Proximal schemes for the estimation of the reproduction number of Covid19:

From convex optimization to Monte Carlo sampling

Variational approaches for signal and image processing

SiSyPhe day, ENS Lyon

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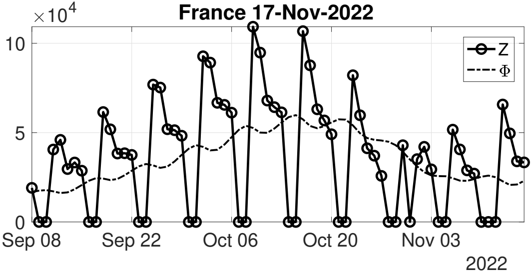
Barbara Pascal

Joint work with P. Abry, N. Pustelnik, S. Roux, R. Gribonval, P. Flandrin;
G. Fort, H. Artigas; Juliana Du

Outline

- Pandemic study: modeling at the service of monitoring
- Reproduction number estimation from minimization of penalized likelihood
- Bayesian framework for credibility interval estimation
- Conclusion & Perspectives

Counts of daily new infections



data from National Health Agencies collected by Johns Hopkins University

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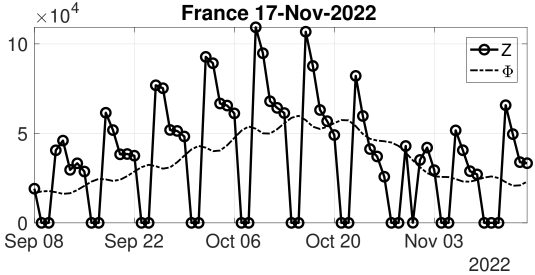

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$\Longrightarrow$ number of cases not informative enough: need to capture the **dynamics**

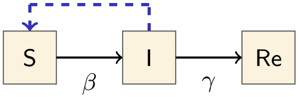
Design adapted counter measures and evaluate their effectiveness

$\longrightarrow$ efficient monitoring tools *epidemiological model,*

$\longrightarrow$ robust to low quality of the data *managing erroneous counts,*

$\longrightarrow$ accompanied by reliable confidence level *credibility intervals.*

Susceptible-Infected-Recovered (SIR), among *compartmental models*



- ODE: $\frac{\mathrm{d}\mathsf{S}_t}{\mathrm{d}t} = -\beta \mathsf{S}_t \mathsf{I}_t$, $\frac{\mathrm{d}\mathsf{I}_t}{\mathrm{d}t} = \beta \mathsf{S}_t \mathsf{I}_t - \gamma \mathsf{I}_t$, $\frac{\mathrm{d}\mathsf{Re}_t}{\mathrm{d}t} = \gamma \mathsf{I}_t$
- Stochastic model: likelihood maximization to infer β, γ

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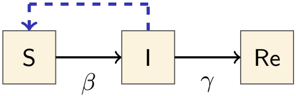

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– Stochastic model: likelihood maximization to infer β, γ

Limitations:

- refinement needed to get socially realistic model
- quadratic increase of the number of parameters
- Bayesian framework: heavy computational burden
- need consolidated and accurate datasets

✗ not adapted to real-time monitoring of Covid19 pandemic

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"averaged number of secondary cases generated by a typical infectious individual"

(Cori et al., 2013, *Am. Journal of Epidemiology*; Liu et al., 2018, *PNAS*)

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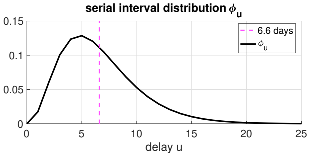
Principle: Z_t new infections at day t

$$\mathbb{E}\left[Z_t\right] = R_t \Phi_t, \quad \Phi_t = \sum_{u=1}^{\tau_\Phi} \phi_u Z_{t-u}$$



with Φ_t global "infectiousness" in the population

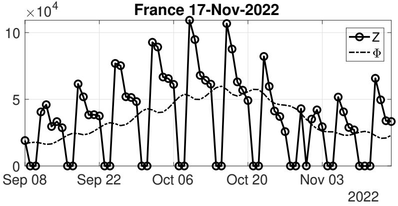
$\{\phi_u\}_{u=1}^{\tau_\Phi}$ distribution of delay between onset of symptoms in primary and secondary cases

Gamma distribution truncated at 25 days, of mean 6.6 days and standard deviation 3.5 days

Data: daily counts $\mathbf{Z} = (Z_1, \ldots, Z_T)$

Model: Poisson distribution

$$\mathbb{P}(\mathsf{Z}_t | \mathbf{Z}_{t-\tau_\Phi:t-1}, \mathsf{R}_t) = \frac{(\mathsf{R}_t \Phi_t)^{\mathsf{Z}_t} \mathrm{e}^{-\mathsf{R}_t \Phi_t}}{\mathsf{Z}_t!}$$



Pandemic study: modeling at the service of monitoring

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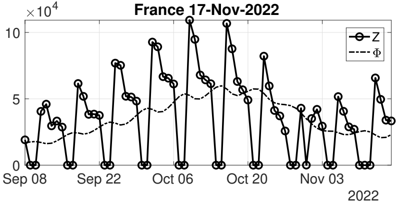

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Maximum Likelihood Estimate (MLE)

$$\begin{aligned}
&\ln\left(\mathbb{P}(Z_t | \mathbf{Z}_{t-\tau_\Phi:t-1}, R_t)\right) \\
&= Z_t \ln(R_t \Phi_t) - R_t \Phi_t - \ln(Z_t!) \\
&\underset{Z_t \gg 1}{\simeq} Z_t \ln(R_t \Phi_t) - R_t \Phi_t - Z_t \ln(Z_t) + Z_t \\
&\underset{\text{(def.)}}{=} -d_{KL}(Z_t | R_t \Phi_t) \quad \text{(Kullback-Leibler)}
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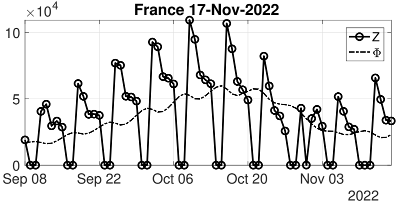

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ratio of moving averages

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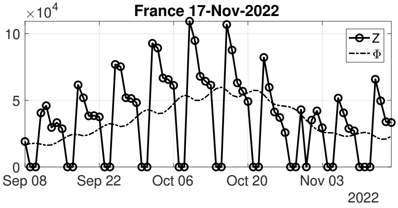

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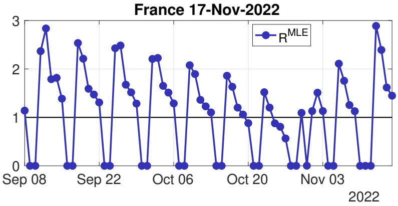

- huge variability along time/ no local trend
- not robust to pseudo-periodicity/ misreported counts

Solution 0: (state-of-the-art) smoothing over a temporal window

$$\widehat{\mathrm{R}}_{t,s}^{\mathrm{MLE}}, \text{ with } s = 7 \text{ days}$$

(Cori et al., 2013, *Am. Journal of Epidemiology*)

$\Longrightarrow$ not able to detect rapid surge, nor fast decrease following sanitary restrictions

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Solution 1: regularization through nonlinear filtering

$$\widehat{\mathbf{R}}^{\mathrm{PKL}} = \underset{\mathbf{R}\in\mathbb{R}_+^T}{\operatorname{argmin}} \sum_{t=1}^{T} \mathrm{d}_{\mathrm{KL}}\left(\mathrm{Z}_t \,|\mathrm{R}_t\Phi_t\right) + \lambda_{\mathrm{R}}\mathcal{P}(\mathbf{R}) \quad \text{(penalized Kullback-Leibler)}$$

with $\mathcal{P}(\mathbf{R})$ favoring some temporal regularity (Abry et al., 2020, *PlosOne*)

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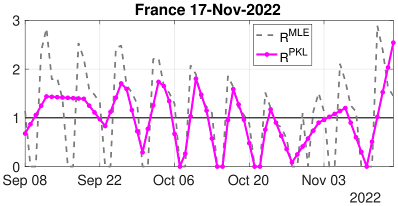
$$\mathcal{P}(\mathbf{R}) = \|\mathbf{D}_2\mathbf{R}\|_1$$

$$(\mathbf{D}_2\mathbf{R})_t = R_{t+1} - 2R_t + R_{t-1}$$

2nd order derivative & ℓ_1-norm

$\Longrightarrow$ piecewise linearity

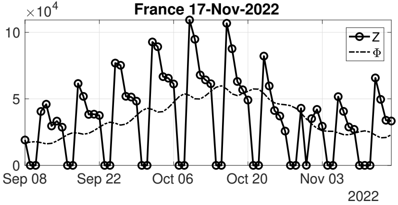


captures global **trend**, more **regular** than MLE, but **pseudo-oscillations**

New infection counts **Z** are corrupted by

- missing samples,
- non meaningful negative counts,
- retrospected cumulated counts,
- pseudo-seasonality effects.

$\Longrightarrow$ full parametric modeling out of reach



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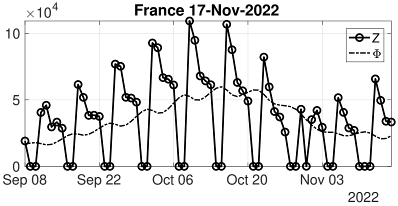

Solution 1': first correct **Z**, then apply penalized Kullback-Leibler on corrected $\mathbf{Z}^{(C)}$

$\Longrightarrow$ two-step procedure not optimal: accumulates correction & regularization biases

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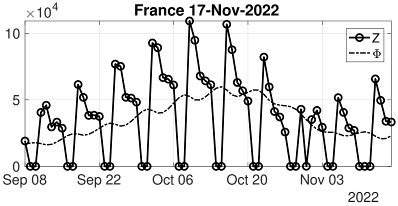

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Solution 2: **one-step** procedure performing jointly

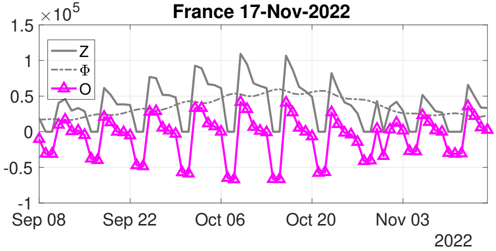
correction of corrupted Z_t & estimation of regularized R_t

(Pascal et al., 2022, *Trans. Sig. Process.*)

Extended Cori Model: additional latent variable O_t accounting for misreport

$$Z_t \sim \text{Poiss}\left(R_t \Phi_t + O_t\right), \quad R_t \Phi_t + O_t \geq 0$$

nonzero values of O_t concentrated on specific days (Sundays, day-offs, ...)



Interpretation:

$$\text{Poiss}\left(R_t \Phi_t + O_t\right) \sim \begin{cases} \text{Poiss}\left(R_t \Phi_t\right) + \text{Poiss}\left(O_t\right) & \text{if } O_t \geq 0, \\ \text{Poiss}\left(\alpha_t R_t \Phi_t\right), \; \alpha_t = 1 - \frac{-O_t}{R_t \Phi_t} \in [0, 1] & \text{if } O_t < 0. \end{cases}$$

Data: reported counts $\mathbf{Z} = (Z_1, \ldots, Z_T)$

Model: corrected Poisson $\mathbb{P}(Z_t | \mathbf{Z}_{t-\tau_\Phi:t-1}, R_t, O_t) = \dfrac{(R_t \Phi_t + O_t)^{Z_t} e^{-(R_t \Phi_t + O_t)}}{Z_t!}$

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Generalized Penalized Kullback-Leibler

$$(\widehat{\mathbf{R}}, \widehat{\mathbf{O}}) = \underset{(\mathbf{R},\mathbf{O}) \in \mathbb{R}_+^T \times \mathbb{R}^T}{\operatorname{argmin}} \sum_{t=1}^{T} d_{KL}\left(Z_t \mid R_t\Phi_t + O_t\right) + \lambda_R \|\mathbf{D}_2\mathbf{R}\|_1 + \iota_{\geq 0}(\mathbf{R}) + \lambda_O \|\mathbf{O}\|_1$$

$\Longrightarrow$ estimates piecewise linear, non-negative R_t and sparse O_t

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properties of the objective function:

- sum of convex functions composed with linear operators $\Longrightarrow$ globally convex;
- feasible domain: $(\forall t, R_t \geq 0)$ & (if $Z_t > 0, R_t\Phi_t + O_t > 0$, else $R_t\Phi_t + O_t \geq 0$);
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Theorem (Pascal et al., 2022, *Trans. Sig. Process.*)

+ The minimization problem has at least one solution $(\widehat{\mathbf{R}}, \widehat{\mathbf{O}})$.
+ The estimated time-varying Poisson intensity $\widehat{p}_t = \widehat{R}_t \Phi_t + \widehat{O}_t$ is unique.

$$\underset{(\mathbf{R},\mathbf{O})\in\mathbb{R}_+^T\times\mathbb{R}^T}{\text{minimize}} \sum_{t=1}^{T} \mathrm{d}_{\mathrm{KL}}\left(\mathrm{Z}_t \,|\, \mathrm{R}_t\Phi_t + \mathrm{O}_t\right) + \lambda_{\mathrm{R}}\|\mathbf{D}_2\mathbf{R}\|_1 + \iota_{\geq 0}(\mathbf{R}) + \lambda_{\mathrm{O}}\|\mathbf{O}\|_1$$

- each term of the functional is convex;
- ℓ_1-norm and indicative functions $\Longrightarrow$ nonsmooth; ✗ gradient descent
- gradient of $\mathrm{p}_t \mapsto \mathrm{d}_{\mathrm{KL}}\left(\mathrm{Z}_t \,|\, \mathrm{p}_t\right)$ is not Lipschitzian; ✗ forward-backward
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$$\Longleftrightarrow \underset{(\mathbf{R},\mathbf{O})\in\mathbb{R}_+^T\times\mathbb{R}^T}{\text{minimize}} \quad f(\mathbf{R},\mathbf{O}|\mathbf{Z}) + h(\mathbf{A}(\mathbf{R},\mathbf{O})), \quad \mathbf{A} \text{ linear; } f, h \text{ proximable}$$

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Primal-dual algorithm (Chambolle et al., 2011, *Int. Conf. Comput. Vis.*)

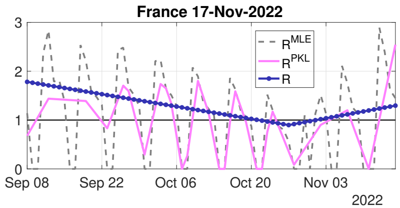
for $k = 1, 2 \ldots$ **do**

$$\mathbf{Q}^{[k+1]} = \mathrm{prox}_{\sigma h^*}(\mathbf{Q}^{[k]} + \sigma\mathbf{A}(\overline{\mathbf{R}}^{[k]}, \overline{\mathbf{O}}^{[k]})) \qquad \text{dual}$$

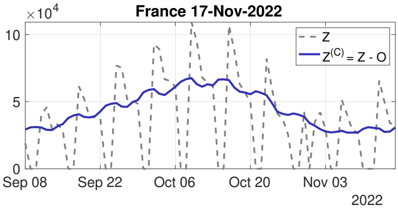
$$(\mathbf{R}^{[k+1]}, \mathbf{O}^{[k+1]}) = \mathrm{prox}_{\tau f(\cdot|\mathbf{Z})}((\mathbf{R}^{[k+1]}, \mathbf{O}^{[k+1]}) - \tau\mathbf{A}^*\mathbf{Q}^{[k+1]}) \qquad \text{primal}$$

$$(\overline{\mathbf{R}}^{[k+1]}, \overline{\mathbf{O}}^{[k+1]}) = 2(\mathbf{R}^{[k+1]}, \mathbf{O}^{[k+1]}) - (\mathbf{R}^{[k]}, \mathbf{O}^{[k]}) \qquad \text{auxiliary}$$

Reproduction number $\widehat{\mathbf{R}}$

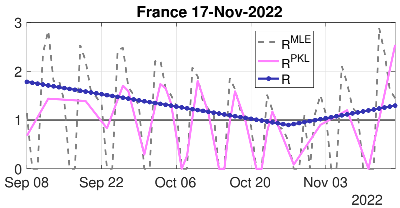


Corrected infection counts $\mathbf{Z}^{(C)}$



$\Longrightarrow$ no more pseudo-seasonality, local trends well captured, smooth behavior

Reproduction number $\widehat{\mathbf{R}}$



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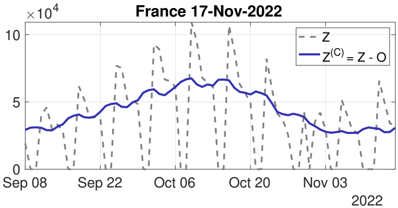
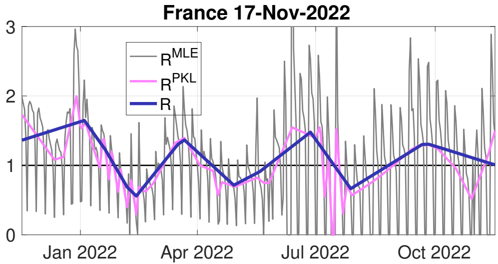


$\Longrightarrow$ no more pseudo-seasonality, local trends well captured, smooth behavior



fast numerical scheme: 15 to 30 sec for 70 days to 1 year

New infection counts per county: $\mathbf{Z} = \left\{ \mathrm{Z}_t^{(d)},\, d \in [1, D],\, t \in [1,\, T] \right\}$

$\Rightarrow$ multivariate time-varying reproduction number $\mathrm{R}_t^{(d)}$

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Multivariate extended penalized Kullback-Leibler

$$\left(\widehat{\mathbf{R}}, \widehat{\mathbf{O}}\right) = \underset{(\mathbf{R},\mathbf{O}) \in \mathbb{R}_+^{D \times T} \times \mathbb{R}^{D \times T}}{\operatorname{argmin}} \sum_{d=1}^{D} \sum_{t=1}^{T} \mathsf{d}_{\mathsf{KL}} \left(\mathsf{Z}_t^{(d)} \,\middle|\, \mathsf{R}_t^{(d)} \Phi_t^{(d)} + \mathsf{O}_t^{(d)} \right) + \lambda_{\mathsf{R}} \|\mathbf{D}_2 \mathbf{R}\|_1 + \iota_{\geq 0}(\mathbf{R}) + \lambda_{\text{space}} \|\mathbf{G}\mathbf{R}\|_1 + \lambda_{\mathsf{O}} \|\mathbf{O}\|_1$$

$\Longrightarrow$ $\|\mathbf{G}\mathbf{R}\|_1$ favors **piecewise constancy** in space

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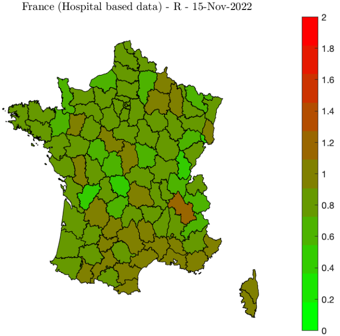
Graph Total Variation

$$\|\mathbf{G}\mathbf{R}\|_1 = \sum_{t=1}^{T} \sum_{d_1 \sim d_2} \left| \mathrm{R}_t^{(d_1)} - \mathrm{R}_t^{(d_2)} \right|$$

sum over neighboring counties

here: $d_1 \sim d_2 \Leftrightarrow$ share terrestrial border

$$\widetilde{\mathbf{A}}(\mathbf{R}, \mathbf{O}) = (\lambda_{\mathrm{R}} \mathbf{D}_2 \mathbf{R}, \mathbf{R}, \lambda_{\text{space}} \mathbf{G}\mathbf{R}, \lambda_{\mathrm{O}} \mathbf{O})$$



Bayesian framework for credibility interval estimation

<u>Pointwise estimate</u> of parameter $\boldsymbol{\theta} = (\mathbf{R}, \mathbf{O})$ from observations $\mathbf{Z}$

$$\underset{(\mathbf{R},\mathbf{O})\in\mathbb{R}_+^T\times\mathbb{R}^T}{\text{minimize}} \quad f(\boldsymbol{\theta}|\mathbf{Z}) + h(\mathbf{A}\boldsymbol{\theta})$$

(Pascal et al., 2022, *Trans. Sig. Process.*)

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Q: what is the value of R today? **A:** solve the minimization problem and output $\widehat{\mathrm{R}}_T$.

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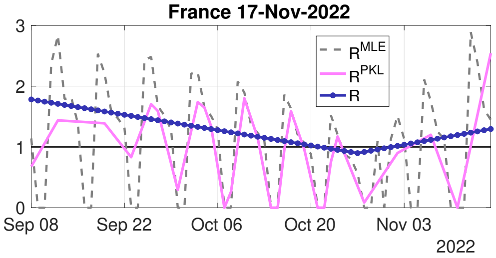

$\widehat{\mathrm{R}}_T = 1.2955$

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Bayesian reformulation: interpret $(\widehat{\mathbf{R}}, \widehat{\mathbf{O}})$ as the MAP of

$$\pi(\boldsymbol{\theta}) \propto \exp(-f(\boldsymbol{\theta}|\mathbf{Z}) - h(\mathbf{A}\boldsymbol{\theta}))$$

- $\exp(-f(\boldsymbol{\theta}|\mathbf{Z})) \sim$ likelihood of the observation
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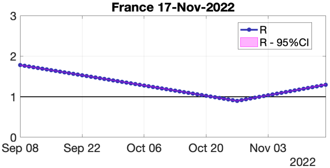
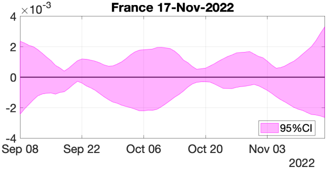
$\Longrightarrow$ instead of focusing on $\widehat{\mathrm{R}}_t$, the **pointwise** MAP, probe π to get

$\mathrm{R}_t \in [\underline{\mathrm{R}}_t, \overline{\mathrm{R}}_t]$ with 95% probability, i.e., **credibility interval** estimates





$$\widehat{\mathrm{R}}_T \in [1.2987, 1.3047]$$

Log-likelihood from Poisson model $\mathcal{D} = \{\boldsymbol{\theta} \,|\, \forall t,\ \mathsf{R}_t\Phi_t + \mathsf{O}_t \geq 0\}$

$$f(\boldsymbol{\theta}\,|\mathbf{Z}) := \begin{cases} -\sum_{t=1}^{T}(\mathsf{Z}_t \ln(\mathsf{R}_t\Phi_t + \mathsf{O}_t) - (\mathsf{R}_t\Phi_t + \mathsf{O}_t) + \mathcal{C}(\mathsf{Z}_t)) & \text{if } \boldsymbol{\theta} \in \mathcal{D}, \\ +\infty & \text{otherwise,} \end{cases}$$

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Prior distribution of $\boldsymbol{\theta} = (\mathbf{R}, \mathbf{O}) = (\mathsf{R}_1, \ldots, \mathsf{R}_T, \mathsf{O}_1, \ldots, \mathsf{O}_T) \in (\mathbb{R}_+)^T \times \mathbb{R}^T$

- reproduction number: $\mathsf{R}_t - 2\mathsf{R}_{t-1} + \mathsf{R}_{t-2} \sim \mathsf{Laplace}(\lambda_\mathsf{R})$

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$$\Rightarrow g(\boldsymbol{\theta}) = \lambda_\mathsf{R}\|\mathbf{D}_2\mathbf{R}\|_1 + \lambda_\mathsf{O}\|\mathbf{O}\|_1, \quad \mathbf{D}_2 = \underbrace{\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & 0 & \ldots & 0 \\ \ldots & & & & & & \ldots \\ 0 & \ldots & & & 1 & -2 & 1 \end{bmatrix}}_{\text{Laplacian}}$$

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Posterior distribution of unknown parameters $\boldsymbol{\theta} = (\mathbf{R}, \mathbf{O})$

$$\pi(\boldsymbol{\theta}) \propto \exp\left(-f(\boldsymbol{\theta}) - g(\boldsymbol{\theta})\right) \mathbb{1}_{\mathcal{D}}(\boldsymbol{\theta})$$

- f, g convex
- f smooth, g nonsmooth

Purpose: sampling the random variable $\boldsymbol{\theta} = (\mathbf{R}, \mathbf{O}) \in \mathbb{R}^{2T}$ according to the posterior[†]

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[†] π is defined up to a normalizing constant

Markov Chain Monte Carlo sampling

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Principle: 1) generate a random sequence $\{\boldsymbol{\theta}^n, \, n \in \mathbb{N}\}$ such that

- $\boldsymbol{\theta}^{n+1}$ only depends on $\boldsymbol{\theta}^n$,
- at convergence, i.e., as $n \to \infty$, $\boldsymbol{\theta}^n \sim \pi$,

2) compute Bayesian estimators, e.g., credibility intervals, on samples $\{\boldsymbol{\theta}^n, \, n \geq N\}$

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State-of-the-art: *Hastings-Metropolis random walk*

(i) propose a random move according to

$$\boldsymbol{\theta}^{n+\frac{1}{2}} = \boldsymbol{\theta}^n + \sqrt{2\gamma}\Gamma\xi^{n+1}, \quad \xi^{n+1} \sim \mathcal{N}_{2T}(0, \mathrm{I})$$

with γ positive step size, $\Gamma \in \mathbb{R}^{2T\times 2T}$

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(ii) accept: $\boldsymbol{\theta}^{n+1} = \boldsymbol{\theta}^{n+\frac{1}{2}}$, with probability $1 \wedge \dfrac{\pi(\boldsymbol{\theta}^{n+\frac{1}{2}})}{\pi(\boldsymbol{\theta}^n)}$, or reject: $\boldsymbol{\theta}^{n+1} = \boldsymbol{\theta}^n$

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Metropolis Adjusted Langevin Algorithm (MALA)

Langevin dynamics: $\boldsymbol{\theta}^{n+\frac{1}{2}} = \mu(\boldsymbol{\theta}^n) + \sqrt{2\gamma}\xi^{n+1}$, (Kent, 1978, *Adv Appl Probab*)

$\mu(\boldsymbol{\theta})$ adapted to $\pi(\boldsymbol{\theta}) = \exp(-f(\boldsymbol{\theta}) - g(\boldsymbol{\theta}))\mathbb{1}_{\mathcal{D}}(\boldsymbol{\theta})$

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Case 1: $g = 0$ and $-\ln \pi = f$ is smooth (Roberts & Tweedie, 1996, *Bernoulli*)

$$\mu(\boldsymbol{\theta}) = \boldsymbol{\theta} - \gamma \Gamma\Gamma^\top \nabla f(\boldsymbol{\theta}) = \boldsymbol{\theta} + \gamma \Gamma\Gamma^\top \nabla \ln \pi(\boldsymbol{\theta})$$

$\Longrightarrow$ move towards areas of higher probability

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Case 2: $-\ln\pi = f + g$ is nonsmooth

$$\mu(\boldsymbol{\theta}) = \mathrm{prox}_{\gamma g}^{\Gamma\Gamma^\top}(\boldsymbol{\theta} - \gamma\Gamma\Gamma^\top\nabla f(\boldsymbol{\theta}))$$

combining *Langevin* and *proximal*[†] approaches

† $\mathrm{prox}_{\gamma g}^{\Gamma\Gamma^\top}(y) = \operatorname*{argmin}_{x\in\mathbb{R}^d}\left(\frac{1}{2}\|x-y\|^2_{\Gamma\Gamma^\top} + \gamma g(x)\right)$: preconditioned proximity operator of g

Posterior density of $\boldsymbol{\theta} = (\mathbf{R}, \mathbf{O})$: $\quad \pi(\boldsymbol{\theta}) \propto \exp\left(-f(\boldsymbol{\theta}) - g(\boldsymbol{\theta})\right) \mathbb{1}_{\mathcal{D}}(\boldsymbol{\theta})$

- smooth negative log-likelihood

$$\text{if } \boldsymbol{\theta} \in \mathcal{D}, \quad f(\boldsymbol{\theta}) = -\sum_{t=1}^{T} \left(\mathsf{Z}_t \ln \mathsf{p}_t(\boldsymbol{\theta}) - \mathsf{p}_t(\boldsymbol{\theta})\right), \quad \mathsf{p}_t(\boldsymbol{\theta}) = \mathsf{R}_t(\Phi \mathsf{Z})_t + \mathsf{O}_t$$

- nonsmooth convex lower-semicontinuous negative a priori log-distribution

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$\mathbf{A} : \boldsymbol{\theta} \mapsto (\mathbf{D}_2\mathbf{R}, \mathbf{O})$ linear operator, $h(\cdot_1, \cdot_2) = \lambda_{\mathsf{R}} \|\cdot_1\|_1 + \lambda_{\mathsf{O}} \|\cdot_2\|_1$

Proximal-Gradient dual sampler `PGdual`

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Case 3: $-\ln \pi = f + h(\mathbf{A}\cdot)$ (Fort et al., 2022, *preprint*)

closed-form expression of $\text{prox}_{\gamma h}$ but not of $\text{prox}_{\gamma h(\mathbf{A}\cdot)}$

1) extend $\mathbf{A}$ into **invertible** $\overline{\mathbf{A}}$, and h in $\bar{h}$ such that $\bar{h}(\overline{\mathbf{A}}\boldsymbol{\theta}) = h(\mathbf{A}\boldsymbol{\theta})$

2) reason on the **dual** variable $\tilde{\boldsymbol{\theta}} = \overline{\mathbf{A}}\boldsymbol{\theta}$

Langevin: drift toward higher probability regions

$$\operatorname*{argmax}_{\boldsymbol{\theta}\in\mathbb{R}^{2T}} \ln \pi(\boldsymbol{\theta}) = \operatorname*{argmin}_{\boldsymbol{\theta}\in\mathbb{R}^{2T}} f(\boldsymbol{\theta}) + \bar{h}(\overline{\mathbf{A}}\boldsymbol{\theta}) = \mathbf{A}^{-1} \operatorname*{argmin}_{\tilde{\boldsymbol{\theta}}\in\mathbb{R}^{2T}} f(\overline{\mathbf{A}}^{-1}\tilde{\boldsymbol{\theta}}) + \bar{h}(\tilde{\boldsymbol{\theta}})$$

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Two strategies to extend $\mathbf{A} = \begin{pmatrix} \mathbf{D}_2 & 0 \\ 0 & \mathbf{I} \end{pmatrix} \in \mathbb{R}^{(2T-1)\times 2T}$ into $\overline{\mathbf{A}} = \begin{pmatrix} \overline{\mathbf{D}} & 0 \\ 0 & \mathbf{I} \end{pmatrix} \in \mathbb{R}^{2T\times 2T}$:

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Invert

$$\overline{\mathbf{D}}_2 := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 & \cdots & 0 \\ & \mathbf{D}_2 & & & \end{bmatrix}$$

Langevin: drift toward higher probability regions

$$\underset{\boldsymbol{\theta}\in\mathbb{R}^{2T}}{\operatorname{argmax}} \ln \pi(\boldsymbol{\theta}) = \underset{\boldsymbol{\theta}\in\mathbb{R}^{2T}}{\operatorname{argmin}}\ f(\boldsymbol{\theta}) + \bar{h}(\overline{\mathbf{A}}\boldsymbol{\theta}) = \mathbf{A}^{-1}\underset{\tilde{\boldsymbol{\theta}}\in\mathbb{R}^{2T}}{\operatorname{argmin}}\ f(\overline{\mathbf{A}}^{-1}\tilde{\boldsymbol{\theta}}) + \bar{h}(\tilde{\boldsymbol{\theta}})$$

$$\Longrightarrow \quad \mu(\boldsymbol{\theta}) = \underbrace{\overline{\mathbf{A}}^{-1}}_{\text{back to } \boldsymbol{\theta}} \underbrace{\operatorname{prox}_{\gamma\bar{h}}\left(\overline{\mathbf{A}}\boldsymbol{\theta} - \gamma\overline{\mathbf{A}}^{-\top}\nabla f(\boldsymbol{\theta})\right)}_{\text{proximal-gradient on } \tilde{\boldsymbol{\theta}}}$$

Two strategies to extend $\mathbf{A} = \begin{pmatrix} \mathbf{D}_2 & 0 \\ 0 & \mathrm{I} \end{pmatrix} \in \mathbb{R}^{(2T-1)\times 2T}$ into $\overline{\mathbf{A}} = \begin{pmatrix} \overline{\mathbf{D}} & 0 \\ 0 & \mathrm{I} \end{pmatrix} \in \mathbb{R}^{2T\times 2T}$:

Invert

$$\overline{\mathbf{D}}_2 := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 & \cdots & 0 \\ & \mathbf{D}_2 & & & \end{bmatrix}$$

Ortho

$$\overline{\mathbf{D}}_o := \begin{bmatrix} v_1^\top \\ v_2^\top \\ \mathbf{D}_2 \end{bmatrix} \qquad \begin{array}{c} v_1, v_2 \in \mathbb{R}^{2T} \\ v_1 \perp v_2,\ v_1, v_2 \in (\mathbf{D}_2^\top)^\perp \end{array}$$

Proximal-Gradient dual sampler `PGdual`

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Proposed `PGdual` **drift terms** on $\boldsymbol{\theta} = (\mathbf{R}, \mathbf{O})$:

reproduction numbers $\mu_{\mathrm{R}}(\boldsymbol{\theta}) = \overline{\mathbf{D}}^{-1} \operatorname{prox}_{\gamma_{\mathrm{R}}\lambda_{\mathrm{R}}\|(\cdot)_{3:T}\|_1}\left(\overline{\mathbf{D}}\,\mathbf{R} - \gamma_{\mathrm{R}}\overline{\mathbf{D}}^{-\top}\nabla_{\mathrm{R}} f(\boldsymbol{\theta})\right)$

outliers $\mu_{\mathrm{O}}(\boldsymbol{\theta}) = \operatorname{prox}_{\gamma_{\mathrm{O}}\lambda_{\mathrm{O}}\|\cdot\|_1}\left(\mathbf{O} - \gamma_{\mathrm{O}}\nabla_{\mathrm{O}} f(\boldsymbol{\theta})\right)$

Data: $\overline{\mathbf{D}} = \overline{\mathbf{D}}_2$ (`Invert`) or $\overline{\mathbf{D}} = \overline{\mathbf{D}}_o$ (`Ortho`)

$\gamma_{\mathrm{R}}, \gamma_{\mathrm{O}} > 0$, $N_{\max} \in \mathbb{N}_\star$, $\boldsymbol{\theta}^0 = (\mathbf{R}^0, \mathbf{O}^0) \in \mathcal{D}$

Result: A $\mathcal{D}$-valued sequence $\{\boldsymbol{\theta}^n = (\mathbf{R}^n, \mathbf{O}^n), n \in 0, \ldots, N_{\max}\}$

for $n = 0, \ldots, N_{\max} - 1$ **do**

Sample $\xi_{\mathrm{R}}^{n+1} \sim \mathcal{N}_T(0, \mathrm{I})$ and $\xi_{\mathrm{O}}^{n+1} \sim \mathcal{N}_T(0, \mathrm{I})$;

Set $\mathbf{R}^{n+\frac{1}{2}} = \mu_{\mathrm{R}}(\boldsymbol{\theta}^n) + \sqrt{2\gamma_{\mathrm{R}}}\,\overline{\mathbf{D}}^{-1}\overline{\mathbf{D}}^{-\top}\,\xi_{\mathrm{R}}^{n+1}$;

$\mathbf{O}^{n+\frac{1}{2}} = \mu_{\mathrm{O}}(\boldsymbol{\theta}^n) + \sqrt{2\gamma_{\mathrm{O}}}\,\xi_{\mathrm{O}}^{n+1}$;

$\boldsymbol{\theta}^{n+\frac{1}{2}} = (\mathbf{R}^{n+\frac{1}{2}}, \mathbf{O}^{n+\frac{1}{2}})$;

Set $\boldsymbol{\theta}^{n+1} = \boldsymbol{\theta}^{n+\frac{1}{2}}$ with probability

$$1 \wedge \frac{\pi(\boldsymbol{\theta}^{n+\frac{1}{2}})}{\pi(\boldsymbol{\theta}^n)} \frac{q_{\mathrm{R}}(\boldsymbol{\theta}^{n+\frac{1}{2}}, \boldsymbol{\theta}_{\mathrm{R}}^n)}{q_{\mathrm{R}}(\boldsymbol{\theta}^n, \boldsymbol{\theta}_{\mathrm{R}}^{n+\frac{1}{2}})} \frac{q_{\mathrm{O}}(\boldsymbol{\theta}^{n+\frac{1}{2}}, \boldsymbol{\theta}_{\mathrm{O}}^n)}{q_{\mathrm{O}}(\boldsymbol{\theta}^n, \boldsymbol{\theta}_{\mathrm{O}}^{n+\frac{1}{2}})},$$

$q_{\mathrm{R}/\mathrm{O}}$: Gaussian kernel stemming from nonsymmetric proposal

and $\boldsymbol{\theta}^{n+1} = \boldsymbol{\theta}^n$ otherwise.

Algorithm 1: Proximal-Gradient dual: `PGdual Invert` and `PGdual Ortho`

Gaussian proposal: $\boldsymbol{\theta}^{n+\frac{1}{2}} = \mu(\boldsymbol{\theta}^n) + \sqrt{2\gamma}\Gamma\xi^{n+1}$

- random walks: $\mu(\boldsymbol{\theta}) = \boldsymbol{\theta}$

  `RW`: $\Gamma = \mathsf{I}$; `RW Invert`: $\Gamma = \overline{\mathbf{D}}_2^{-1}\overline{\mathbf{D}}_2^{-\top}$; `RW Ortho`: $\Gamma = \overline{\mathbf{D}}_o^{-1}\overline{\mathbf{D}}_o^{-\top}$

- Proximal-Gradient dual: $\mu_{\mathsf{R}}(\boldsymbol{\theta})$, $\mu_{\mathsf{O}}(\boldsymbol{\theta})$, $\Gamma = \overline{\mathbf{D}}^{-1}\overline{\mathbf{D}}^{-\top}$

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Practical settings: $N_{\max} = 10^7$ iterations, 15 independent runs

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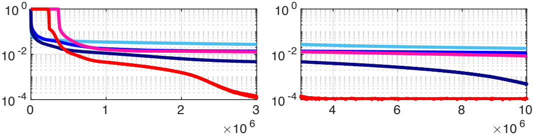
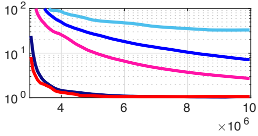
log-density

burn in | post burn in



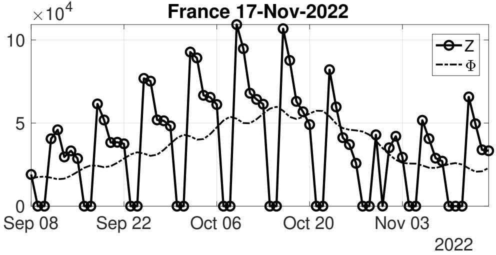
$\left(\ln \pi(\boldsymbol{\theta}^n) - \max \ln \pi\right) / \left(\ln \pi(\boldsymbol{\theta}^0) - \max \ln \pi\right)$

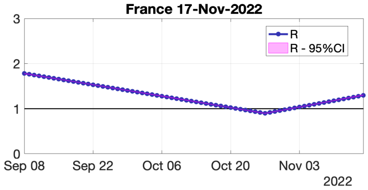
Gelman-Rubin

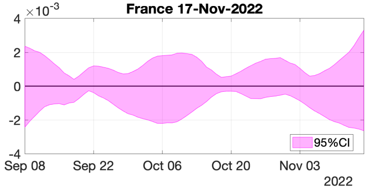


ANOVA-type criterion

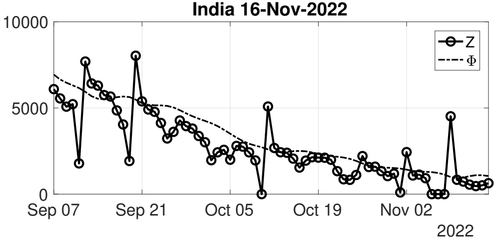
Sanitary situation in France

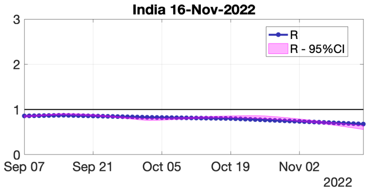


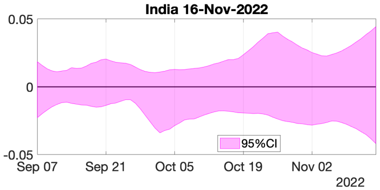




Worldwide Covid19 monitoring







Why not United Kingdom?

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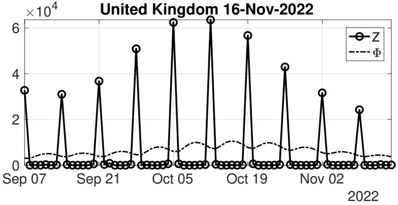
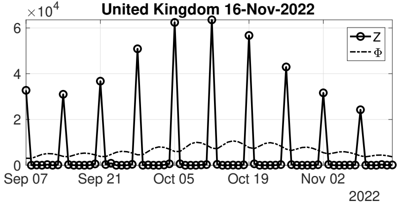


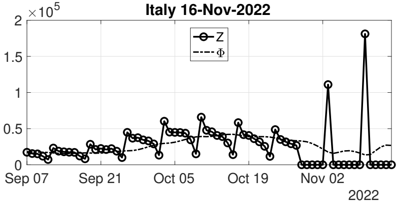
rate of erroneous counts: $6/7$!

PGdual credibility interval estimation of the reproduction number

Why not United Kingdom?

And Italy?





rate of erroneous counts: $6/7$!

seems to adopt the same reporting rate ...

$\Longrightarrow$ call for new tools, robust to very scarce data

Conclusion

✓ Extended Cori model handling erroneous reported counts via a latent variable

$$\mathrm{Z}_t | \mathbf{Z}_{t-\tau_\Phi:t-1}, \mathrm{R}_t, \mathrm{O}_t \sim \mathrm{Poiss}(\mathrm{R}_t \Phi_t + \mathrm{O}_t)$$

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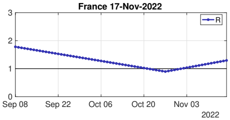
✓ Estimation of piecewise linear R_t and corrected counts via convex optimization

$$\underset{(\mathbf{R},\mathbf{O}) \in \mathbb{R}_+^T \times \mathbb{R}^T}{\mathrm{minimize}} \sum_{t=1}^{T} \mathrm{d}_{\mathrm{KL}} \left(\mathrm{Z}_t \,|\, \mathrm{R}_t \Phi_t + \mathrm{O}_t \right) + \lambda_{\mathrm{R}} \|\mathbf{D}_2 \mathbf{R}\|_1 + \iota_{\geq 0}(\mathbf{R}) + \lambda_{\mathrm{O}} \|\mathbf{O}\|_1$$



$\widehat{\mathrm{R}}_T = 1.1959$

(Pascal et al., 2022, *Trans. Sig. Process.*;)

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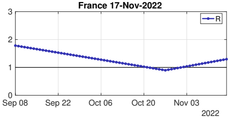
✓ Bayesian credibility interval estimates via proximal Langevin MCMC samplers



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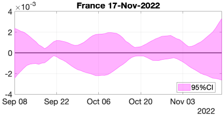


$\widehat{\mathrm{R}}_T \in [1.1978, 1.2016]$

(Pascal et al., 2022, *Trans. Sig. Process.*; Fort et al., 2022, *arXiv:2203.09142*)

$\longrightarrow$ Avoid mixing errors O_t with the pandemic mechanism $R_t\Phi_t$: anomaly models

$$Z_t|\mathbf{Z}_{t-\tau_\Phi:t-1}, R_t, O_t \sim \text{Poiss}((1-e_t)R_t\Phi_t + e_t O_t), \quad e_t \in \{0,1\}$$

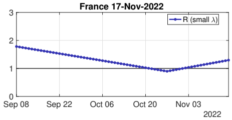
Perspectives

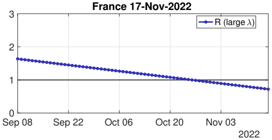
⟶ Avoid mixing errors O_t with the pandemic mechanism $\mathsf{R}_t\Phi_t$: anomaly models

$$\mathsf{Z}_t|\mathbf{Z}_{t-\tau_\Phi:t-1}, \mathsf{R}_t, \mathsf{O}_t \sim \mathrm{Poiss}((1-\mathsf{e}_t)\mathsf{R}_t\Phi_t + \mathsf{e}_t\mathsf{O}_t), \quad \mathsf{e}_t \in \{0,1\}$$

⟶ Selection of regularization parameters λ_R, λ_O

$$\underset{(\mathbf{R},\mathbf{O})\in\mathbb{R}_+^T\times\mathbb{R}^T}{\text{minimize}} \sum_{t=1}^{T} \mathsf{d}_{\mathsf{KL}}\left(\mathsf{Z}_t \,|\, \mathsf{R}_t\Phi_t + \mathsf{O}_t\right) + \lambda_\mathsf{R}\|\mathbf{D}_2\mathbf{R}\|_1 + \iota_{\geq 0}(\mathbf{R}) + \lambda_\mathsf{O}\|\mathbf{O}\|_1$$





Juliana Du PhD thesis

⟶ Synthetic data