\Phi}\right)^{-1} \left((\boldsymbol{h}, \boldsymbol{v}) + \tau \boldsymbol{\Phi}^\top \log \boldsymbol{\mathcal{L}}\right)$$

$$\iff \begin{cases} \widetilde{\boldsymbol{h}} = \mathcal{D}^{-1}\left((1 + \tau S_0)(\tau \boldsymbol{\mathcal{G}} + \boldsymbol{h}) - \tau S_1(\tau \boldsymbol{\mathcal{T}} + \boldsymbol{v})\right) \\ \widetilde{\boldsymbol{v}} = \mathcal{D}^{-1}\left((1 + \tau S_2)(\tau \boldsymbol{\mathcal{T}} + \boldsymbol{v}) - \tau S_1(\tau \boldsymbol{\mathcal{G}} + \boldsymbol{h})\right) \end{cases}$$

Accelerated algorithm based on strong-convexity

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



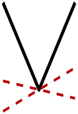
Primal-dual algorithm *(Chambolle, 2011)*

δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$

Accelerated algorithm based on strong-convexity

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



Primal-dual algorithm *(Chambolle, 2011)*

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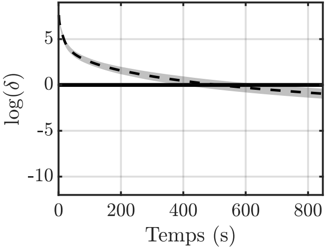
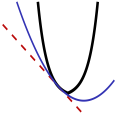


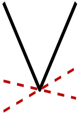
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



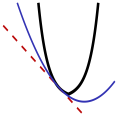
$\boldsymbol{\mu}$-strongly convex

nonsmooth



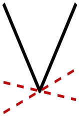
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\text{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\text{Total Variation}}$$



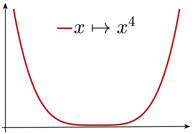
$\boldsymbol{\mu}$-strongly convex

nonsmooth

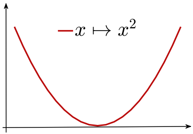


Strong-convexity

- φ $\boldsymbol{\mu}$-strongly convex iff $\varphi - \frac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convex



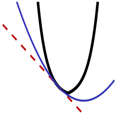
✓ strictly convex
✗ non strongly convex



✓ strictly convex
✓ 1-strongly convex

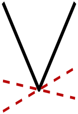
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \; \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



$\boldsymbol{\mu}$-strongly convex

nonsmooth

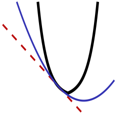


Strong-convexity

- φ $\boldsymbol{\mu}$-strongly convex iff $\varphi - \frac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convex
- φ $\mathcal{C}^2$ with Hessian matrix $\boldsymbol{H}\varphi \succeq 0 \implies \boldsymbol{\mu} = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

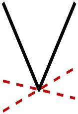
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



μ-strongly convex

nonsmooth



Strong-convexity

- φ μ-strongly convex iff $\varphi - \frac{\mu}{2}\|\cdot\|^2$ convex
- φ $\mathcal{C}^2$ with Hessian matrix $\boldsymbol{H}\varphi \succeq 0 \implies \mu = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

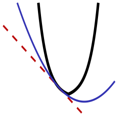
Proposition *(Pascal, 2019)*

$\sum_a \|\log \boldsymbol{\mathcal{L}} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2$ est μ-strongly convex.

$a_{\min} = 2^1$, $a_{\max}$	2^2	2^3	2^4	2^5	2^6
$\mu = \min \mathrm{Sp}\left(2\boldsymbol{\Phi}^\top\boldsymbol{\Phi}\right)$	0.29	**0.72**	1.20	1.69	2.20

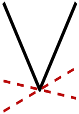
Accelerated algorithm based on strong-convexity

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



μ-strongly convex

nonsmooth



Accelerated Primal-dual algorithm *(Chambolle, 2011)*

for $n = 0, 1, \ldots$ $\qquad \boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})$

$$\boldsymbol{y}^{n+1} = \text{prox}_{\sigma_n(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma_n \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

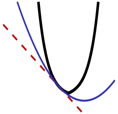
$$\boldsymbol{x}^{n+1} = \text{prox}_{\tau_n\|\boldsymbol{\mathcal{L}}-\Phi\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau_n \mathbf{D}^\top \boldsymbol{y}^{n+1}\right)$$

$$\theta_n = \sqrt{1 + 2\mu\tau_n}, \quad \tau_{n+1} = \tau_n/\theta_n, \quad \sigma_{n+1} = \theta_n\sigma_n$$

$$\bar{\boldsymbol{x}}^{n+1} = \boldsymbol{x}^{n+1} + \theta_n^{-1}\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

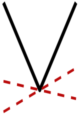
Algorithme accéléré par forte-convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_{a} \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



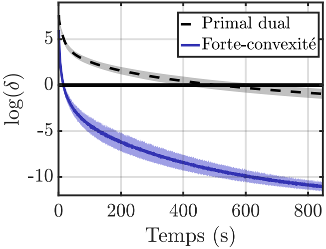
μ-strongly convex

nonsmooth



Accelerated Primal-dual algorithm *(Chambolle, 2011)*

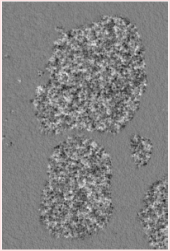
δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$



Segmentation *via* iterated thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_{a} \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

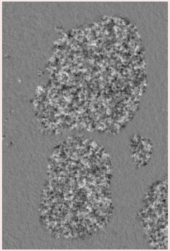
Textured image



Segmentation *via* iterated thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image

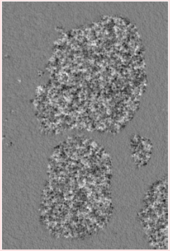


Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Segmentation *via* iterated thresholding

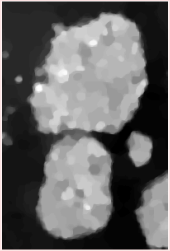
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image



Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

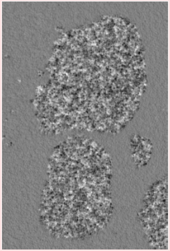
Co-localized contours $\widehat{\boldsymbol{h}}^{\mathrm{C}}$



Segmentation *via* iterated thresholding

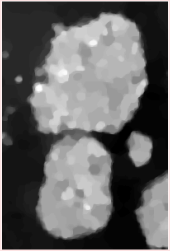
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image



Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

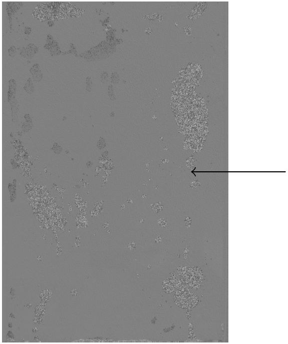


Threshold estimate[†] $T\widehat{\boldsymbol{h}}^{\mathrm{C}}$

† *(Cai, 2013)*

Multiphase flow through porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



Multiphase flow through porous media

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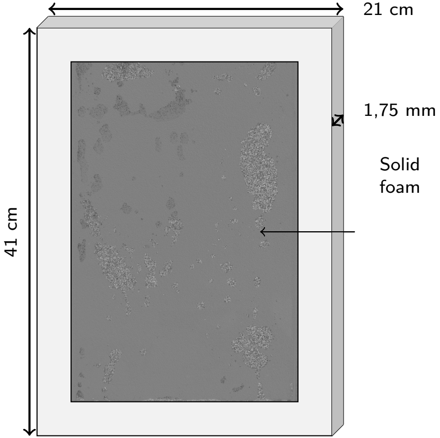

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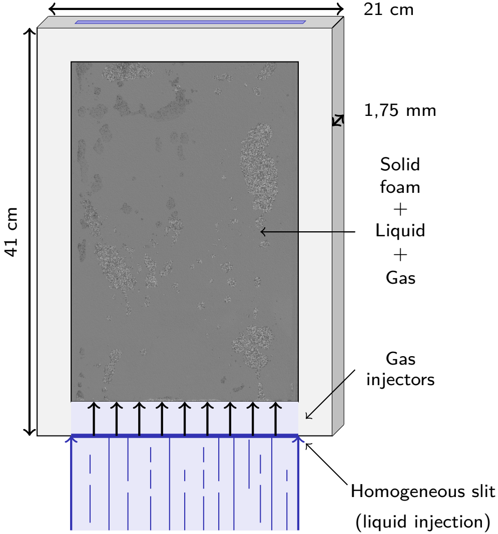
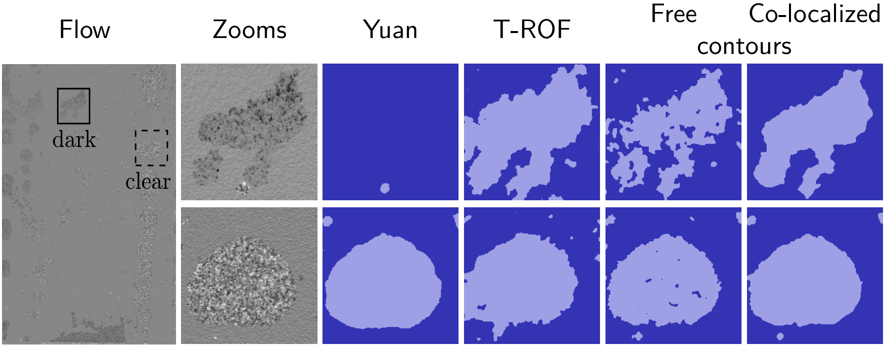
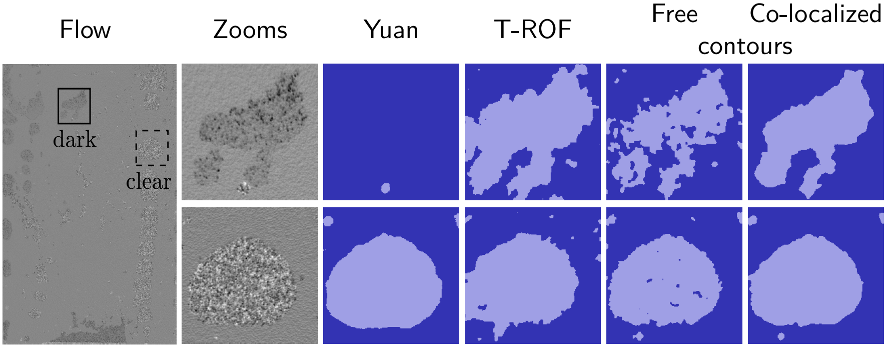


- → 1600 × 1100 pixels
- → video: ∼ 1000 images
- → phase diagram: ∼ 10 flow rates

Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$



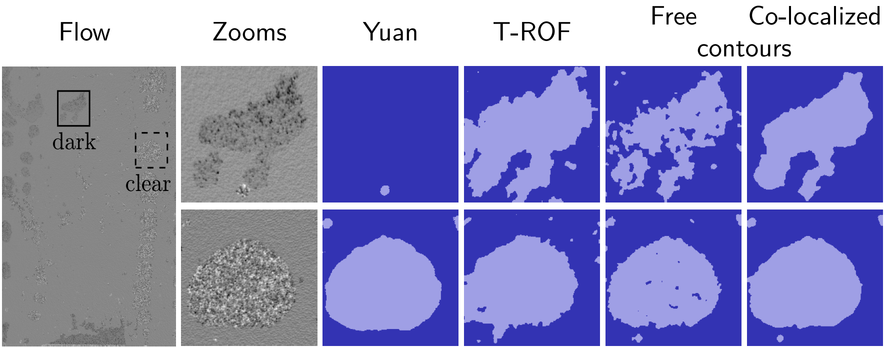
Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$



Liquid: $h_{\mathrm{L}} = 0.4$

Gas: $h_{\mathrm{G}} = 0.9$

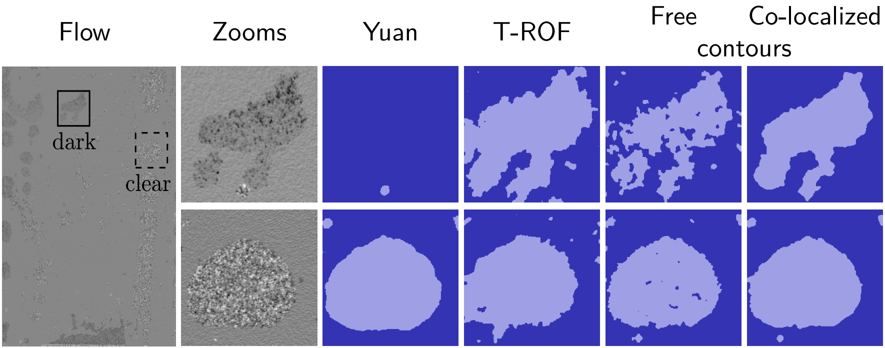
Low activity: $Q_{\mathrm{G}} = 300\text{mL/min}$ - $Q_{\mathrm{L}} = 300\text{mL/min}$



Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$

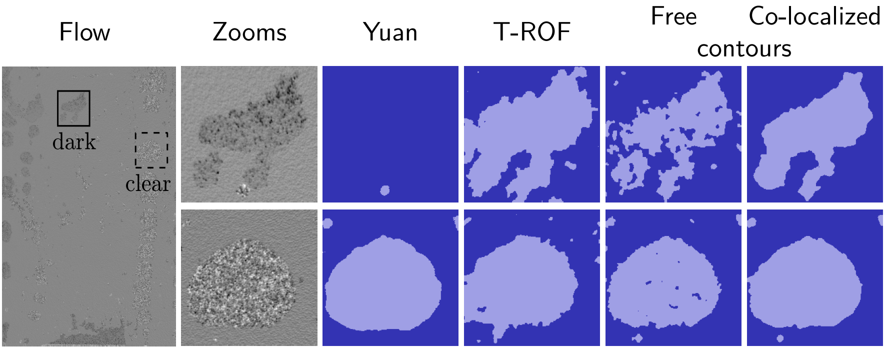
Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$



Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$

Low activity: $Q_{\rm G} = 300$mL/min - $Q_{\rm L} = 300$mL/min

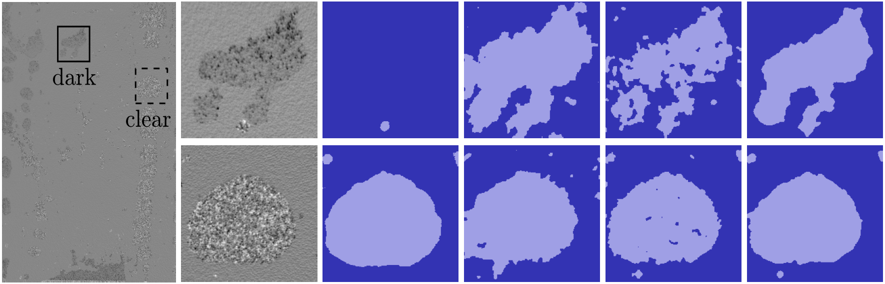


Liquid: $h_{\rm L} = 0.4$ $\sigma^2_{\rm dark} = 10^{-2}$

Gas: $h_{\rm G} = 0.9$ | $\sigma^2_{\rm dark} = 10^{-2}$ (dark bubbles)

Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$

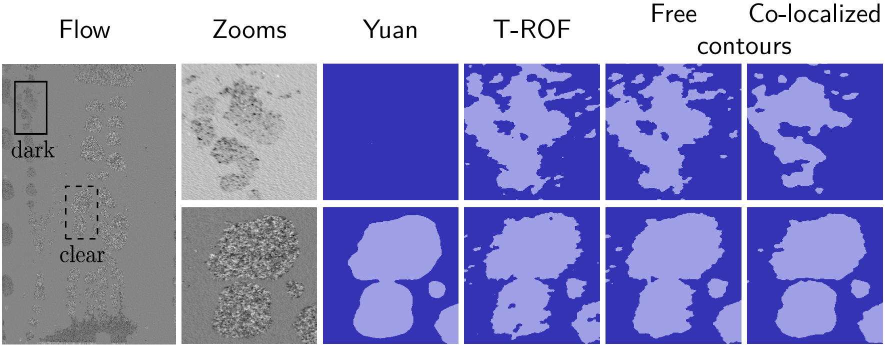
Flow | Zooms | Yuan | T-ROF | Free | Co-localized
contours



Liquid: $h_{\mathrm{L}} = 0.4$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$ (dark bubbles)
$\sigma^2_{\mathrm{clear}} = 10^{-1}$ (clear bubbles)

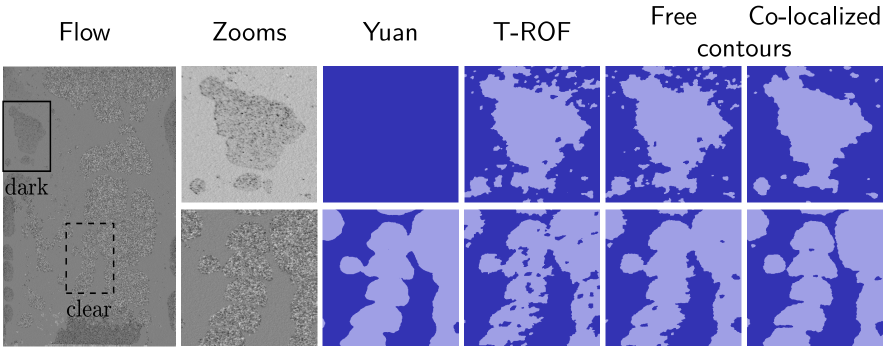
Transition: $Q_{\mathrm{G}} = 400\text{mL}/\text{min}$ - $Q_{\mathrm{L}} = 700\text{mL}/\text{min}$



Liquid: $h_{\mathrm{L}} = 0.4 \qquad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9 \quad \left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

High activity: $Q_{\mathrm{G}} = 1200\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$

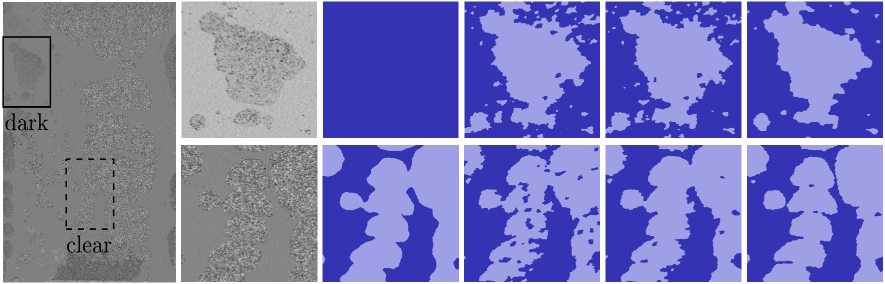


Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

High activity: $Q_{\mathrm{G}} = 1200\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$

Flow	Zooms	Yuan	T-ROF	Free contours	Co-localized contours



Computational load	1s	12s	700s	**2100s**

Liquid: $h_{\mathrm{L}} = 0.4$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\quad \left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\mathcal{L}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

$(\lambda, \alpha) = (0, 0)$

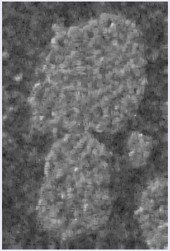
Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

Regularization parameters selection

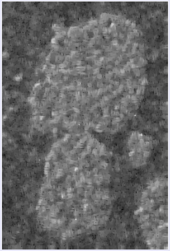
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\mathcal{L}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

$(\lambda, \alpha) = (500, 500)$

too large

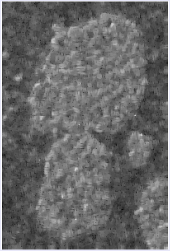
Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ — Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

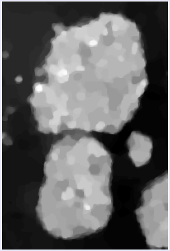
$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

$(\lambda^\dagger, \alpha^\dagger) = (11.5, 0.8)$



optimal

$(\lambda, \alpha) = (500, 500)$

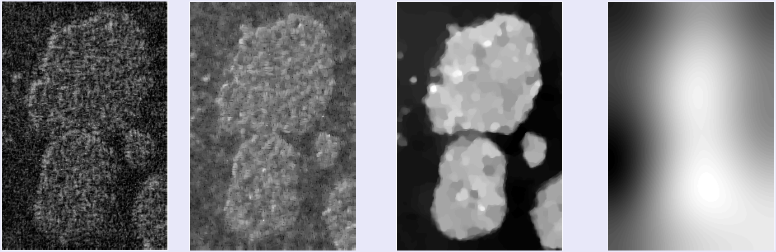
too large

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \| \log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v} \|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$ | $(\lambda, \alpha) = (0.5, 0.5)$ | $(\lambda^{\dagger}, \alpha^{\dagger}) = (11.5, 0.8)$ | $(\lambda, \alpha) = (500, 500)$



too small | optimal | too large

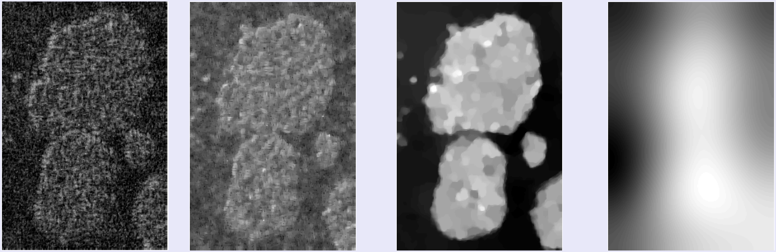
What *optimal* means?

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ — Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$ | $(\lambda, \alpha) = (0.5, 0.5)$ | $(\lambda^\dagger, \alpha^\dagger) = (11.5, 0.8)$ | $(\lambda, \alpha) = (500, 500)$



too small | optimal | too large

What *optimal* means? How to determine $\lambda^\dagger$ and $\alpha^\dagger$?

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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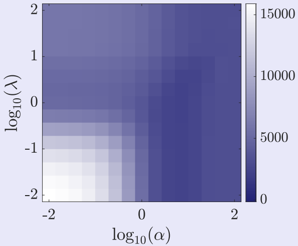
$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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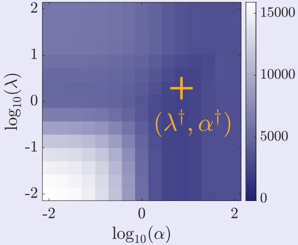
$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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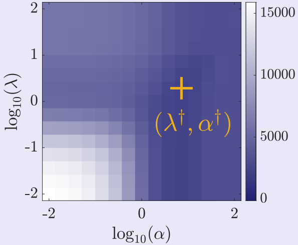
$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

?

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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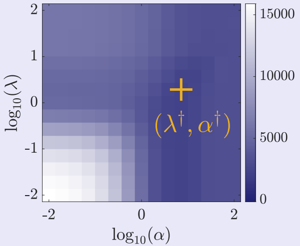
$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

?

Stein Unbiased Risk Estimate
(SURE)

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

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Quadratic error $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \stackrel{?}{=} \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ unknown

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Quadratic error $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \overset{?}{=} \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ unknown

Theorem *(Stein, 1981)*

Let $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\zeta \mapsto \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \zeta; \lambda), \zeta \rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$.

$$\widehat{R}(\boldsymbol{y}; \lambda) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\rho^2 \operatorname{tr}\left(\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)\right) - \rho^2 P$$
$$\Longrightarrow R(\lambda) = \mathbb{E}_\zeta[\widehat{R}(\boldsymbol{y}; \lambda)].$$

Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \Phi \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\Phi : \mathbb{R}^{P \times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \Phi \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\Phi : \mathbb{R}^{P \times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \Phi(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$

$\Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

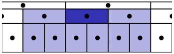
Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \Phi \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\Phi : \mathbb{R}^{P \times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \Phi(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$ $\qquad \zeta \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

$\Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



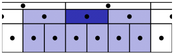
Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \Phi \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\Phi : \mathbb{R}^{P \times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

E.g. the estimators $\widehat{h}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \mathcal{L} = \Phi(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$ $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\Pi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

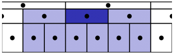
Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \Phi\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\Phi : \mathbb{R}^{P\times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

E.g. the estimators $\widehat{h}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \Phi(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$ $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\Pi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $R_{\Pi}(\Lambda) \triangleq \mathbb{E}_{\zeta}\|\Pi\widehat{\boldsymbol{x}}(\boldsymbol{y}; \Lambda) - \Pi\bar{\boldsymbol{x}}\|^2$

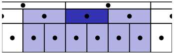
Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$ $\qquad$ $\zeta \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$ $\qquad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Theorem *(Pascal, 2020)*

Let $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\zeta \mapsto \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \zeta; \lambda), \mathbf{A}\zeta \rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$.

$$\widehat{R}(\boldsymbol{\Lambda}) \triangleq \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right) - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

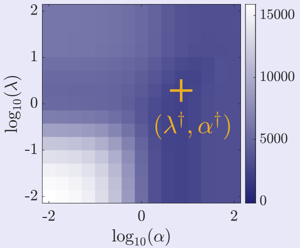
$$\Longrightarrow R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \mathbb{E}_\zeta[\widehat{R}(\boldsymbol{\Lambda})].$$

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

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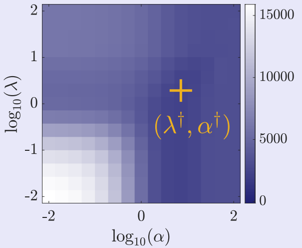

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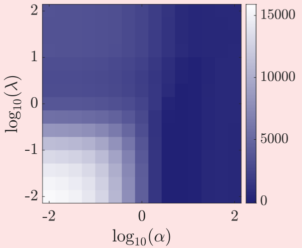

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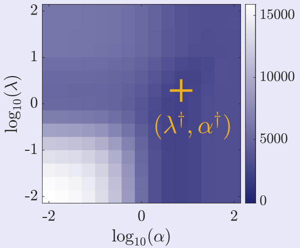

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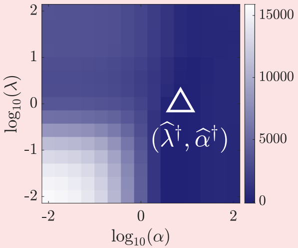

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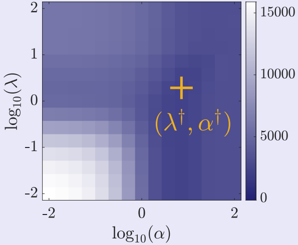

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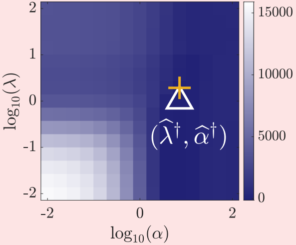
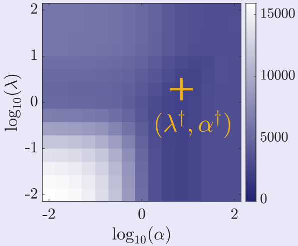


Parameter tuning (Automatic selection)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

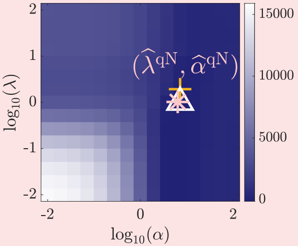
$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

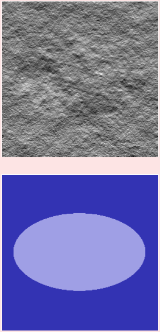
$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$



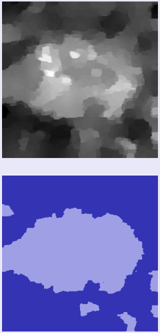
Automated selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}^{\mathsf{F}}, \widehat{\boldsymbol{v}}^{\mathsf{F}}\right)(\boldsymbol{\mathcal{L}};\boldsymbol{\Lambda}) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathsf{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v};\alpha)$$

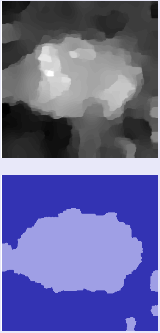
Example



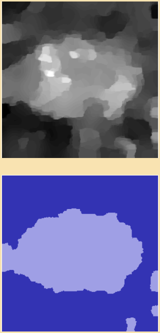
$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}};\lambda^{\dagger},\alpha^{\dagger})$ (grid)



$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}};\widehat{\lambda}^{\dagger},\widehat{\alpha}^{\dagger})$ (grid)



$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}};\widehat{\lambda}^{\mathrm{qN}},\widehat{\alpha}^{\mathrm{qN}})$ (quasi-Newton)



40 calls of the estimator v.s. **225** over a grid

Part I: Fractal texture segmentation

Take home messages

- Fractal texture model based on local *regularity* and *variance*
  - ∗ appropriate for real-world texture characterization
  - ∗ complementary attributes able to finely discriminate

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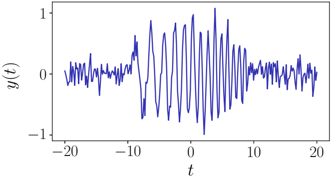
- Fractal texture model based on local *regularity* and *variance*
  - ∗ appropriate for real-world texture characterization
  - ∗ complementary attributes able to finely discriminate
- Simultaneous estimation and regularization
  - ∗ significant decrease of the estimation error
  - ∗ accurate and regular contours thanks to co-localized penalization
- Fast algorithms for automated tuning of hyperparameters
  - ∗ possibility to manage huge amount of data
  - ∗ amenable to process data corrupted by *correlated* Gaussian noise
  - ∗ ensured objectivity and reproducibility

Part II: Point processes in time-frequency analysis

Harmonic analysis of temporal signals

Standard modeling of a "signal": $y : \mathbb{R} \to \mathbb{C}$ function of time t.



- electrical cardiac activity,
- audio recording,
- seismic activity,
- light intensity on a photosensor
- ...

Information of interest:

– time events, e.g., an earthquake and its replica
– frequency content, e.g., monitoring of the heart beating rate

time	**frequency**
ever-changing world	waves, oscillations, rhythms
marker of events and evolutions	intrinsic mechanisms

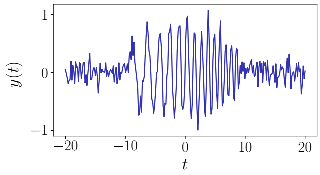
Harmonic analysis of temporal signals

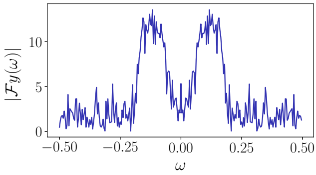
Noisy chirp: transient waveform modulated in amplitude and frequency

$$y(t) = e_\nu(t) \sin\left(2\pi\left(f_1 + (f_2 - f_1)\frac{t+\nu}{2\nu}\right)t\right) + \sigma n(t)$$

Time or frequency

Fourier transform: $\mathcal{F}y(\omega) \triangleq \int_{\mathbb{R}} \overline{y(t)} \exp(-\mathrm{i}\omega t)\,\mathrm{d}t$



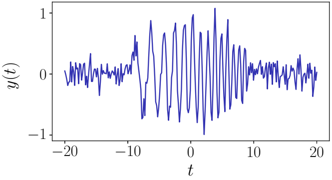


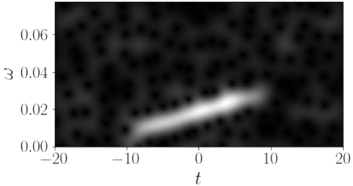
In the Fourier representation, the temporal information is **lost**.

Time-frequency analysis

Time *and* frequency Short-Time Fourier Transform with window h:

$$V_h y(t,\omega) \triangleq \int_{-\infty}^{\infty} \overline{y(u)} h(u-t) \exp(-\mathrm{i}\omega u)\,\mathrm{d}u$$





Energy density interpretation $S_h y(t,\omega) = |V_h y(t,\omega)|^2$ the *spectrogram*

$$\int \int_{-\infty}^{+\infty} S_h y(t,\omega)\,\mathrm{d}t \frac{\mathrm{d}\omega}{2\pi} = \int_{-\infty}^{+\infty} |x(t)|^2\,\mathrm{d}t \quad \text{if} \quad \|h\|_2^2 = 1$$

Signal, i.e., information of interest: regions of maximal energy.

Standard: denoising based on the spectrogram maxima

Inversion formula $$y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u,\omega)} h(t-u) \exp(\mathrm{i}\omega u) \, \mathrm{d}u \frac{\mathrm{d}\omega}{2\pi}$$

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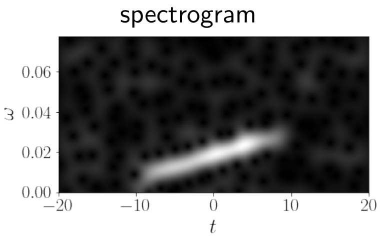

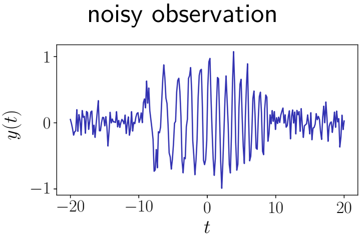

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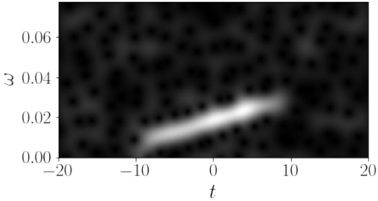
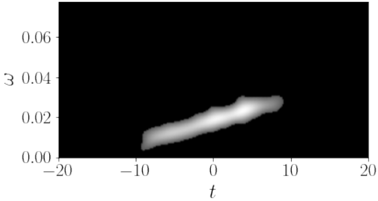
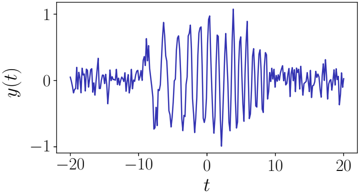
spectrogram



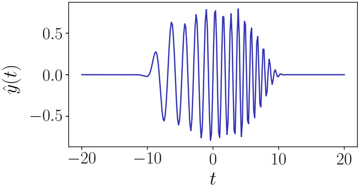
thresholded



noisy observation



denoised signal

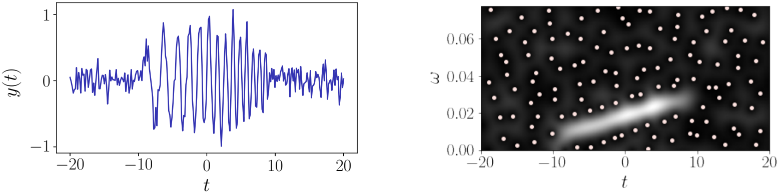


Unorthodox: focus on the spectrogram zeros

Restriction to the *circular Gaussian window*: $g(t) = \pi^{-1/4} e^{-t^2/2}$

Look for the (t_i, ω_i) such that $S_g(t_i, \omega_i) = 0$.

[Flandrin, 2015]

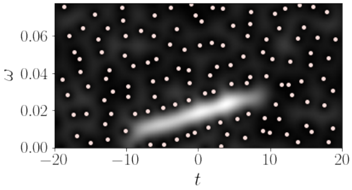


Observations:

- zeros are "repelled" by the signal,
- in the "noise" region, zeros are evenly spread,
- short-range repulsion between zeros.

Unorthodox: theoretical study of the spectrogram zeros

Idea assimilate the time-frequency plane with $\mathbb{C}$ through $z = \omega + \mathrm{i}t$



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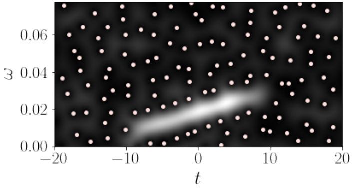


<u>Bargmann factorization</u>

$$V_g y(t,\omega) = \mathrm{e}^{-|z|^2/4}\mathrm{e}^{-\mathrm{i}\omega t/2}\mathcal{B}y(z/\sqrt{2})$$

where the Bargmann transform of the signal y, defined as

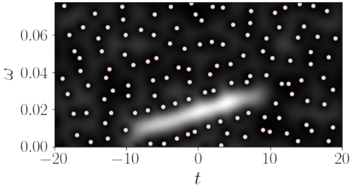
$$\mathcal{B}y(z) \triangleq \pi^{-1/4}\mathrm{e}^{-z^2/2}\int_{\mathbb{R}} \overline{y(u)}\exp\left(\sqrt{2}uz - u^2/2\right)\,\mathrm{d}u,$$

is an **entire** function, almost characterized by its infinitely many zeros:

$$\mathcal{B}y(z) = z^m \mathrm{e}^{C_0+C_1z+C_2z^2}\prod_{n\in\mathbb{N}}\left(1-\frac{z}{z_n}\right)\exp\left(\frac{z}{z_n}+\frac{1}{2}\left(\frac{z}{z_n}\right)^2\right).$$

Unorthodox: theoretical study of the spectrogram zeros

Idea assimilate the time-frequency plane with $\mathbb{C}$ through $z = \omega + \mathrm{i}t$



Bargmann factorization

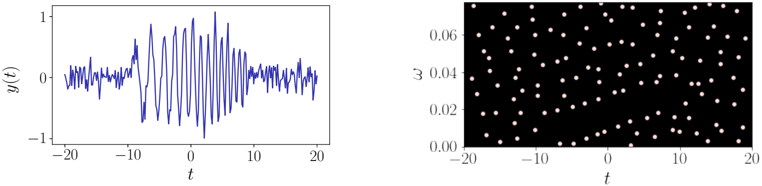
$$V_g y(t, \omega) = \mathrm{e}^{-|z|^2/4} \mathrm{e}^{-\mathrm{i}\omega t/2} \mathcal{B}y(z/\sqrt{2})$$

Theorem The zeros of the Gaussian spectrogram $V_g y(t, \omega)$

- coincide with the zeros of $\mathcal{B}y(\cdot/\sqrt{2})$, which is an **entire** function
- hence are **isolated** and constitute a **random point process**,
- which almost completely **characterizes** the spectrogram.

[Flandrin, 2015]

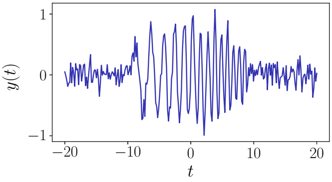
Unorthodox: the point pattern of the spectrogram zeros

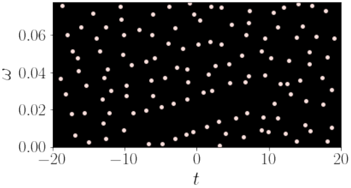


Advantages of working with the zeros

- easy to find compared to *relative* maxima,
- require little memory space for storage,
- use of the tools of **stochastic geometry**.

Unorthodox: the point pattern of the spectrogram zeros





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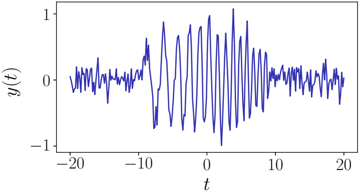
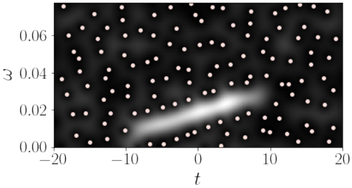
Application: hypothesis testing for signal detection

- H_0 white noisy only, i.e., $y(t) = n(t)$
- H_1 presence of a signal i.e., $y(t) = x(t) + \sigma n(t)$

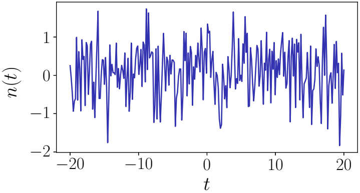
Unorthodox path: signal detection from the zeros

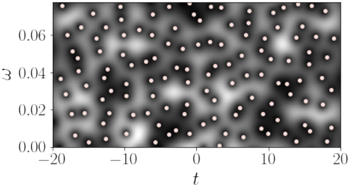
Noisy chirp H_1



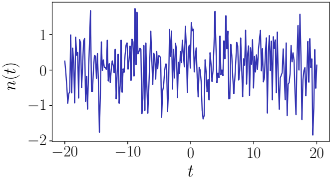


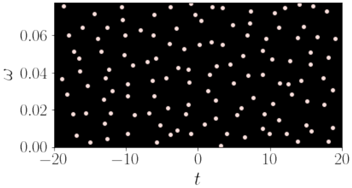
White noise only H_0





Unorthodox path: zeros of the spectrogram of white noise





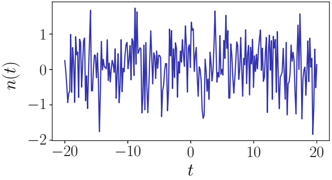
Complex white noise $\quad \xi(t) = \sum_{k=0}^{\infty} \xi_k h_k(t),\ \xi_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$

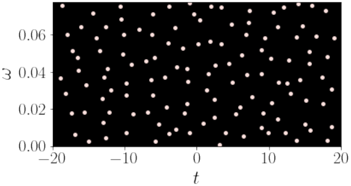
$\{h_k, k = 0, 1, \ldots\}$ the Hermite functions, Hilbertian basis of $L^2(\mathbb{R})$

Theorem $\quad V_g\xi(t,\omega) = \mathrm{e}^{-|z|^2/4}\mathrm{e}^{-\mathrm{i}\omega t/2} \sum_{k=0}^{\infty} \xi_k \frac{1}{\sqrt{k!}} \left(\frac{z}{\sqrt{2}}\right)^k$

[Bardenet & Hardy, 2021]

Unorthodox path: zeros of Gaussian Analytic Functions





$$V_g\xi(t,\omega) = \mathrm{e}^{-|z|^2/4}\mathrm{e}^{-\mathrm{i}\omega t/2}\mathrm{GAF}_{\mathbb{C}}(z/\sqrt{2})$$

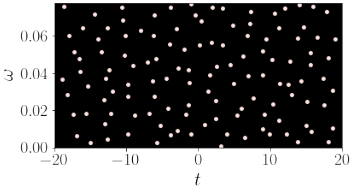
where $\mathrm{GAF}_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}, \quad \xi_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$

Zeros of the *Planar* Gaussian Analytic Function (GAF)

$$\mathcal{Z}(\mathrm{GAF}_{\mathbb{C}}) \overset{\text{(def.)}}{=} \{z_i, \text{ s.t. } \mathrm{GAF}_{\mathbb{C}}(z_i) = 0\}$$

Spatial statistics of the point process $\mathcal{Z}(\mathrm{GAF}_{\mathbb{C}})$ known explicitly.

Unorthodox path: zeros of Gaussian Analytic Functions



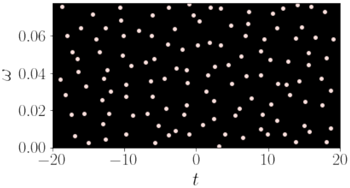
$$V_g\xi(t,\omega) \propto \mathrm{GAF}_{\mathbb{C}}(z/\sqrt{2})$$
$$z = \omega + \mathrm{i}t$$

Properties of the point process $\mathcal{Z}(\mathrm{GAF}_{\mathbb{C}})$:

- invariant under the isometries of $\mathbb{C}$, i.e., **stationary**,
- has a uniform density $\rho^{(1)}(z) = \rho^{(1)} = 1/\pi$,
- explicit pair correlation function $\rho^{(2)}(z, z') = g_0(|z - z'|)$,
- scaling of the *hole probability*: $r^{-4} \log p_r \to -3\mathrm{e}^2/4$, as $r \to \infty$

$$p_r = \mathbb{P}\,(\text{no point in the disk of center 0 and radius } r)$$

Unorthodox path: zeros of Gaussian Analytic Functions



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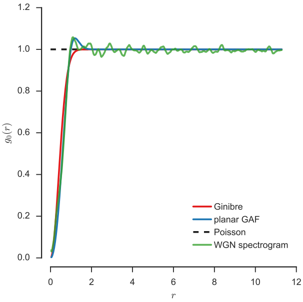

Pair correlation function *informally*

$$\rho^{(2)}(z,z')\mathrm{d}z\mathrm{d}z' =$$

$$\mathbb{P}\,(\text{one point in } B(z,\mathrm{d}z) \text{ and } B(z',\mathrm{d}z'))$$

Unorthodox: other GAF, other transforms

Spherical Gaussian Analytic Function

$$\mathrm{GAF}_{\mathbb{S}}(z) = \sum_{k=0}^{N} \xi_k \sqrt{\binom{N}{k}} z^k, \quad \xi_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

" **Kravchuk transform**" of a **discrete** signal $y = \{y_k,\ k = 0, 1, \ldots, N\}$

$$K_y(\vartheta, \varphi) = \sum_{n=0}^{N} Ty_n \sqrt{\binom{N}{n}} \left(\cos\frac{\vartheta}{2}\right)^n \left(\sin\frac{\vartheta}{2}\right)^{N-n} \mathrm{e}^{\mathrm{i}\varphi n}, \ z = \cot\vartheta/2\mathrm{e}^{\mathrm{i}\varphi}$$

with $Ty_n = \langle y, k_n \rangle$, $\{k_n, n = 0, 1, ..., N\}$ the *Kravchuk functions*.