# Segmentation de textures à partir d'attributs fractals par minimisation de fonctionnelle, avec réglage automatique des hyperparamètres.

**Barbara Pascal**

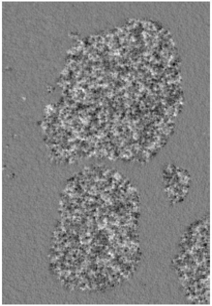
*4 février 2021*

**Séminaire Cristollien d'Analyse Multifractale**

Sous la direction de **Patrice Abry** et **Nelly Pustelnik**

En collaboration avec **Valérie Vidal** et **Samuel Vaiter**

# Segmentation d'image



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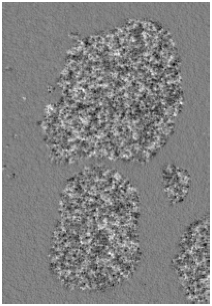

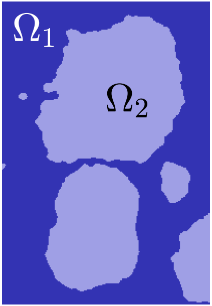


**Objectif** : obtenir une partition de l'image en $K$ régions homogènes

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_K$$

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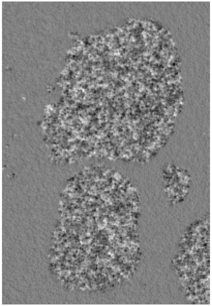

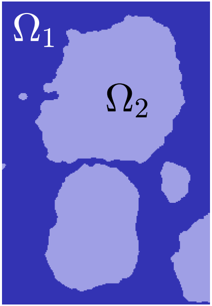


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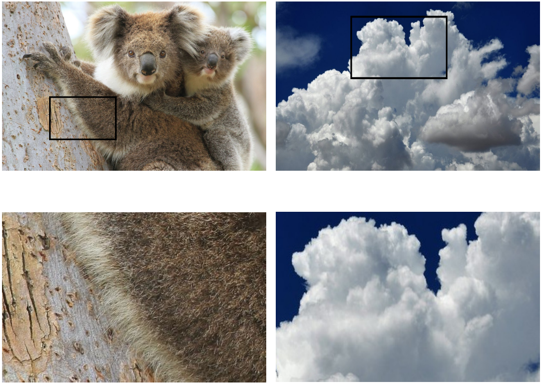

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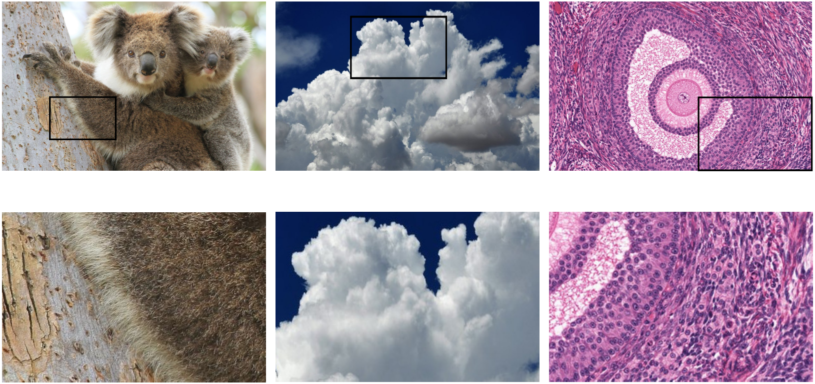

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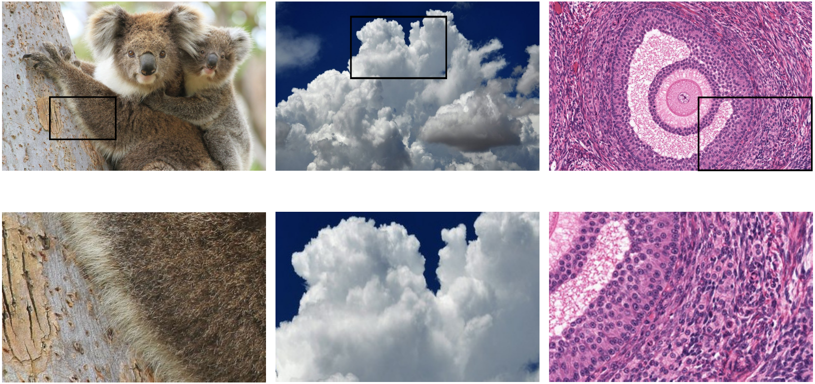
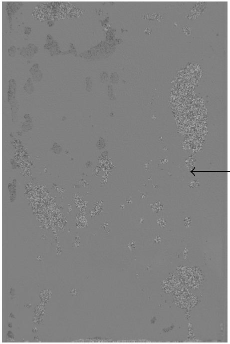


Crucial pour décrire les images réelles

# Écoulement multiphasiques en milieu poreux

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



Mousse solide

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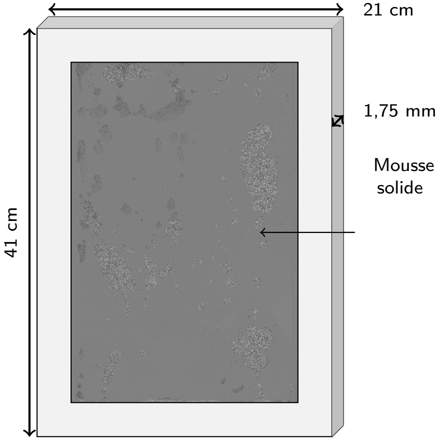


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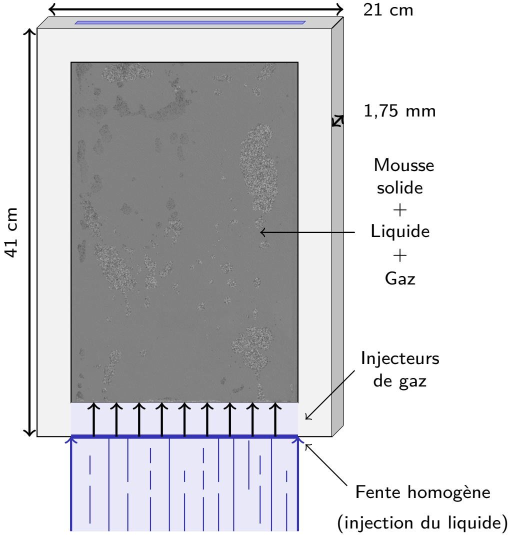


→ 1600 × 1100 pixels
→ vidéo : ∼ 1000 images
→ plan de phases : ∼ 10 débits

# Plan de l'exposé

## 1. Caractérisation de textures

[Filtres de Gabor (Dunn, 1995)]
[Amplitude et fréquence locales (Havlicek, 1996)]
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## 4. Réglage des hyperparamètres

→ SURE avec bruit gaussien corrélé
- erreur d'estimation projetée
- minimisation par quasi-Newton
  ↪ SUGAR généralisé

[SURE (Stein, 1981)]
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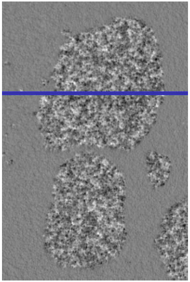
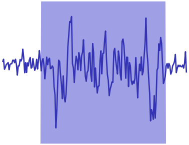
# Modèle monofractal par morceaux

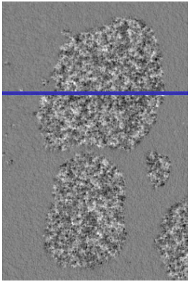


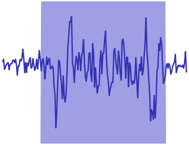


# Modèle monofractal par morceaux

**Attributs fractals**

- variance $\sigma^2$ *amplitude des variations*





# Modèle monofractal par morceaux

**Attributs fractals**

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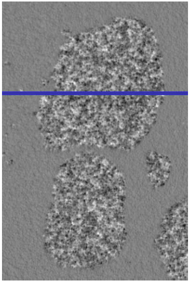
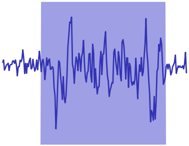
- régularité locale $h$ — *invariance d'échelle*





# Modèle monofractal par morceaux

**Attributs fractals**

- variance $\sigma^2$ — *amplitude des variations*
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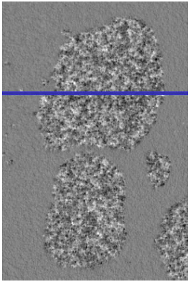
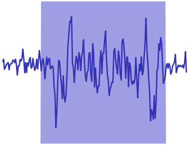
$$|f(x) - f(y)| \leq \sigma(x)|x - y|^{h(x)}$$





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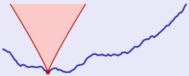
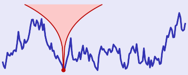
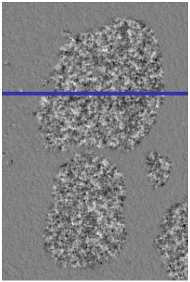
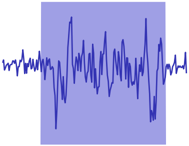


$h(x) \equiv h_1 = 0.9$



$h(x) \equiv h_2 = 0.3$





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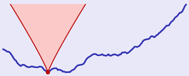

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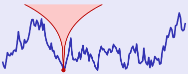

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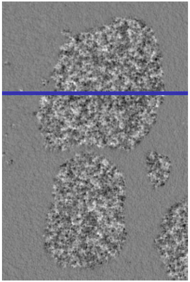
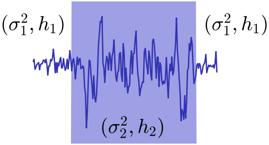
## Segmentation

- $h$ et $\sigma^2$ constants par morceaux





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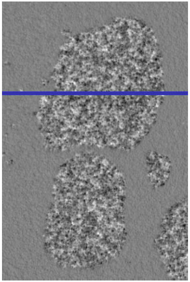
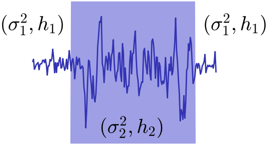
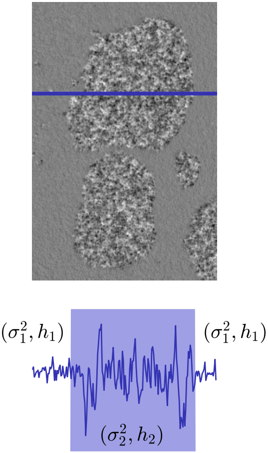





## Segmentation

- $h$ et $\sigma^2$ constants par morceaux
- région $\Omega_k$ caractérisée par $(h_k, \sigma_k^2)$

# Intermède : synthèse de texture fractales *par morceaux*



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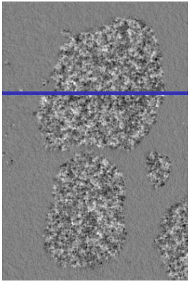

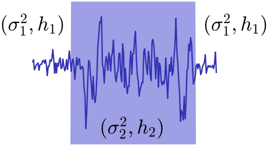


Quel modèle de textures synthétiques

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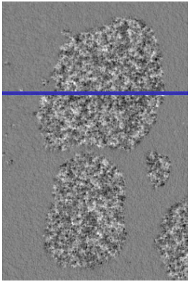

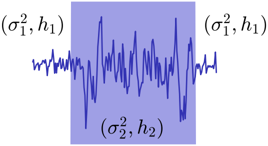


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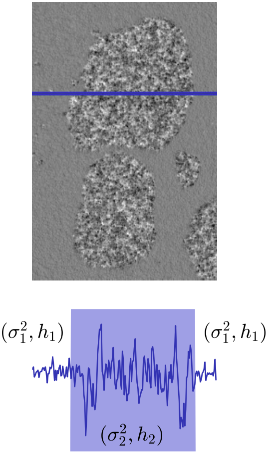


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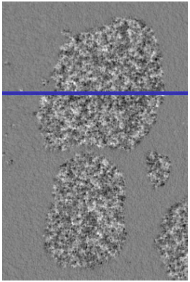

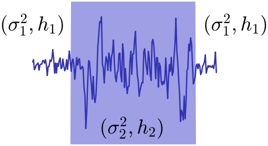


Quel modèle de textures synthétiques

- ressemblant aux textures réelles,
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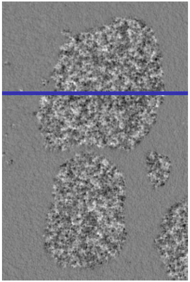

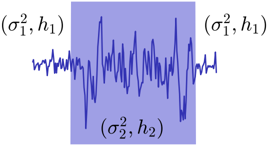


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Proposition d'un champ aléatoire gaussien

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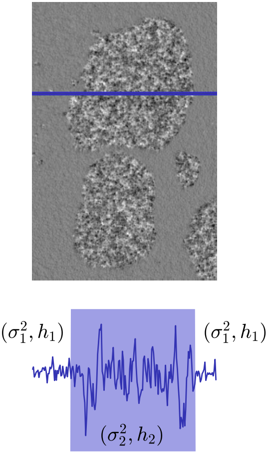


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- isotrope,

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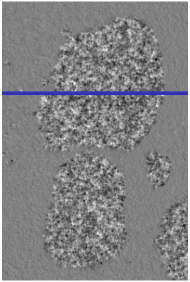

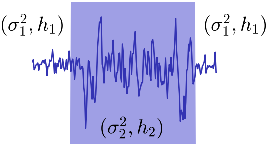


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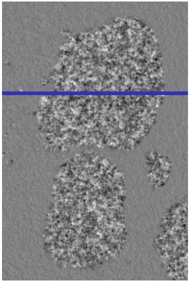

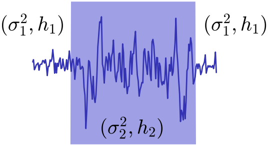


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- stationnaire, de variance $\sigma^2$

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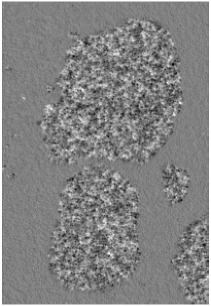
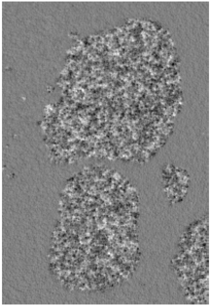
Texture réelle

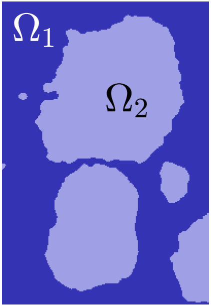


# Intermède : synthèse de texture fractales *par morceaux*

Texture réelle



Masque




$\Omega_1 : (\sigma_1^2, h_1) \quad \Omega_2 : (\sigma_2^2, h_2)$

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Champ brownien fractionnaire

$$b_h(\underline{x}) = \sigma \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-\mathrm{i}\langle \underline{x}, \underline{k} \rangle} - 1}{C_h^{1/2} \|\underline{k}\|^{H+1}} \, \widehat{w}(\mathrm{d}\underline{k})$$

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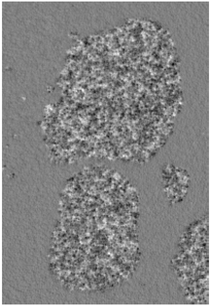

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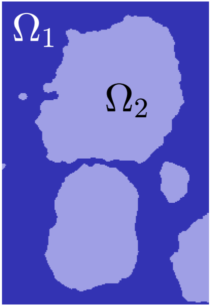


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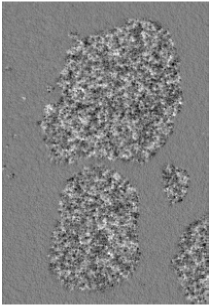

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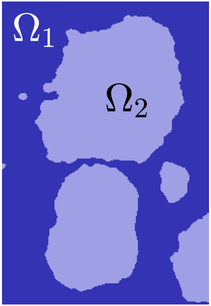
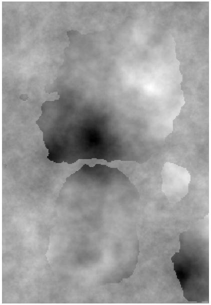



Texture synthétique



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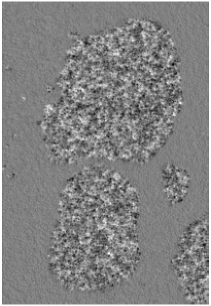
# Intermède : synthèse de texture fractales *par morceaux*

Champ gaussien fractionnaire *stationnaire*

$$g_h(\underline{x}) = \frac{1}{2} \underbrace{\left(b_h(\underline{x} + \underline{e}_1) - b_h(\underline{x})\right)}_{\text{accroissement horizontal}} + \frac{1}{2} \underbrace{\left(b_h(\underline{x} + \underline{e}_2) - b_h(\underline{x})\right)}_{\text{accroissement vertical}}$$

Texture réelle



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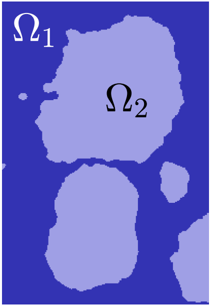
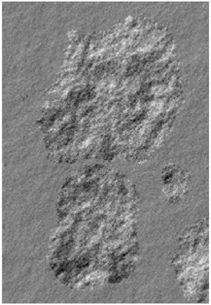



Texture synthétique



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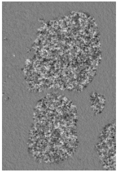
# Analyse multi-échelle

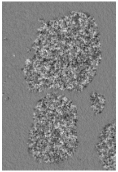
Image texturée

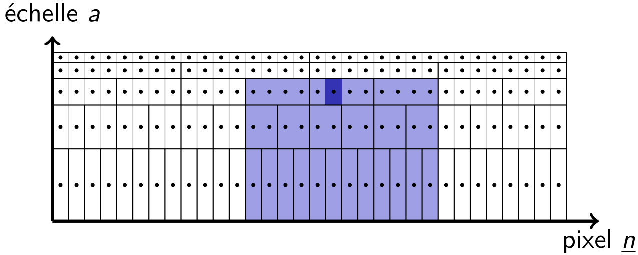


# Analyse multi-échelle

Image texturée

Maximum local des coefficients d'ondelettes : $\mathcal{L}_{a,\cdot}$





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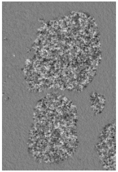
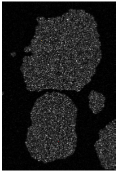


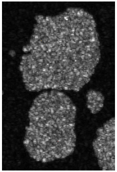
Maximum local des coefficients d'ondelettes : $\mathcal{L}_{a,\cdot}$

**Échelle**

$a = 2^1$

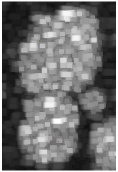


$a = 2^2$

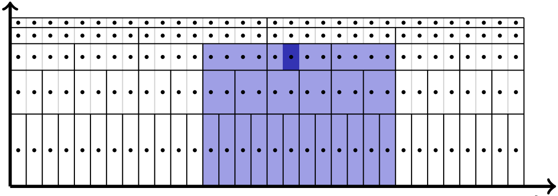


...

$a = 2^5$



échelle $a$



pixel $\underline{n}$

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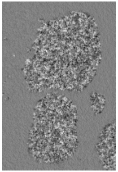

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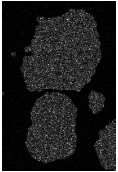

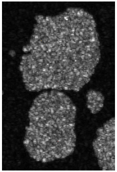

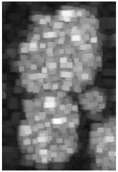

**Proposition** *(Jaffard, 2004), (Wendt, 2008)*

$$\log\left(\boldsymbol{\mathcal{L}}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underset{\text{régularité}}{\boldsymbol{h}} + \underset{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}{\boldsymbol{v}}$$

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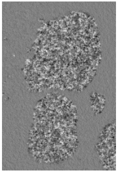

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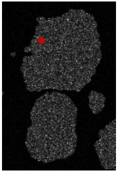

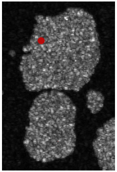

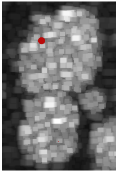

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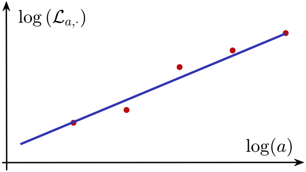

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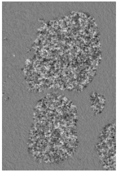

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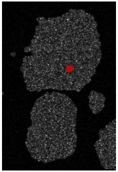

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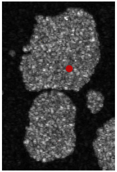

$\dots$

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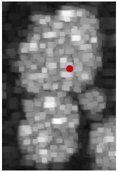

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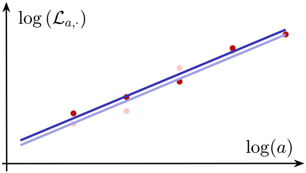

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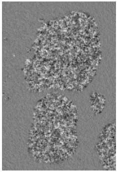

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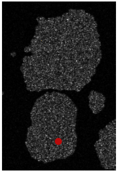

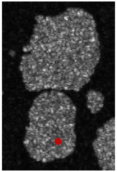

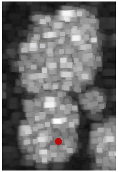

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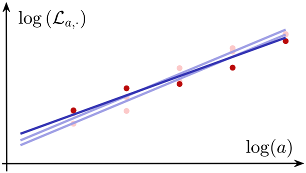

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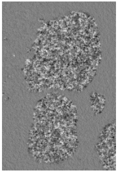

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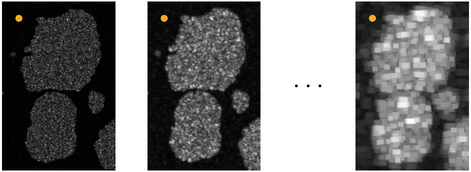

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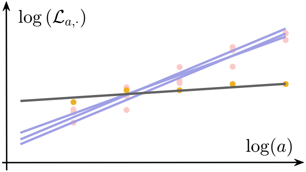

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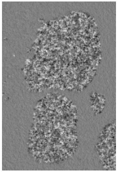

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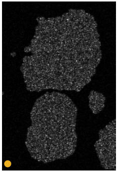
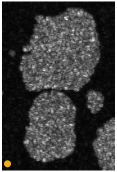
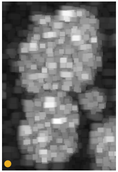

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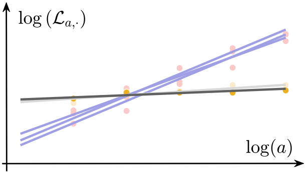

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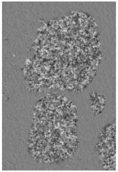

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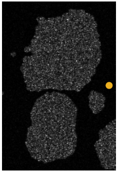

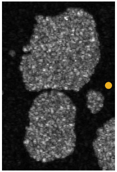

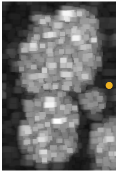

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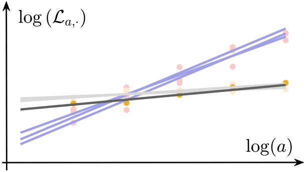
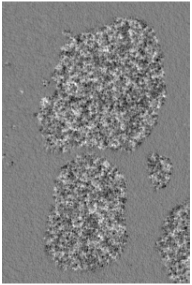


# Estimation directe ponctuelle

**Régression linéaire** $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{régularité}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

Image texturée



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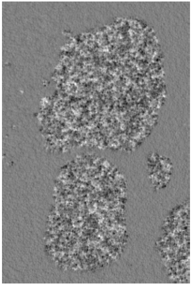
$$\left(\widehat{\boldsymbol{h}}^{\mathrm{RL}}, \widehat{\boldsymbol{v}}^{\mathrm{RL}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\| \log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v} \right\|^2$$

Image texturée



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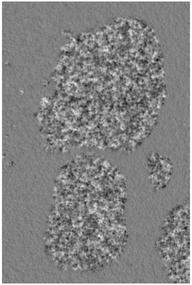
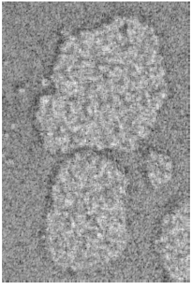
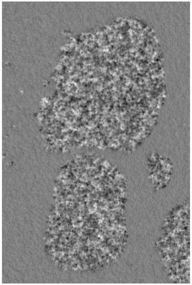
| Image texturée | Régularité locale $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$ | Puissance locale $\widehat{\boldsymbol{v}}^{\mathrm{RL}}$ |
|---|---|---|
|  |  |  |

# Estimation directe ponctuelle

**Régression linéaire** $\underbrace{\mathbb{E}\log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{espérance}} = \log(a)\underbrace{\bar{\boldsymbol{h}}}_{\text{régularité}} + \underbrace{\bar{\boldsymbol{v}}}_{\propto\log(\boldsymbol{\sigma}^2)}$

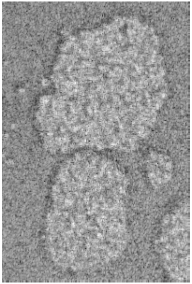
$$\left(\widehat{\boldsymbol{h}}^{\mathrm{RL}}, \widehat{\boldsymbol{v}}^{\mathrm{RL}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Image texturée



Régularité locale $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Puissance locale $\widehat{\boldsymbol{v}}^{\mathrm{RL}}$



$\longrightarrow$ variance d'estimation élevée

# Régularisation *a posteriori*

Régression linéaire $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

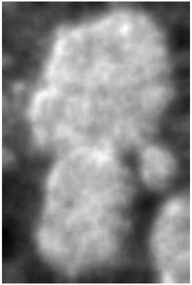
# Régularisation *a posteriori*

**Lissage par filtrage** (linéaire)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{RL}}$$

Régression linéaire $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Lissage



# Régularisation *a posteriori*

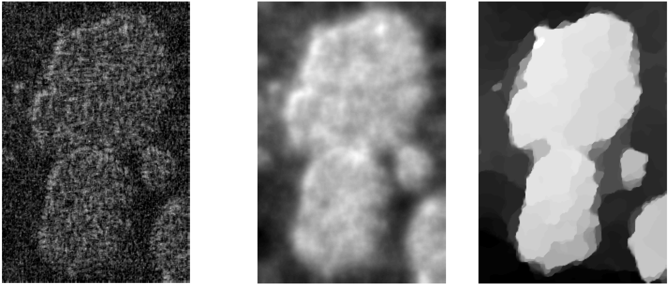
**Lissage par filtrage** (linéaire)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{RL}}$$

**Débruitage ROF** (non linéaire)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \; \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{RL}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Régression linéaire $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$ Lissage ROF



# Régularisation *a posteriori*

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Régression linéaire $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

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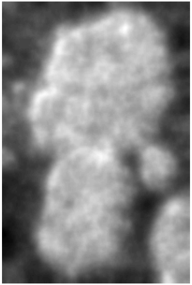
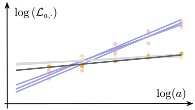


ROF

$\longrightarrow$ cumul de la variance d'estimation et du biais de régularisation

# Fonctionnelles à contours libres et à contours co-localisés

$$\sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\substack{\textbf{Moindres Carrés} \\ \to \text{ fidélité au modèle log-linéaire}}}$$

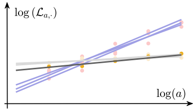


# Fonctionnelles à contours libres et à contours co-localisés

$$\sum_a \underbrace{\|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

→ fidélité au modèle log-linéaire

→ favorise la constance par morceaux




# Fonctionnelles à contours libres et à contours co-localisés

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

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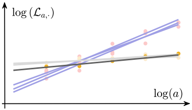


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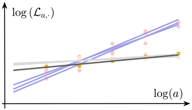


→ favorise la constance par morceaux


**Différences finies** $\mathbf{D}_1\boldsymbol{x}$ (horizontales), $\mathbf{D}_2\boldsymbol{x}$ (verticales) en chaque pixel

# Fonctionnelles à contours libres et à contours co-localisés

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

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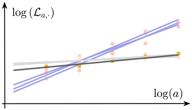


→ favorise la constance par morceaux


**Différences finies** $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

<u>libres :</u> $\boldsymbol{h}$, $\boldsymbol{v}$ sont **indépendamment** constantes par morceaux

$$\mathcal{Q}_{\mathrm{L}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

# Fonctionnelles à contours libres et à contours co-localisés

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \; + \; \underbrace{\lambda \, \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

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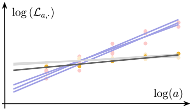



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$$\mathcal{Q}_{\mathrm{L}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

co-localisés : $\boldsymbol{h}$, $\boldsymbol{v}$ sont **concomitamment** constantes par morceaux

$$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \|[\alpha\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}]\|_{2,1}$$

# Fonctionnelles à contours libres et à contours co-localisés

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

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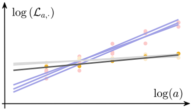



Contours disjoints


$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

Contours communs


$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

# Fonctionnelles à contours libres et à contours co-localisés

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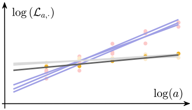


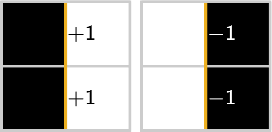


## Contours disjoints


$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

## Contours communs




$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

# Fonctionnelles à contours libres et à contours co-localisés

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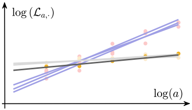




## Contours disjoints


$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad \boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

$\mathcal{Q}_C(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 2 + \sqrt{2} \simeq 3,4$

## Contours communs

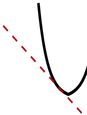

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad \boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

$\mathcal{Q}_C(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 2\sqrt{2} \simeq 2,8$

# Minimisation de fonctionnelle

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



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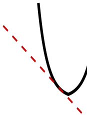

- descente de gradient $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$

# Minimisation de fonctionnelle

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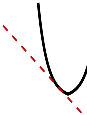
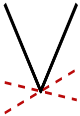


non lisse

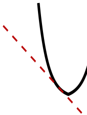


- descente de gradient $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau\nabla\varphi(\boldsymbol{x}^n)$
- descente de sous-gradient implicite : algorithme du point proximal

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \boldsymbol{u}^n,\ \ \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \ \Leftrightarrow\ \boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

# Minimisation de fonctionnelle

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \ \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \ + \ \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



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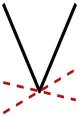

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- algorithme proximal scindé

$$\boldsymbol{y}^{n+1} = \mathrm{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma\mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

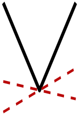
$$\boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}}-\boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau\mathbf{D}^\top\boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h},\boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

# Minimisation de fonctionnelle

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \ \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda\ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

non lisse



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- algorithme proximal scindé $\qquad \operatorname{prox}_{\tau\varphi}(\boldsymbol{x}) = \underset{\boldsymbol{u}}{\operatorname{argmin}} \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{u}\|^2 + \tau\varphi(\boldsymbol{u})$

$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma\mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

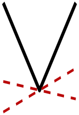
$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau\mathbf{D}^\top\boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

# Calcul des opérateurs proximaux

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \qquad \lambda\ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

non lisse



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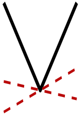
non lisse

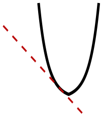


**Ex. Norme mixte :** pour $\boldsymbol{z} = [\boldsymbol{z}_1; \ldots, ; \boldsymbol{z}_I]$

$$\mathcal{Q}(\boldsymbol{z}) = \|\boldsymbol{z}\|_{2,1} = \sum_{\underline{n}\in\Omega} \sqrt{\sum_{i=1}^{I} z_i^2(\underline{n})} = \sum_{\underline{n}\in\Omega} \|\boldsymbol{z}(\underline{n})\|_2$$

# Calcul des opérateurs proximaux

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



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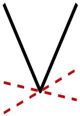

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$$\boldsymbol{p} = \mathrm{prox}_{\lambda\|\cdot\|_{2,1}}(\boldsymbol{z}) \quad \Leftrightarrow \quad p_i(\underline{n}) = \max\left(0, 1 - \frac{\lambda}{\|\boldsymbol{z}(\underline{n})\|_2}\right) z_i(\underline{n})$$

# Calcul des opérateurs proximaux

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

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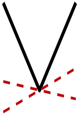
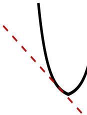


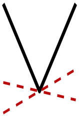
**Moindres carrés** : $\|\log \mathcal{L} - \Phi(\boldsymbol{h}, \boldsymbol{v})\|^2, \quad \Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

# Calcul des opérateurs proximaux

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



non lisse



**Moindres carrés** : $\|\log \boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}(\boldsymbol{h}, \boldsymbol{v})\|^2, \quad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

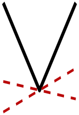
**Proposition** *(Pascal, 2019)*

$$(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|^2}(\boldsymbol{h}, \boldsymbol{v}) \iff (\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \left(\mathbf{I} + \tau \boldsymbol{\Phi}^\top \boldsymbol{\Phi}\right)^{-1} \left((\boldsymbol{h}, \boldsymbol{v}) + \tau \boldsymbol{\Phi}^\top \log \boldsymbol{\mathcal{L}}\right)$$

# Calcul des opérateurs proximaux

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \; \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

non lisse



**Moindres carrés** : $\|\log \boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}(\boldsymbol{h},\boldsymbol{v})\|^2, \quad \boldsymbol{\Phi} : (\boldsymbol{h},\boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

**Proposition** *(Pascal, 2019)*

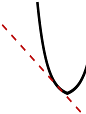
Soit $S_m = \sum_a \log^m(a), \;\; \mathcal{D} = (1+\tau S_2)(1+\tau S_0) - \tau^2 S_1^2,$
$\boldsymbol{\mathcal{T}} = \sum_a \log \boldsymbol{\mathcal{L}}_a$ et $\boldsymbol{\mathcal{G}} = \sum_a \log(a) \log \boldsymbol{\mathcal{L}}_a$, alors

$$(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}}-\boldsymbol{\Phi}\cdot\|^2}(\boldsymbol{h},\boldsymbol{v}) \iff (\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \left(\mathbf{I} + \tau\boldsymbol{\Phi}^\top\boldsymbol{\Phi}\right)^{-1}\left((\boldsymbol{h},\boldsymbol{v}) + \tau\boldsymbol{\Phi}^\top \log\boldsymbol{\mathcal{L}}\right)$$

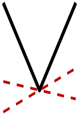
$$\iff \begin{cases} \widetilde{\boldsymbol{h}} = \mathcal{D}^{-1}\left((1+\tau S_0)(\tau\boldsymbol{\mathcal{G}} + \boldsymbol{h}) - \tau S_1(\tau\boldsymbol{\mathcal{T}} + \boldsymbol{v})\right) \\ \widetilde{\boldsymbol{v}} = \mathcal{D}^{-1}\left((1+\tau S_2)(\tau\boldsymbol{\mathcal{T}} + \boldsymbol{v}) - \tau S_1(\tau\boldsymbol{\mathcal{G}} + \boldsymbol{h})\right) \end{cases}$$

# Algorithme accéléré par forte-convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



non lisse

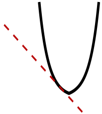


Algorithme primal-dual *(Chambolle, 2011)*

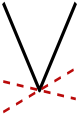
$$\delta \text{ : gap de dualité, } \delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to+\infty}{\longrightarrow} 0$$

# Algorithme accéléré par forte-convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \underbrace{\lambda \; \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



non lisse



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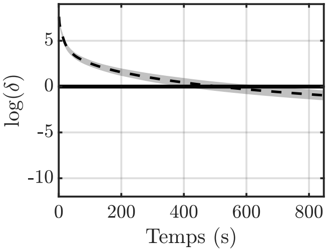
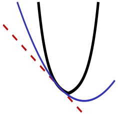


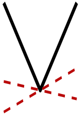
# Propriétés de convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



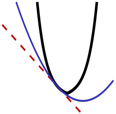
$\mu$-fortement convexe

non lisse



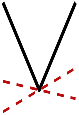
# Propriétés de convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



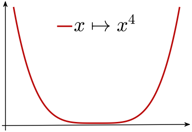
$\mu$-fortement convexe

non lisse

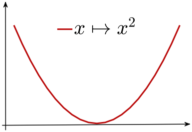


**Forte convexité**

- $\varphi$ $\mu$-fortement convexe ssi $\varphi - \dfrac{\mu}{2}\|\cdot\|^2$ convexe



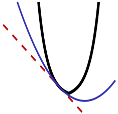

✓ strictement convexe
✗ non fortement convexe




✓ strictement convexe
✓ 1-fortement convexe

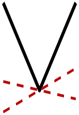
# Propriétés de convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \ + \ \lambda \ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



$\boldsymbol{\mu}$-fortement convexe

non lisse

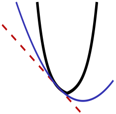


**Forte convexité**

- $\varphi$ $\boldsymbol{\mu}$-fortement convexe ssi $\varphi - \dfrac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convexe
- $\varphi$ $\mathcal{C}^2$ de hessienne $\boldsymbol{H}\varphi \succeq 0 \implies \boldsymbol{\mu} = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

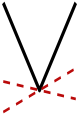
# Propriétés de convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



$\mu$-fortement convexe

non lisse



**Forte convexité**

- $\varphi$ $\mu$-fortement convexe ssi $\varphi - \frac{\mu}{2}\|\cdot\|^2$ convexe
- $\varphi$ $\mathcal{C}^2$ de hessienne $\boldsymbol{H}\varphi \succeq 0 \implies \mu = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

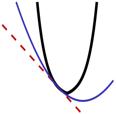
**Proposition** *(Pascal, 2019)*

$\sum_a \|\log \boldsymbol{\mathcal{L}} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2$ est $\mu$-fortement convexe.

| $a_{\min} = 2^1$, $a_{\max}$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ |
|---|---|---|---|---|---|
| $\mu = \min \mathrm{Sp}\left(2\boldsymbol{\Phi}^\top\boldsymbol{\Phi}\right)$ | 0.29 | **0.72** | 1.20 | 1.69 | 2.20 |

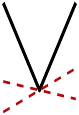
# Algorithme accéléré par forte-convexité

$$\operatorname*{minimiser}_{\boldsymbol{h},\boldsymbol{v}} \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



$\mu$-fortement convexe

non lisse

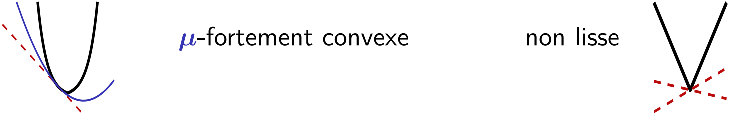


Algorithme primal-dual **accéléré** *(Chambolle, 2011)*

$$\begin{aligned}
&\textbf{for } n = 0, 1, \ldots \qquad\qquad \boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})\\
&\quad \boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma_n(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma_n \mathbf{D}\bar{\boldsymbol{x}}^n\right)\\
&\quad \boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau_n\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau_n \mathbf{D}^\top \boldsymbol{y}^{n+1}\right)\\
&\quad \theta_n = \sqrt{1 + 2\mu\tau_n}, \quad \tau_{n+1} = \tau_n/\theta_n, \quad \sigma_{n+1} = \theta_n \sigma_n\\
&\quad \bar{\boldsymbol{x}}^{n+1} = \boldsymbol{x}^{n+1} + \theta_n^{-1}\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)
\end{aligned}$$

# Algorithme accéléré par forte-convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

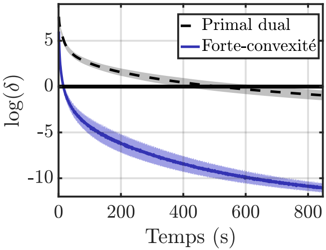


$\boldsymbol{\mu}$-fortement convexe

non lisse

Algorithme primal-dual **accéléré** *(Chambolle, 2011)*

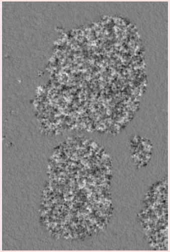
$$\delta : \text{gap de dualité}, \ \delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to+\infty}{\longrightarrow} 0$$



# Segmentation par seuillage itéré

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_{a} \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

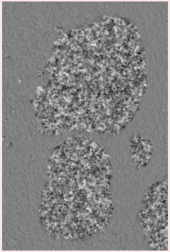
Image texturée



# Segmentation par seuillage itéré

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

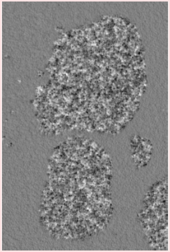
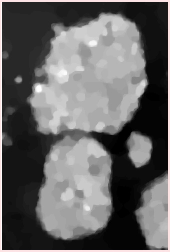
Image texturée



Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

# Segmentation par seuillage itéré

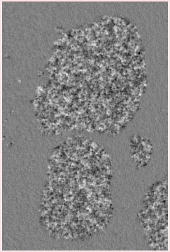
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

| Image texturée | Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$ | Contours co-localisés $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ |
|---|---|---|
|  |  |  |

# Segmentation par seuillage itéré

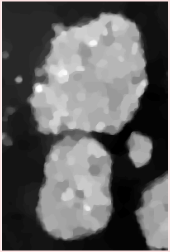
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

Image texturée



Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Contours co-localisés $\widehat{\boldsymbol{h}}^{\mathrm{C}}$



Estimée seuillée† $S\widehat{\boldsymbol{h}}^{\mathrm{C}}$

† *(Cai, 2013)*

# Méthodes de l'état-de-l'art en segmentation de texture

**ROF-Seuillé sur $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$**

*(Nafornita, 2014), (Pustelnik, 2016)*

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \; \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{RL}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Rég. lin.

ROF

Seuillage

S'appuie uniquement sur la régularité $\boldsymbol{h}$.

# Méthodes de l'état-de-l'art en segmentation de texture

## ROF-Seuillé sur $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

*(Nafornita, 2014), (Pustelnik, 2016)*

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{RL}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Rég. lin.

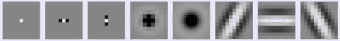
ROF

Seuillage

S'appuie uniquement sur la régularité $\boldsymbol{h}$.

## Segmentation par factorisation matricielle† *(Yuan, 2015)*

(i) histogrammes locaux



(ii) factorisation matricielle

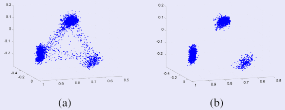


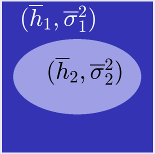

Fig. 2. Scatterplot of features in subspace. (a) Scatterplot of features projected onto the 3-d subspace. (b) Scatterplot after removing features with high edgeness.

† `https://sites.google.com/site/factorizationsegmentation/`

# Performances comparées sur des textures synthétiques

**Synthèse de texture monofractale par morceaux** *(Pascal, 2019)*

- masque : $\Omega = \Omega_1 \sqcup \Omega_2$,
- attributs : $(\bar{h}_k, \bar{\sigma}_k^2)_{k=1,2}$



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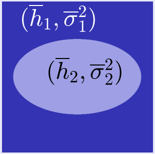
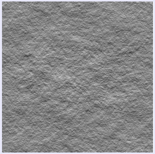
**Ex.** $\bar{h}_1 = 0{,}5$, $\bar{\sigma}_1^2 = 0{,}6$
$\bar{h}_2 = 0{,}6$, $\bar{\sigma}_2^2 = 0{,}7$



# Performances comparées sur des textures synthétiques

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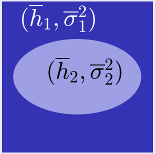
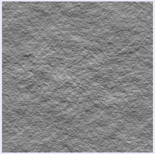
$\overline{h}_2 = 0{,}6,\ \bar{\sigma}_2^2 = 0{,}7$



## Performances de segmentation moyennées sur 5 réalisations

Yuan

$71{,}1 \pm 1{,}3\%$

ROF-S

$78{,}5 \pm 1{,}1\%$

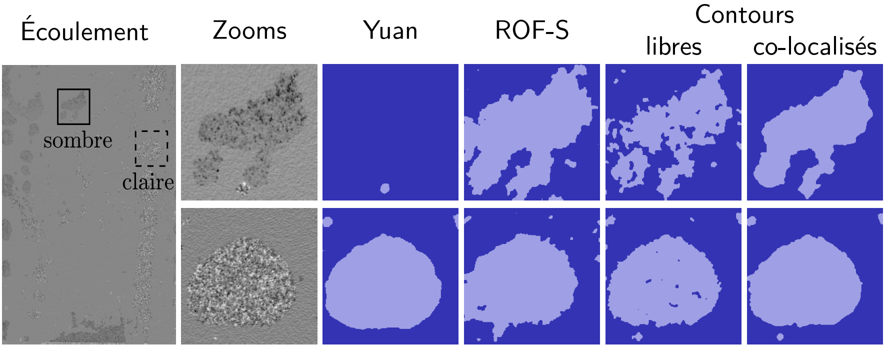
Contours libres

$90{,}2 \pm 1{,}9\%$

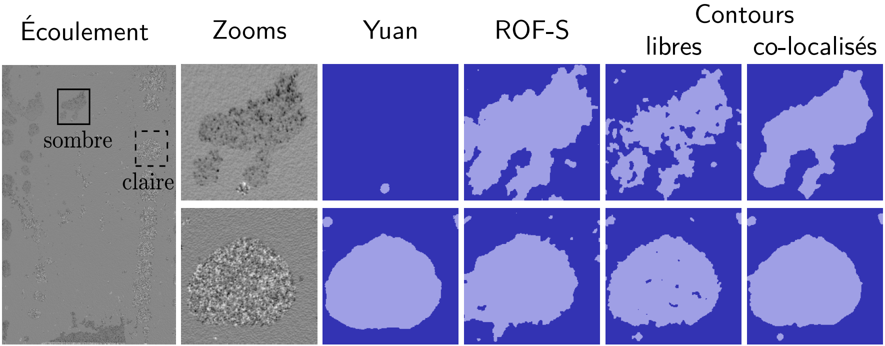
Contours co-localisés

$91{,}1 \pm 1{,}5\%$

# Faible activité : $Q_{\mathrm{G}} = 300\mathrm{mL}/\min$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\min$



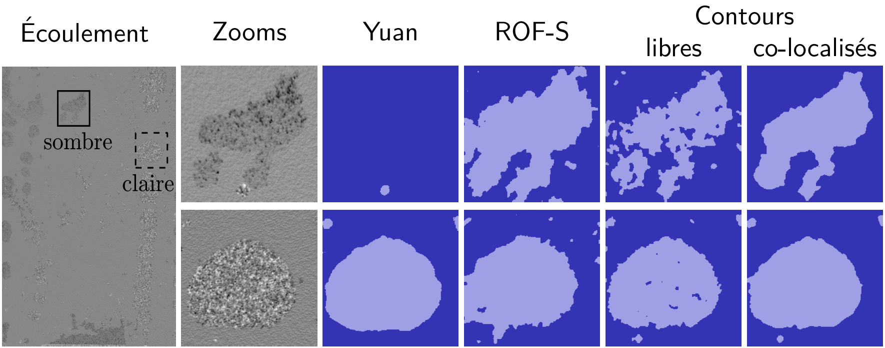
# Faible activité : $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$




Liquide : $h_{\mathrm{L}} = 0{,}4$

Gaz : $h_{\mathrm{G}} = 0{,}9$

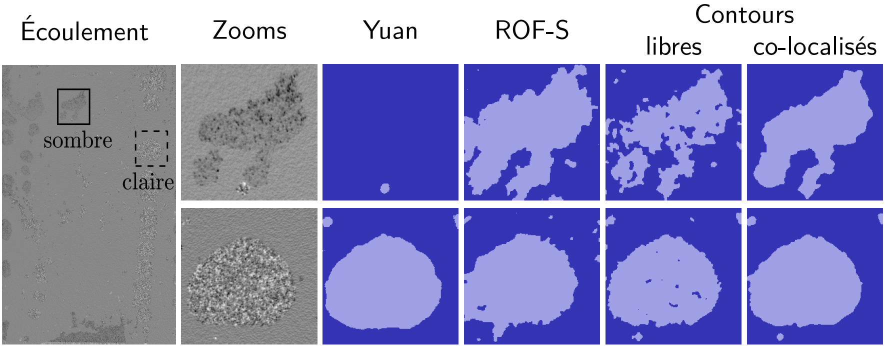
# Faible activité : $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$




Liquide : $h_{\mathrm{L}} = 0{,}4$ $\quad \sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$

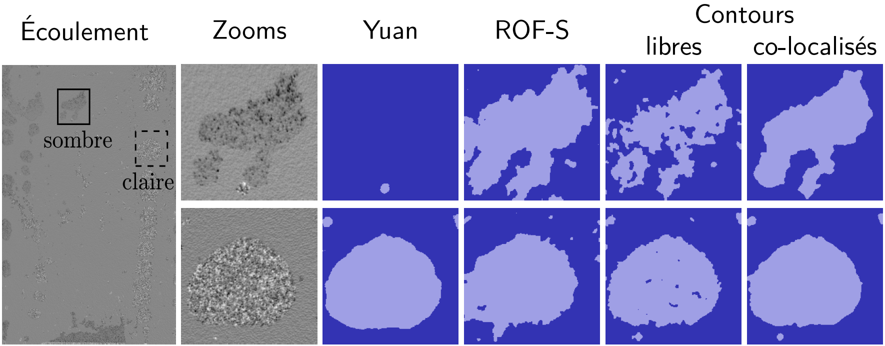
# Faible activité : $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$




Liquide : $h_{\mathrm{L}} = 0{,}4$ $\sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$

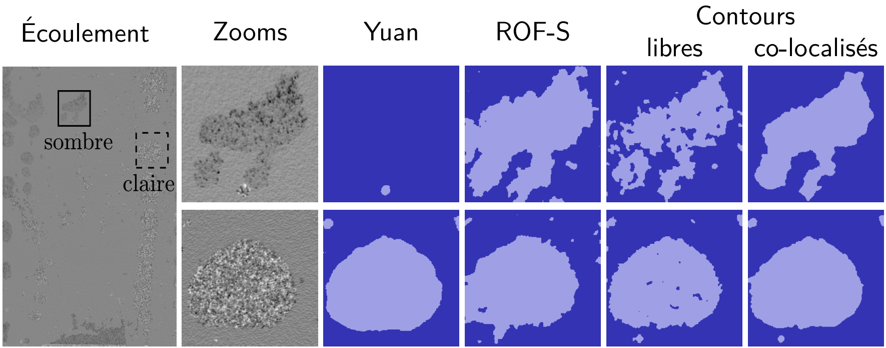
# Faible activité : $Q_{\mathrm{G}} = 300\text{mL/min}$ - $Q_{\mathrm{L}} = 300\text{mL/min}$




Liquide : $h_{\mathrm{L}} = 0{,}4$ $\quad \sigma^2_{\text{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$ $\quad \sigma^2_{\text{sombre}} = 10^{-2}$ (bulles sombres)

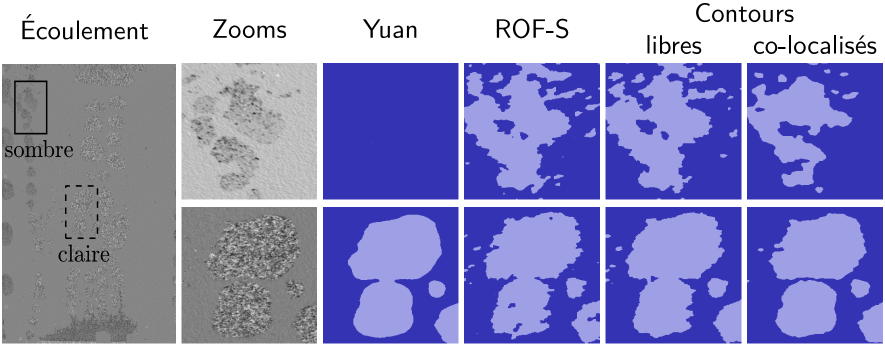
# Faible activité : $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$




Liquide : $h_{\mathrm{L}} = 0{,}4 \qquad \sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$ $\Bigg|$ $\sigma^2_{\mathrm{sombre}} = 10^{-2}$ (bulles sombres)
$\sigma^2_{\mathrm{claire}} = 10^{-1}$ (bulles claires)

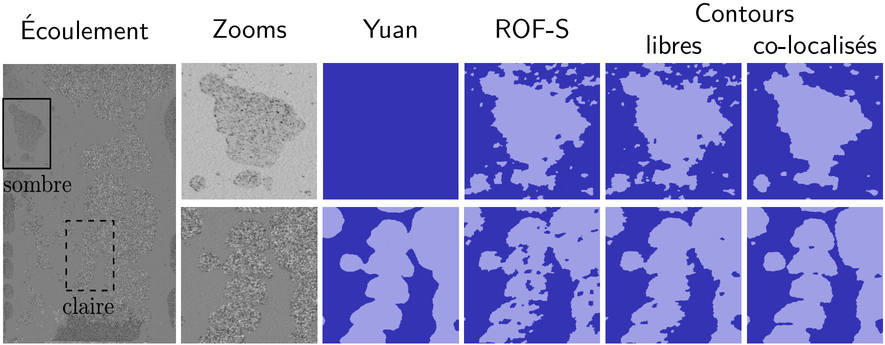
# Transition : $Q_{\mathrm{G}} = 400\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 700\mathrm{mL/min}$




Liquide : $h_{\mathrm{L}} = 0{,}4$ $\quad \sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$ $\quad \left| \begin{array}{lll} \sigma^2_{\mathrm{sombre}} & = 10^{-2} & \text{(bulles sombres)} \\ \sigma^2_{\mathrm{claire}} & = 10^{-1} & \text{(bulles claires).} \end{array} \right.$

# Forte activité : $Q_{\mathrm{G}} = 1200\text{mL/min}$ - $Q_{\mathrm{L}} = 300\text{mL/min}$

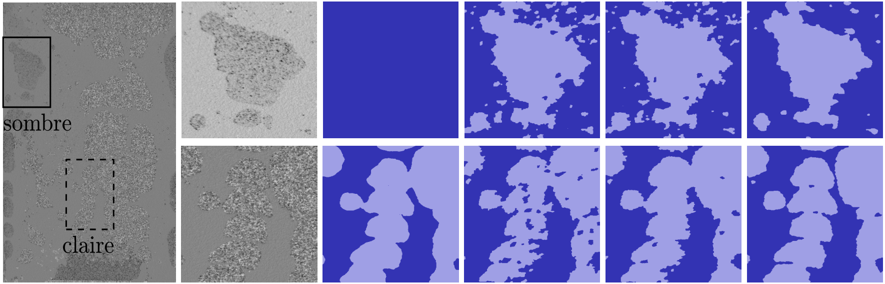



Liquide : $h_{\mathrm{L}} = 0{,}4 \qquad \sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$ $\left| \begin{array}{ll} \sigma^2_{\mathrm{sombre}} = 10^{-2} & \text{(bulles sombres)} \\ \sigma^2_{\mathrm{claire}} = 10^{-1} & \text{(bulles claires).} \end{array} \right.$

# Forte activité : $Q_{\mathrm{G}} = 1200\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$

| Écoulement | Zooms | Yuan | ROF-S | Contours libres | Contours co-localisés |
|---|---|---|---|---|---|




| Temps de calcul | 1s | 12s | 700s | **2100s** |
|---|---|---|---|---|

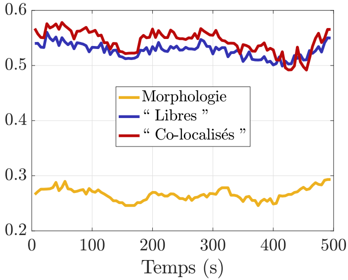
Liquide : $h_{\mathrm{L}} = 0{,}4$ $\quad \sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$ $\left| \begin{array}{ll} \sigma^2_{\mathrm{sombre}} = 10^{-2} & \text{(bulles sombres)} \\ \sigma^2_{\mathrm{claire}} = 10^{-1} & \text{(bulles claires).} \end{array} \right.$

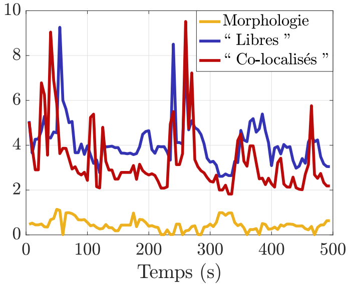
# Écoulement multiphasiques en milieu poreux

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)

Fraction de gaz dans la cellule




Périmètre d'interface



# Réglage des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

# Réglage des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

$(\lambda; \alpha) = (0; 0)$

# Réglage des paramètres de régularisation

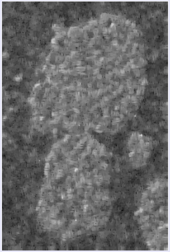
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Estimée $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ à contours co-localisés

$(\lambda; \alpha) = (0; 0)$

$(\lambda, \alpha) = (0{,}5; 0{,}5)$



trop faible

# Réglage des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Estimée $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ à contours co-localisés

$(\lambda; \alpha) = (0; 0)$

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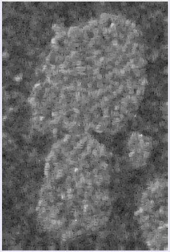


trop faible

$(\lambda; \alpha) = (500; 500)$

trop grand

# Réglage des paramètres de régularisation

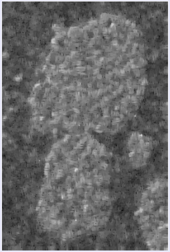
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Estimée $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ à contours co-localisés

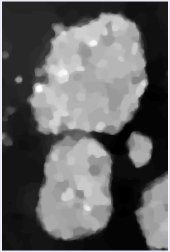
$(\lambda; \alpha) = (0; 0)$

$(\lambda, \alpha) = (0{,}5; 0{,}5)$



trop faible

$(\lambda^\dagger, \alpha^\dagger) = (11{,}5; 0{,}8)$



optimal

$(\lambda; \alpha) = (500; 500)$

trop grand

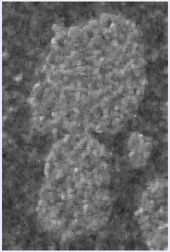
# Réglage des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$ | Estimée $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ à contours co-localisés

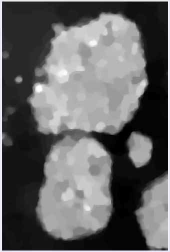
$(\lambda; \alpha) = (0; 0)$

$(\lambda, \alpha) = (0{,}5; 0{,}5)$



trop faible

$(\lambda^\dagger, \alpha^\dagger) = (11{,}5; 0{,}8)$



optimal

$(\lambda; \alpha) = (500; 500)$

trop grand

Que signifie *optimal* ?

# Réglage des paramètres de régularisation

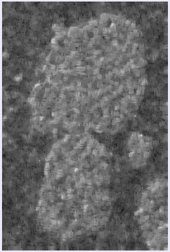
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Estimée $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ à contours co-localisés

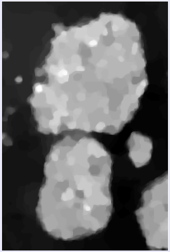
$(\lambda; \alpha) = (0; 0)$

$(\lambda, \alpha) = (0{,}5; 0{,}5)$



trop faible

$(\lambda^\dagger, \alpha^\dagger) = (11{,}5; 0{,}8)$



optimal

$(\lambda; \alpha) = (500; 500)$

trop grand

Que signifie *optimal* ? Comment déterminer $\lambda^\dagger$ et $\alpha^\dagger$ ?

# Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \| \log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v} \|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$ : *discriminant*, $\boldsymbol{v}$ : *auxiliaire*

# Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$ : *discriminant*, $\boldsymbol{v}$ : *auxiliaire*

$\bar{\boldsymbol{h}}$ : *vraie* régularité

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$

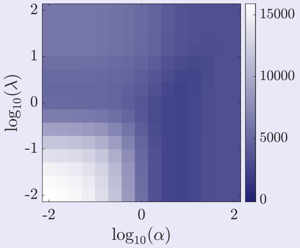
# Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$ : *discriminant*, $\boldsymbol{v}$ : *auxiliaire*

$\bar{\boldsymbol{h}}$ : *vraie* régularité

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



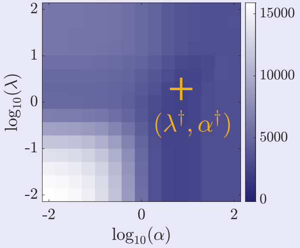
# Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \left\| \log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v} \right\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$ : *discriminant*, $\boldsymbol{v}$ : *auxiliaire*

$\bar{\boldsymbol{h}}$ : *vraie* régularité

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



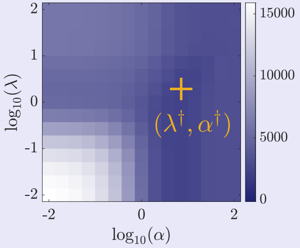
# Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$ : *discriminant*, $\boldsymbol{v}$ : *auxiliaire*

$\bar{\boldsymbol{h}}$ : *vraie* régularité

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$




$\bar{\boldsymbol{h}}$ : inconnue !

?

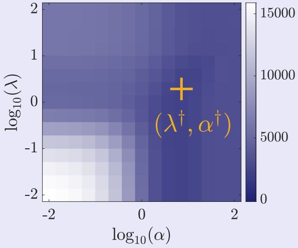
# Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$ : *discriminant*, $\boldsymbol{v}$ : *auxiliaire*

$\bar{\boldsymbol{h}}$ : *vraie* régularité

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$




$\bar{\boldsymbol{h}}$ : inconnue !

?

*Stein Unbiased Risk Estimate*
(SURE)

# *Stein Unbiased Risk Estimate* (Principe)

**Observations** $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$ : vérité et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

# *Stein Unbiased Risk Estimate* (Principe)

**Observations** $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$ : vérité et $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

**Estimateur paramétrique** $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

**Ex.** $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$$

# *Stein Unbiased Risk Estimate* (Principe)

**Observations** $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$ : vérité et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

**Estimateur paramétrique** $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$$

**Erreur quadratique** $R(\lambda) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \stackrel{?}{=} \mathbb{E}_{\boldsymbol{\zeta}} \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ inconnue

# *Stein Unbiased Risk Estimate* (Principe)

**Observations** $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$ : vérité et $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

**Estimateur paramétrique** $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$$

**Erreur quadratique** $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \stackrel{?}{=} \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ inconnue

**Théorème** *(Stein, 1981)*

Soit $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$ un estimateur de $\bar{\boldsymbol{x}}$

- différentiable au sens faible par rapport à $\boldsymbol{y}$,
- tel que $\zeta \mapsto \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \zeta; \lambda), \zeta \rangle$ est intégrable par rapport à $\mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$.

$$\widehat{R}(\boldsymbol{y}; \lambda) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\rho^2 \operatorname{tr}\left(\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)\right) - \rho^2 P$$
$$\Longrightarrow R(\lambda) = \mathbb{E}_\zeta[\widehat{R}(\boldsymbol{y}; \lambda)].$$

# *Stein Unbiased Risk Estimate* généralisé

**Observations** $\boldsymbol{y} = \Phi\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\Phi : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

# *Stein Unbiased Risk Estimate* généralisé

**Observations** $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

**Ex. des estimateurs $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ à contours libres ou co-localisés**

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

# *Stein Unbiased Risk Estimate* généralisé

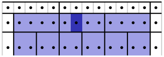
**Observations** $\boldsymbol{y} = \Phi \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N,\ \Phi : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

**Ex. des estimateurs $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ à contours libres ou co-localisés**

$\log \boldsymbol{\mathcal{L}} = \Phi(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$

$\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

$\Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



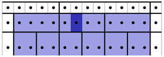
# *Stein Unbiased Risk Estimate* généralisé

**Observations** $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N,\ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

**Ex. des estimateurs $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ à contours libres ou co-localisés**

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\qquad$ $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$ $\qquad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

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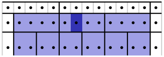

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**Erreur d'estimation projetée** $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

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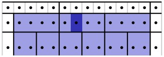

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**Théorème** *(Pascal, 2020)*

Soit $\mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top$ et $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ un estimateur de $\bar{\boldsymbol{x}}$

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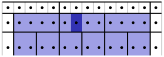
$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$  $\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \boldsymbol{0})$

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# Stein Unbiased Risk Estimate généralisé

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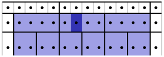

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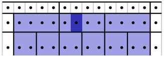

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$$\Longrightarrow R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \mathbb{E}_\zeta[\widehat{R}(\boldsymbol{\Lambda})].$$

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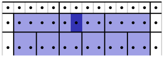

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# Calcul des degrés de liberté

**Degrés de liberté** $\quad \mathrm{dof} \triangleq \mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi} \partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right)$

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- **Stratégie de Monte Carlo** (MC) **$\mathbf{M} \in \mathbb{R}^{P \times P}$ de grande taille**

$$\mathrm{tr}(\mathbf{M}) = \mathbb{E}_{\boldsymbol{\varepsilon}} \langle \mathbf{M}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} \rangle, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_P)$$

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- **Différences Finies** (DF) **Jacobienne inaccessible**

$$\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}\left[\boldsymbol{\varepsilon}\right] \underset{\nu\to 0}{\simeq} \frac{1}{\nu}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)$$

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**Proposition** *(Pascal, 2020)*

Soit $(\boldsymbol{y};\boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})$ un estimateur de $\bar{\boldsymbol{x}}$

- uniformément lipschitzien par rapport à $\boldsymbol{y}$,
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$$\mathbb{E}_{\boldsymbol{\zeta}}\left[\mathrm{dof}\right] = \lim_{\nu \to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}}\left[\frac{1}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y} + \nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle\right]$$

# *Stein Unbiased Risk Estimate* (Calcul)

**Observations** $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N,\ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

**Erreur d'estimation projetée** $R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

**SURE généralisé Différences Finies Monte Carlo**

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) \triangleq \left\|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\right\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

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**Théorème** *(Pascal, 2020)*

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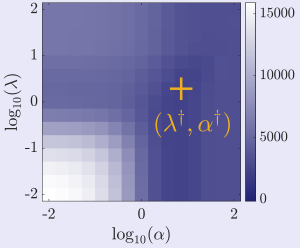
$$R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \lim_{\nu\to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}}\left[\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}})\right]$$

# Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\bar{\boldsymbol{h}}$ : *vraie* régularité

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$




$\bar{\boldsymbol{h}}$ : inconnue !

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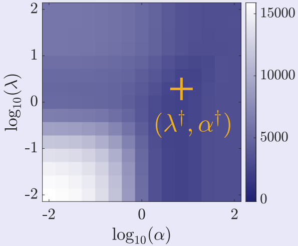


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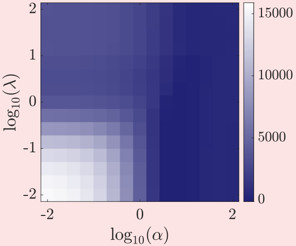

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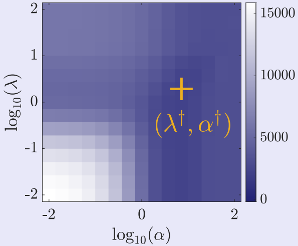


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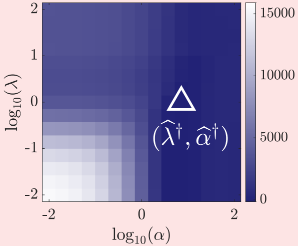

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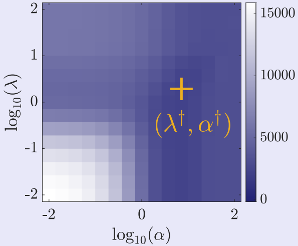
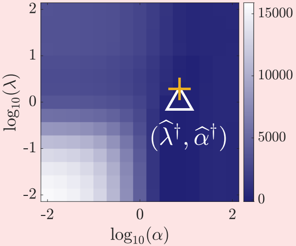
$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$




$\bar{\boldsymbol{h}}$ : inconnue !

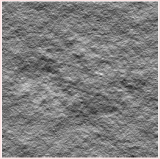
$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

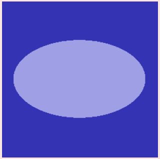


# Recherche systématique des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}}^{\mathsf{L}}, \widehat{\boldsymbol{v}}^{\mathsf{L}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathsf{L}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

**Exemple**





# Recherche systématique des paramètres de régularisation

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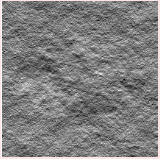
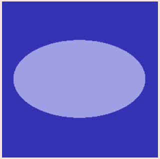
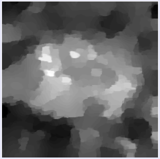
**Exemple**





$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}}; \lambda^\dagger, \alpha^\dagger)$
(grille)



# Recherche systématique des paramètres de régularisation

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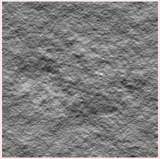
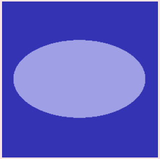
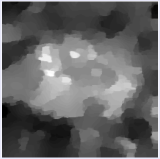
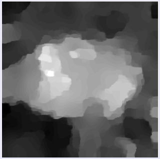




$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}}; \lambda^{\dagger}, \alpha^{\dagger})$ (grille)



$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\dagger}, \widehat{\alpha}^{\dagger})$ (grille)



# Recherche systématique des paramètres de régularisation

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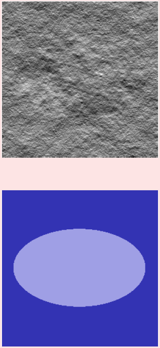
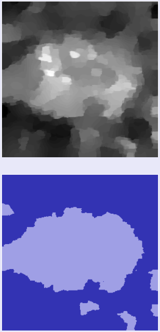
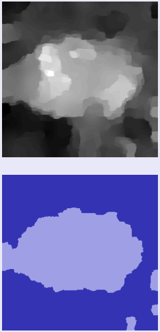
**Exemple**



$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}}; \lambda^\dagger, \alpha^\dagger)$
(grille)



$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^\dagger, \widehat{\alpha}^\dagger)$
(grille)



$15 \times 15 =$ **225** paramètres $\longrightarrow$ recherche sur grille très coûteuse

# Minimisation automatique de l'estimateur SURE

**Observations** $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

**SURE généralisé DFMC** $\lim_{\nu \to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) = R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda})$

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**But** : $\underset{\boldsymbol{\Lambda}}{\text{minimiser}}\ \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}})$ pour $\boldsymbol{y}$, $\boldsymbol{\mathcal{S}}$ donnés

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Quasi-Newton de Broyden-Fletcher-Goldfarb-Shanno *(Nocedal, 2006)*

**for** $t = 0, 1, \ldots$

$\boldsymbol{d}^{[t]} = -\boldsymbol{H}^{[t]}\partial_{\boldsymbol{\Lambda}}\widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *direction de descente*

$\alpha^{[t]} \in \underset{\alpha\in\mathbb{R}}{\text{Argmin}}\ \widehat{R}(\boldsymbol{\Lambda}^{[t]} + \alpha \boldsymbol{d}^{[t]})$ *recherche sur une ligne*

$\boldsymbol{\Lambda}^{[t+1]} = \boldsymbol{\Lambda}^{[t]} + \alpha^{[t]}\boldsymbol{d}^{[t]}$

$\boldsymbol{u}^{[t]} = \partial_{\boldsymbol{\Lambda}}\widehat{R}(\boldsymbol{\Lambda}^{[t+1]}) - \partial_{\boldsymbol{\Lambda}}\widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *variation du gradient*

$\boldsymbol{H}^{[t+1]} = \mathrm{BFGS}(\boldsymbol{H}^{[t]}, \boldsymbol{d}^{[t]}, \boldsymbol{u}^{[t]})$ *mise à jour "hessienne inverse"*

# Minimisation automatique de l'estimateur SURE

**Observations** $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

**SURE généralisé DFMC** $\lim_{\nu\to 0} \mathbb{E}_{\zeta,\boldsymbol{\varepsilon}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) = R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda})$

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# *Stein Unbiased GrAdient Risk estimate*

**SURE généralisé DFMC**

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = \left\|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\right\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

# *Stein Unbiased GrAdient Risk estimate*

**SURE généralisé DFMC**

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = \|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

**SUGAR généralisé Différences Finies Monte Carlo**

$$\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = 2\left(\mathbf{A}\boldsymbol{\Phi}\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)^\top\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right) + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle$$

# *Stein Unbiased GrAdient Risk estimate*

**SURE généralisé DFMC**

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = \|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\|^2 +$$

$$\frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

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$$+\frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle$$

**Théorème** *(Pascal, 2020)*

Soit $(\boldsymbol{y};\boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})$ un estimateur de $\bar{\boldsymbol{x}}$

- uniformément lipschitzien par rapport à $\boldsymbol{y}$
- tel que $\forall\boldsymbol{\Lambda}\in\mathbb{R}^L$, $\widehat{\boldsymbol{x}}(\boldsymbol{0}_P;\boldsymbol{\Lambda}) = \boldsymbol{0}_N$,
- uniformément $L$-lipschitzien par rapport à $\boldsymbol{\Lambda}$, $L$ indép. de $\boldsymbol{y}$. Alors

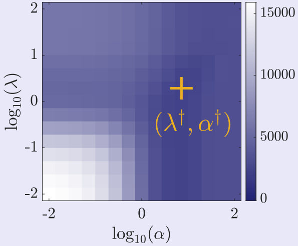
$$\partial_{\boldsymbol{\Lambda}}R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \lim_{\nu\to 0}\mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}}\left[\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}})\right]$$

# Réglage des paramètres (Recherche automatique)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}\left(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha\right)$$

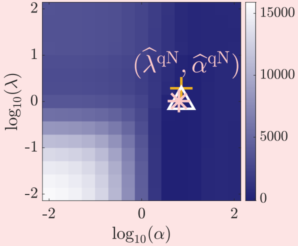
$\bar{\boldsymbol{h}}$ : *vraie* régularité

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$




$\bar{\boldsymbol{h}}$ : inconnue !

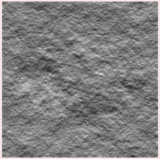
$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

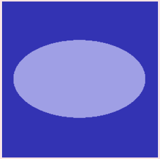


# Recherche automatique des paramètres de régularisation

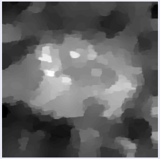
$$\left(\widehat{\boldsymbol{h}}^{\mathrm{L}}, \widehat{\boldsymbol{v}}^{\mathrm{L}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \operatorname*{argmin}_{\boldsymbol{h},\boldsymbol{v}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathrm{L}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v};\alpha)$$

**Exemple**





$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}};\lambda^\dagger,\alpha^\dagger)$
(grille)



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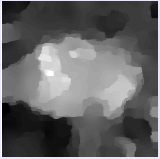

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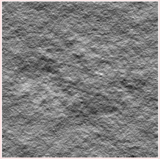
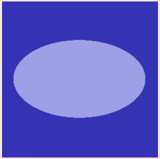
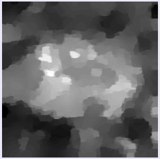
**Exemple**

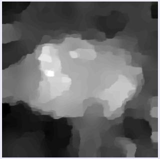




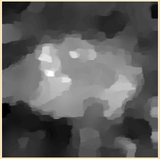
$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}}; \lambda^{\dagger}, \alpha^{\dagger})$ (grille)



$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\dagger}, \widehat{\alpha}^{\dagger})$ (grille)



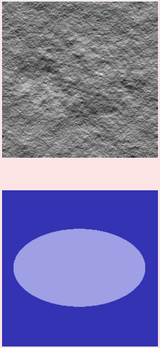
$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\mathrm{qN}}, \widehat{\alpha}^{\mathrm{qN}})$ (quasi-Newton)



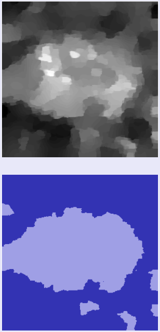
# Recherche automatique des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}}^{\mathsf{L}}, \widehat{\boldsymbol{v}}^{\mathsf{L}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathsf{L}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

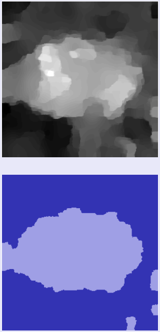
**Exemple**



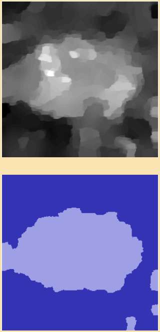
$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}}; \lambda^\dagger, \alpha^\dagger)$
(grille)



$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^\dagger, \widehat{\alpha}^\dagger)$
(grille)



$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\mathrm{qN}}, \widehat{\alpha}^{\mathrm{qN}})$
(quasi-Newton)



**40** appels de l'estimateurs v.s. **225** sur une grille

# Bilan de la présentation

- Régularité et variance locale [ICIP, 2018]

# Bilan de la présentation

- Régularité et variance locale [ICIP, 2018]
  - aptes à caractériser des textures réelles

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- Régularité et variance locale [ICIP, 2018]
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  - attributs complémentaires → capacité à discriminer finement

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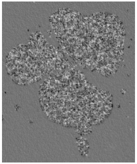
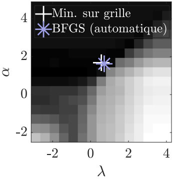
$\longrightarrow$ **En cours** : traitement automatisé de séries temporelles issues de l'étude des écoulements multiphasiques [Ann. Telecom, 2020]



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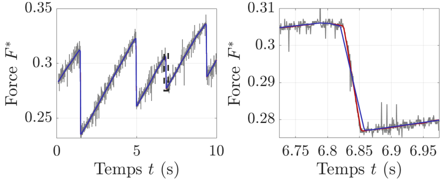
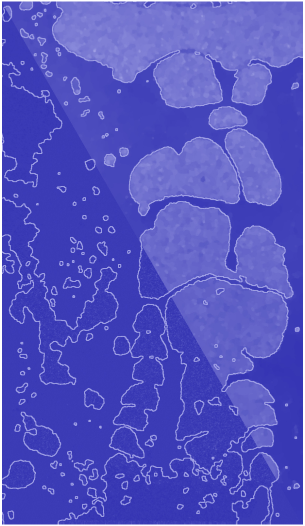
  - débruitage linéaire par morceaux → frottement solide



Merci



Grazie

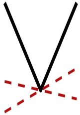
Gracias

Thank you

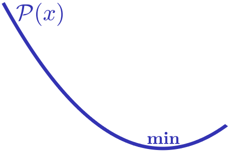
# Définition du gap de dualité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \sum_a \underbrace{\left\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \; \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

non lisse



**Primal** $\quad \min_{\boldsymbol{x}} \; \mathrm{MC}(\boldsymbol{x}) + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x})$



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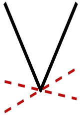
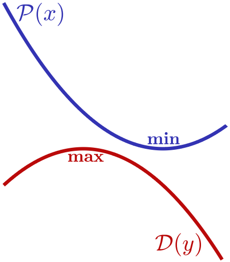


**Primal** $\quad \min_{\boldsymbol{x}} \ \mathrm{MC}(\boldsymbol{x}) + \lambda\mathcal{Q}(\mathbf{D}\boldsymbol{x})$

**Dual** $\quad \max_{\boldsymbol{y}} \ -\mathrm{MC}^*(-\mathbf{D}^\top \boldsymbol{y}) - (\lambda\mathcal{Q})^*(\boldsymbol{y})$



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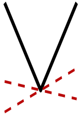

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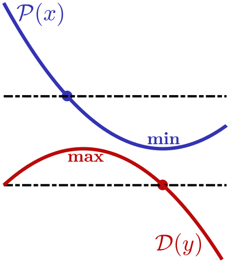
**Proposition** *(Bauschke, 2011)*

Soit $\delta(\boldsymbol{x}; \boldsymbol{y}) \triangleq \mathcal{P}(\boldsymbol{x}) - \mathcal{D}(\boldsymbol{y})$ le gap de dualité,



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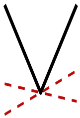

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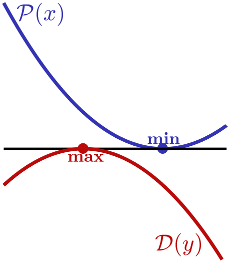


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Soit $\delta(\boldsymbol{x};\boldsymbol{y}) \triangleq \mathcal{P}(\boldsymbol{x}) - \mathcal{D}(\boldsymbol{y})$ le gap de dualité,

$$\delta(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{y}}) = \mathcal{P}(\widehat{\boldsymbol{x}}) - \mathcal{D}(\widehat{\boldsymbol{y}}) = 0$$

# Conjuguée convexe des moindres carrés

$$\mathrm{MC}^*(\boldsymbol{h},\boldsymbol{v}) = \sup_{\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}} \langle \widetilde{\boldsymbol{h}}, \boldsymbol{h}\rangle + \langle \widetilde{\boldsymbol{v}}, \boldsymbol{v}\rangle - \mathrm{MC}(\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) \underset{\text{(si le sup est atteint)}}{=} \langle \bar{\boldsymbol{h}}, \boldsymbol{h}\rangle + \langle \bar{\boldsymbol{v}}, \boldsymbol{v}\rangle - \mathrm{MC}(\bar{\boldsymbol{h}},\bar{\boldsymbol{v}}).$$

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## Condition d'optimalité

$$\begin{cases} \boldsymbol{h} - 2\sum_a \log(a)\left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.}\right) = 0 \\ \boldsymbol{v} - 2\sum_a \left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.}\right) = 0 \end{cases}$$

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$$\mathcal{T} = \sum_a \log \mathcal{L}_{a,.} \quad \text{et} \quad \mathcal{G} = \sum_a \log(a)\log \mathcal{L}_{a,.},$$

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$$\mathcal{T} = \sum_a \log \mathcal{L}_{a,.} \quad \text{et} \quad \mathcal{G} = \sum_a \log(a) \log \mathcal{L}_{a,.},$$

$$\forall m = \{0, 1, 2\},\ S_m = \sum_a (\log a)^m, \quad \boldsymbol{\Phi}^* \boldsymbol{\Phi} = \begin{pmatrix} S_2 \boldsymbol{I} & S_1 \boldsymbol{I} \\ S_1 \boldsymbol{I} & S_0 \boldsymbol{I} \end{pmatrix}$$

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$$\mathrm{MC}^*(\boldsymbol{h},\boldsymbol{v}) = \sup_{\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}} \langle \widetilde{\boldsymbol{h}}, \boldsymbol{h}\rangle + \langle \widetilde{\boldsymbol{v}}, \boldsymbol{v}\rangle - \mathrm{MC}(\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) \underset{\text{(si le sup est atteint)}}{=} \langle \bar{\boldsymbol{h}}, \boldsymbol{h}\rangle + \langle \bar{\boldsymbol{v}}, \boldsymbol{v}\rangle - \mathrm{MC}(\bar{\boldsymbol{h}},\bar{\boldsymbol{v}}).$$

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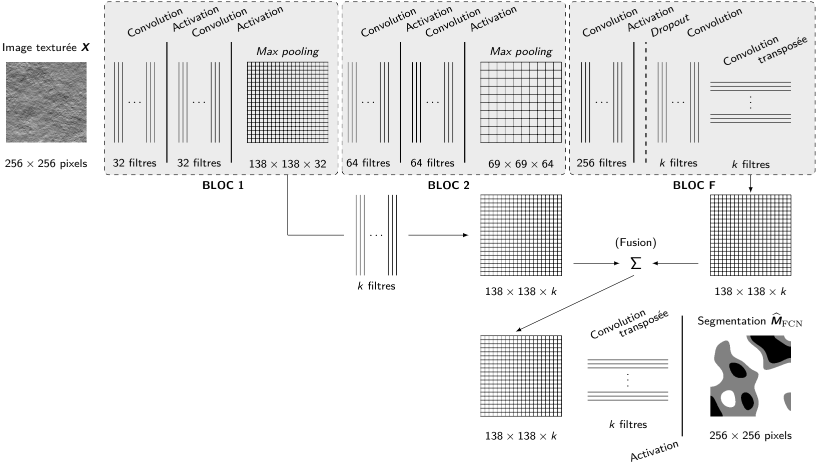
$$\mathrm{MC}^*(\boldsymbol{h},\boldsymbol{v}) = \frac{1}{4}\langle(\boldsymbol{h},\boldsymbol{v}), (\boldsymbol{\Phi}^*\boldsymbol{\Phi})^{-1}(\boldsymbol{h},\boldsymbol{v})\rangle + \langle(\boldsymbol{\mathcal{G}},\boldsymbol{\mathcal{T}}), (\boldsymbol{\Phi}^*\boldsymbol{\Phi})^{-1}(\boldsymbol{h},\boldsymbol{v})\rangle + \mathcal{C}$$

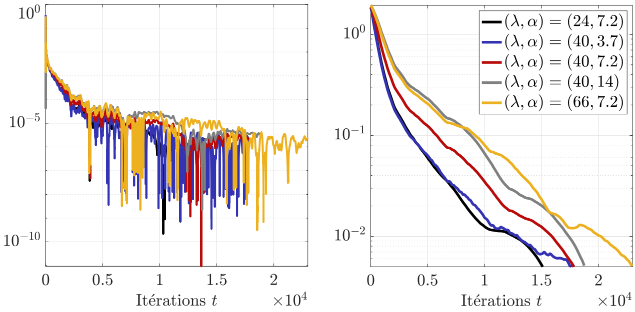
où $\mathcal{C}$ est une constante dépendant uniquement de $\boldsymbol{\mathcal{L}}$.

# Architecture pour la segmentation de texture

## Avec connexions *résiduelles*



# Critère d'arrêt



# Performances de segmentation

## Configuration I

| | 2 classes | 3 classes | 4 classes |
|---|---|---|---|
| Entraîné sur la Config. I, testé sur la Config. I | | | |
| Segmentation à contours « libres » | $93,2 \pm 0,8\%$ | $69,3 \pm 2,8\%$ | $58,6 \pm 1,5\%$ |
| Entraîné sur 2000 images | | | |
| Réseau à $8 \cdot 10^7$ poids / $P = 2000$ | $97.3 \pm 0{,}6\%$ | $97,8 \pm 0,3\%$ | $97,1 \pm 0{,}4\%$ |
| Réseau à $2 \cdot 10^6$ poids / $P = 2000$ | $97,4 \pm 0{,}6\%$ | $98.1 \pm 0,3\%$ | $96,8 \pm 0{,}5\%$ |
| Réseau à $4 \cdot 10^5$ poids / $P = 2000$ | $96,9 \pm 0{,}7\%$ | $98,0 \pm 0,3\%$ | $96.5 \pm 0{,}5\%$ |
| Entraîné sur 20 images | | | |
| Réseau à $8 \cdot 10^7$ poids / $P = 20$ | $95,5 \pm 0{,}9\%$ | $97,5 \pm 0{,}4\%$ | $95,4 \pm 0,8\%$ |
| Réseau à $2 \cdot 10^6$ poids / $P = 20$ | $95,4 \pm 1,1\%$ | $97,4 \pm 0{,}5\%$ | $95,9 \pm 0{,}7\%$ |
| Réseau à $4 \cdot 10^5$ poids / $P = 20$ | $96,6 \pm 0{,}7\%$ | $98,0 \pm 0{,}4\%$ | $96,5 \pm 0{,}5\%$ |

# Performances de segmentation
## Configuration II

| | 2 classes | 3 classes | 4 classes |
|---|---|---|---|
| Entraîné sur la Config. II, testé sur la Config. II | | | |
| Segmentation à contours ≪ libres ≫ | $97,8 \pm 0,2\%$ | $95,2 \pm 3,1\%$ | $64,9 \pm 1,4\%$ |
| Entraîné sur 2000 images | | | |
| Réseau à $8 \cdot 10^7$ poids / $P = 2000$ | $99,1 \pm 0,2\%$ | $98,3 \pm 0,3\%$ | $95,7 \pm 0{,}5\%$ |
| Réseau à $2 \cdot 10^6$ poids / $P = 2000$ | $99,0 \pm 0,2\%$ | $98,5 \pm 0,3\%$ | $95,6 \pm 0{,}5\%$ |
| Réseau à $4 \cdot 10^5$ poids / $P = 2000$ | $99,1 \pm 0,2\%$ | $98,4 \pm 0,3\%$ | $95,2 \pm 0{,}6\%$ |
| Entraîné sur 20 images | | | |
| Réseau à $8 \cdot 10^7$ poids / $P = 20$ | $98,8 \pm 0,2\%$ | $97,9 \pm 0,3\%$ | $94,5 \pm 0{,}7\%$ |
| Réseau à $2 \cdot 10^6$ poids / $P = 20$ | $98,6 \pm 0,3\%$ | $97,4 \pm 0{,}4\%$ | $93,0 \pm 0{,}9\%$ |
| Réseau à $4 \cdot 10^5$ poids / $P = 20$ | $98,8 \pm 0,3\%$ | $98,3 \pm 0,3\%$ | $94,8 \pm 0{,}6\%$ |

# Robustesse

## Entraîné sur la Config. I, testé sur la Config. II

| | 2 classes | 3 classes | 4 classes |
|---|---|---|---|
| Entraîné sur la Config. I, testé sur la Config. II | | | |
| Segmentation à contours « libres » | $79,2 \pm 2,9\%$ | $95,2 \pm 1,2\%$ | $66,3 \pm 1,1\%$ |
| Entraîné sur 2000 images | | | |
| Réseau à $8 \cdot 10^7$ poids / $P = 2000$ | $91,2 \pm 2,1\%$ | $65,7 \pm 7,2\%$ | $55,6 \pm 3,4\%$ |
| Réseau à $2 \cdot 10^6$ poids / $P = 2000$ | $87,9 \pm 2,5\%$ | $69,0 \pm 7,6\%$ | $50,8 \pm 4,0\%$ |
| Réseau à $4 \cdot 10^5$ poids / $P = 2000$ | $81,8 \pm 3,8\%$ | $65,2 \pm 7,2\%$ | $46,4 \pm 3,7\%$ |
| Entraîné sur 20 images | | | |
| Réseau à $8 \cdot 10^7$ poids / $P = 20$ | $91,4 \pm 1,6\%$ | $63,3 \pm 7,1\%$ | $54,7 \pm 3,3\%$ |
| Réseau à $2 \cdot 10^6$ poids / $P = 20$ | $92,4 \pm 1,6\%$ | $65,6 \pm 7,4\%$ | $44,4 \pm 3,4\%$ |
| Réseau à $4 \cdot 10^5$ poids / $P = 20$ | $86,3 \pm 2,6\%$ | $64,9 \pm 7,2\%$ | $48,4 \pm 3,8\%$ |

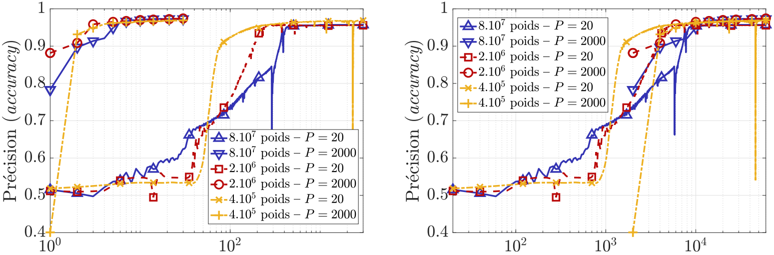
# Robustesse

## Entraîné sur la Config. II, testé sur la Config. I

| | 2 classes | 3 classes | 4 classes |
|---|---|---|---|
| Entraîné sur la Config. II, testé sur la Config. I | | | |
| Segmentation à contours « libres » | $90{,}9 \pm 2{,}8\%$ | $66{,}7 \pm 2,5\%$ | $52{,}0 \pm 1{,}5\%$ |
| Entraîné sur 2000 images | | | |
| Réseau à $8 \cdot 10^7$ poids / $P = 2000$ | $56{,}2 \pm 13{,}5\%$ | $73{,}5 \pm 8{,}2\%$ | $50{,}9 \pm 3{,}9\%$ |
| Réseau à $2 \cdot 10^6$ poids / $P = 2000$ | $55{,}1 \pm 14{,}0\%$ | $74{,}9 \pm 8{,}2\%$ | $51{,}3 \pm 4{,}3\%$ |
| Réseau à $5\,10^5$ poids / $P = 2000$ | $55,5 \pm 13,8\%$ | $72,6 \pm 8{,}1\%$ | $50{,}2 \pm 3{,}8\%$ |
| Entraîné sur 20 images | | | |
| Réseau à $8 \cdot 10^7$ poids / $P = 20$ | $57,1 \pm 13,3\%$ | $71,1 \pm 8,2\%$ | $52,6 \pm 3,8\%$ |
| Réseau à $2 \cdot 10^6$ poids / $P = 20$ | $55,3 \pm 14,0\%$ | $71,7 \pm 8,4\%$ | $49,6 \pm 4,2\%$ |
| Réseau à $5\,10^5$ poids / $P = 20$ | $62,3 \pm 11,5\%$ | $71,0 \pm 8,2\%$ | $54,1 \pm 3,7\%$ |

# Convergence de la phase d'entraînement

Évolution du score de segmentation des trois réseaux au cours de l'entraînement sur la Config. I, avec deux classes $k = 2$

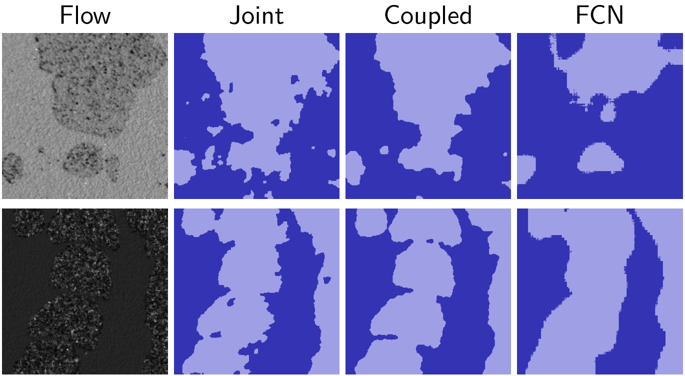


## Comparaison effort de calcul $\mathcal{C}$

| | $\mathcal{W}$ | $P$ | $\mathcal{I}$ | $\mathcal{C}$ |
|---|---|---|---|---|
| Segmentation à contours ≪ libres ≫ | 2 | 1 | $10^7$ | $2\,10^7$ |
| Réseau à $8 \cdot 10^7$ poids / $P = \{20, 2000\}$ | $8 \cdot 10^7$ | $\{20, 2000\}$ | $\{3000, 30\}$ | $4,8 \cdot 10^{12}$ |
| Réseau à $2 \cdot 10^6$ poids / $P = \{20, 2000\}$ | $2 \cdot 10^6$ | $\{20, 2000\}$ | $\{3000, 30\}$ | $1,2 \cdot 10^{11}$ |
| Réseau à $4 \cdot 10^5$ poids / $P = \{20, 2000\}$ | $4 \cdot 10^5$ | $\{20, 2000\}$ | $\{3000, 30\}$ | $2,4 \cdot 10^{10}$ |

- $\mathcal{W}$ : nombre de poids
- $P$ : taille base d'entraînement
- $\mathcal{I}$ : nombre d'*epochs*

# Réseaux de neurones convolutionnels[†]




[†] V. Andrearczyk, https://arxiv.org/abs/1703.05230

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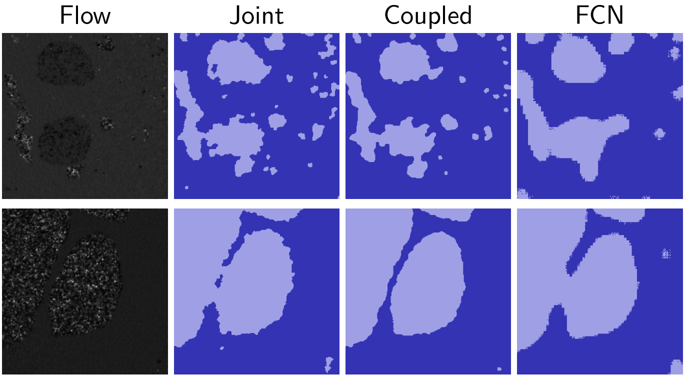


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# *Stein Unbiased Risk Estimate* (Principe)

**Observations** $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$ : vérité et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

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$$

# Estimateur séquentiel et différentiation récursive

$$\boldsymbol{\Lambda} \triangleq (\lambda, \alpha), \qquad \mathbf{U}_{\boldsymbol{\Lambda}} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \lambda[\alpha \mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}]$$

**Primal-dual accéléré**

$$\widetilde{\boldsymbol{z}}^n = \boldsymbol{z}^n + \tau_n \mathbf{U}_{\boldsymbol{\Lambda}} \boldsymbol{w}^n$$
$$\boldsymbol{z}^{n+1} = \mathrm{prox}_{\tau_n (\|\cdot\|_{2,1})^*} \left(\widetilde{\boldsymbol{z}}^n\right)$$
$$\widetilde{\boldsymbol{x}}^n = \boldsymbol{x}^n - \sigma_n \mathbf{U}_{\boldsymbol{\Lambda}}^* \boldsymbol{z}^{n+1}$$
$$\boldsymbol{x}^{n+1} = \mathrm{prox}_{\sigma_n \|\mathbf{D}\mathcal{L} - \boldsymbol{\Phi}\cdot\|_2^2} \left(\widetilde{\boldsymbol{x}}^n\right)$$
$$\theta_n = (1 + 2\mu\sigma_n)^{-\frac{1}{2}},$$
$$\tau_{n+1} = \tau_n / \theta_n, \;\; \sigma_{n+1} = \theta_n \sigma_n$$
$$\boldsymbol{w}^{n+1} = \boldsymbol{x}^n + \theta^n \left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

**Primal-dual accéléré différentié**

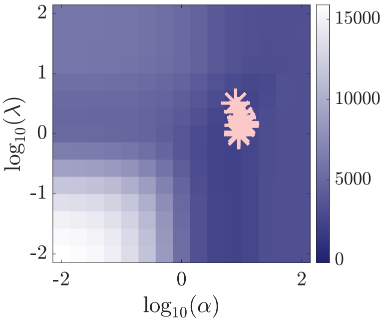
$$\partial_{\boldsymbol{\Lambda}} \widetilde{\boldsymbol{z}}^n = \partial_{\boldsymbol{\Lambda}} \boldsymbol{z}^n + \tau_n \mathbf{U}_{\boldsymbol{\Lambda}} \partial_{\boldsymbol{\Lambda}} \boldsymbol{w}^n + \tau_n \partial_{\boldsymbol{\Lambda}} \mathbf{U}_{\boldsymbol{\Lambda}} \boldsymbol{w}^n$$
$$\partial_{\boldsymbol{\Lambda}} \boldsymbol{z}^{n+1} = \partial_{\tilde{\mathbf{z}}} \mathrm{prox}_{\tau_n (\|\cdot\|_{2,1})^*} \left(\widetilde{\boldsymbol{z}}^n\right) [\partial_{\boldsymbol{\Lambda}} \widetilde{\boldsymbol{z}}^n]$$
$$\partial_{\boldsymbol{\Lambda}} \widetilde{\boldsymbol{x}}^n = \partial_{\boldsymbol{\Lambda}} \boldsymbol{x}^n - \sigma_n \mathbf{U}_{\boldsymbol{\Lambda}}^* \partial_{\boldsymbol{\Lambda}} \boldsymbol{z}^{n+1} - \sigma_n \partial_{\boldsymbol{\Lambda}} \mathbf{U}_{\boldsymbol{\Lambda}} \boldsymbol{z}^{n+1}$$
$$\partial_{\boldsymbol{\Lambda}} \boldsymbol{x}^{n+1} = \partial_{\tilde{\mathbf{x}}} \mathrm{prox}_{\sigma_n \|\mathbf{D}\mathcal{L} - \boldsymbol{\Phi}\cdot\|_2^2} \left(\widetilde{\boldsymbol{x}}^n\right) [\partial_{\boldsymbol{\Lambda}} \widetilde{\boldsymbol{x}}^n]$$

$$\partial_{\boldsymbol{\Lambda}} \boldsymbol{w}^{n+1} = \partial_{\boldsymbol{\Lambda}} \boldsymbol{x}^n + \theta^n \left(\partial_{\boldsymbol{\Lambda}} \boldsymbol{x}^{n+1} - \partial_{\boldsymbol{\Lambda}} \boldsymbol{x}^n\right)$$

# Recherche automatique des paramètres de régularisation

## Moyenne sur dix réalisations de texture

Matrice de covariance estimée $\boldsymbol{\mathcal{S}}$



# Recherche automatique des paramètres de régularisation

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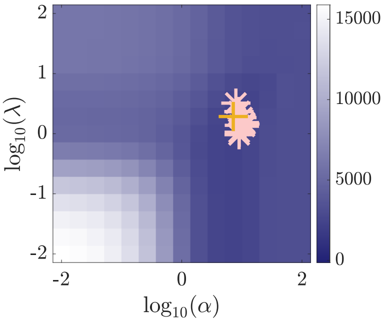

# Recherche automatique des paramètres de régularisation

## Moyenne sur dix réalisations de texture

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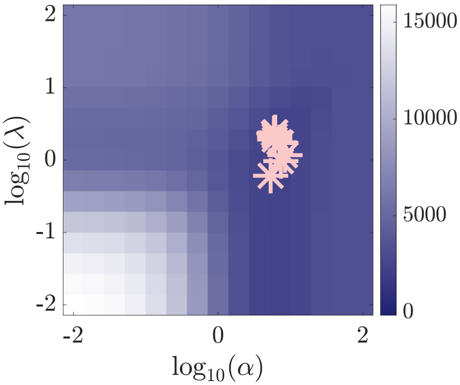


$\widehat{\lambda}^{\mathsf{qN}} = 1{,}68 \pm 0{,}37 \qquad \widehat{\alpha}^{\mathsf{qN}} = 6{,}70 \pm 0{,}58$

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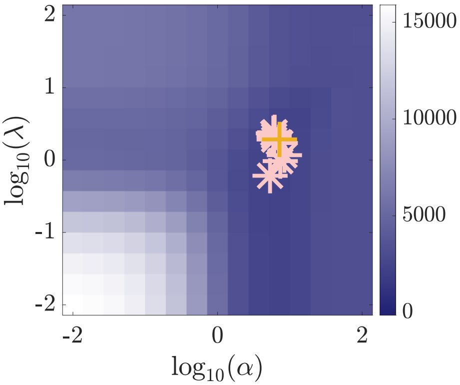


$\widehat{\lambda}^{\mathrm{qN}} = 1{,}68 \pm 0{,}37 \qquad \widehat{\alpha}^{\mathrm{qN}} = 6{,}70 \pm 0{,}58$

# Initialisation quasi-Newton

- Hyperparamètres $\boldsymbol{\Lambda}^{[0]} = \left(\lambda^{[0]}, \alpha^{[0]}\right)$, avec

$$\lambda^{[0]} = \frac{\mathrm{tr}(\boldsymbol{\mathcal{S}})}{2\,\mathrm{TV}(\widehat{\boldsymbol{v}}^{\mathrm{RL}}(\boldsymbol{\mathcal{L}}))}, \quad \text{et} \quad \alpha^{[0]} = \frac{\mathrm{TV}(\widehat{\boldsymbol{v}}^{\mathrm{RL}}(\boldsymbol{\mathcal{L}}))}{\mathrm{TV}(\widehat{\boldsymbol{h}}^{\mathrm{RL}}(\boldsymbol{\mathcal{L}}))}.$$

- Approximation de l'inverse de la hessienne

$$\boldsymbol{H}^{[0]} = \mathrm{diag}\left(\left|\frac{\kappa\lambda^{[0]}}{\partial_\lambda \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \boldsymbol{\Lambda}^{[0]}|\boldsymbol{\mathcal{S}})}\right|, \left|\frac{\kappa\alpha^{[0]}}{\partial_\alpha \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \boldsymbol{\Lambda}^{[0]}|\boldsymbol{\mathcal{S}})}\right|\right).$$