Laboratoire des Sciences du Numérique de Nantes

Nantes Université

cnrs

The Kravchuk transform:

A novel covariant representation for discrete signals amenable to zero-based detection tests

November 18, 2026

Barbara Pascal, Rémi Bardenet, Julien Flamant

SOFTA Workshop

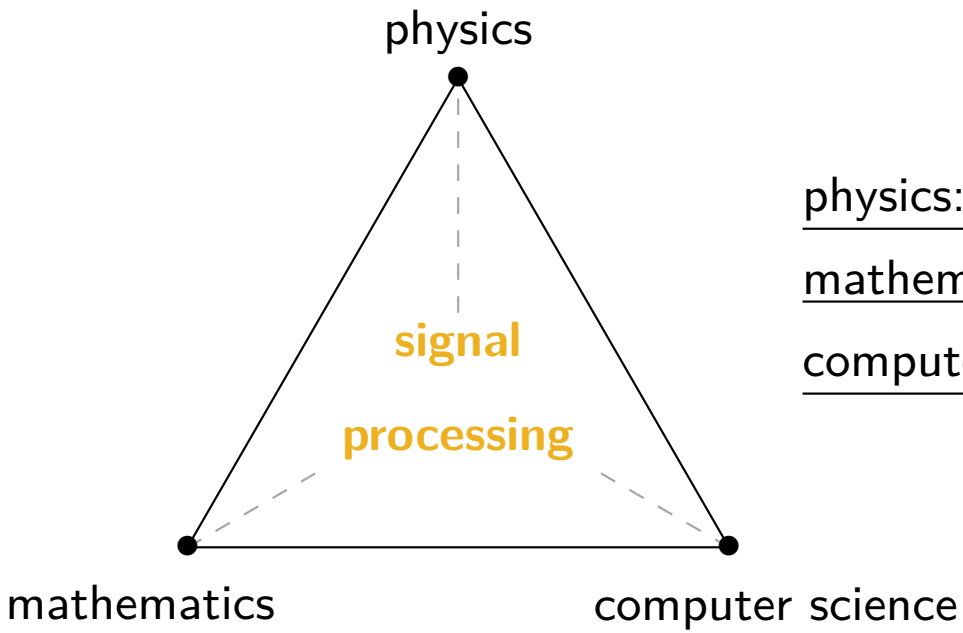
Centre International de Rencontres Mathématiques (CIRM), Marseille

Signal processing aims to extract **information** from **data**.

Data of very diverse types:

- measurements of a physical quantity,
- biological or epidemiological indicators,
- data produced by human activities.

The Golden triangle of signal processing



physics: modeling of phenomena

mathematics: formalization & evaluation

computer science: efficient implementation

inspired from P. Flandrin

Outline of the presentation

- Signal detection: the role of representations

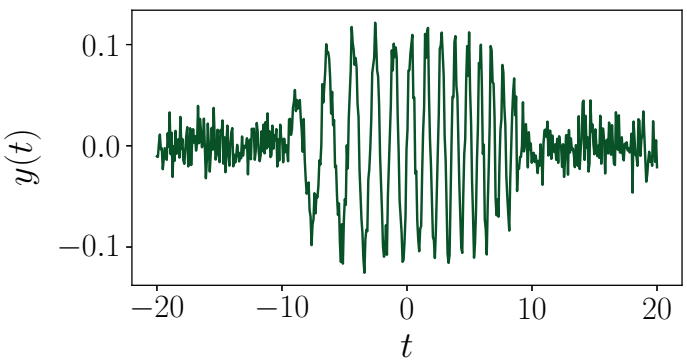
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- Covariance principle and stationary point processes

- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

- Bridging Kravchuk transform and quantum momentum theory

Time and frequency: two dual descriptions of temporal signals

A continuous finite energy **signal** is a function of time $y(t)$ with $y \in L^2_{\mathbb{C}}(\mathbb{R})$.



- electrical cardiac activity,
- audio recording,
- seismic activity,
- light intensity on a photosensor
- ...

Information of interest:

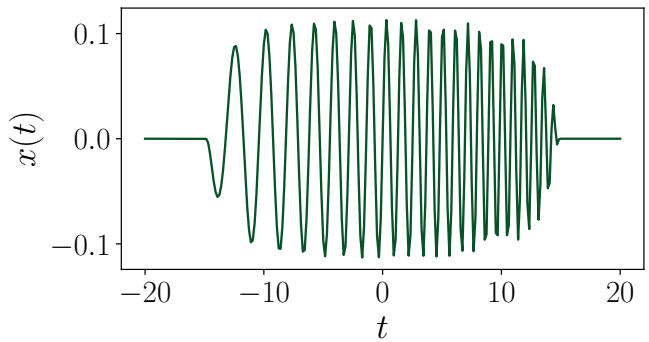
- time events, e.g., an earthquake and its replica
- frequency content, e.g., monitoring of the heart beating rate

time	**frequency**
ever-changing world	waves, oscillations, rhythms
marker of events and evolutions	intrinsic mechanisms

Signal-plus-noise observation model

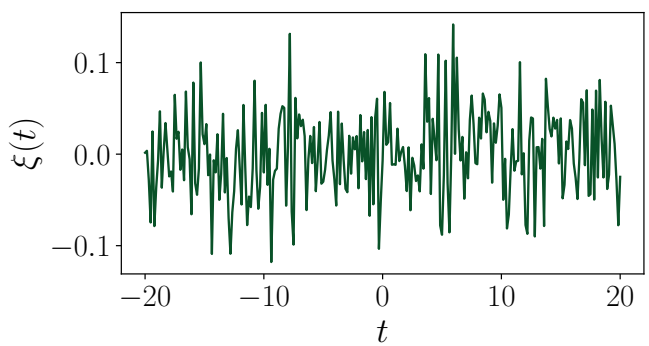
A **chirp** is a transient waveform modulated in amplitude and frequency:

$$x(t) = A_\nu(t) \sin\left(2\pi\left(f_1 + (f_2 - f_1)\frac{t+\nu}{2\nu}\right)t\right)$$



White noise is a random variable $\xi(t)$ such that

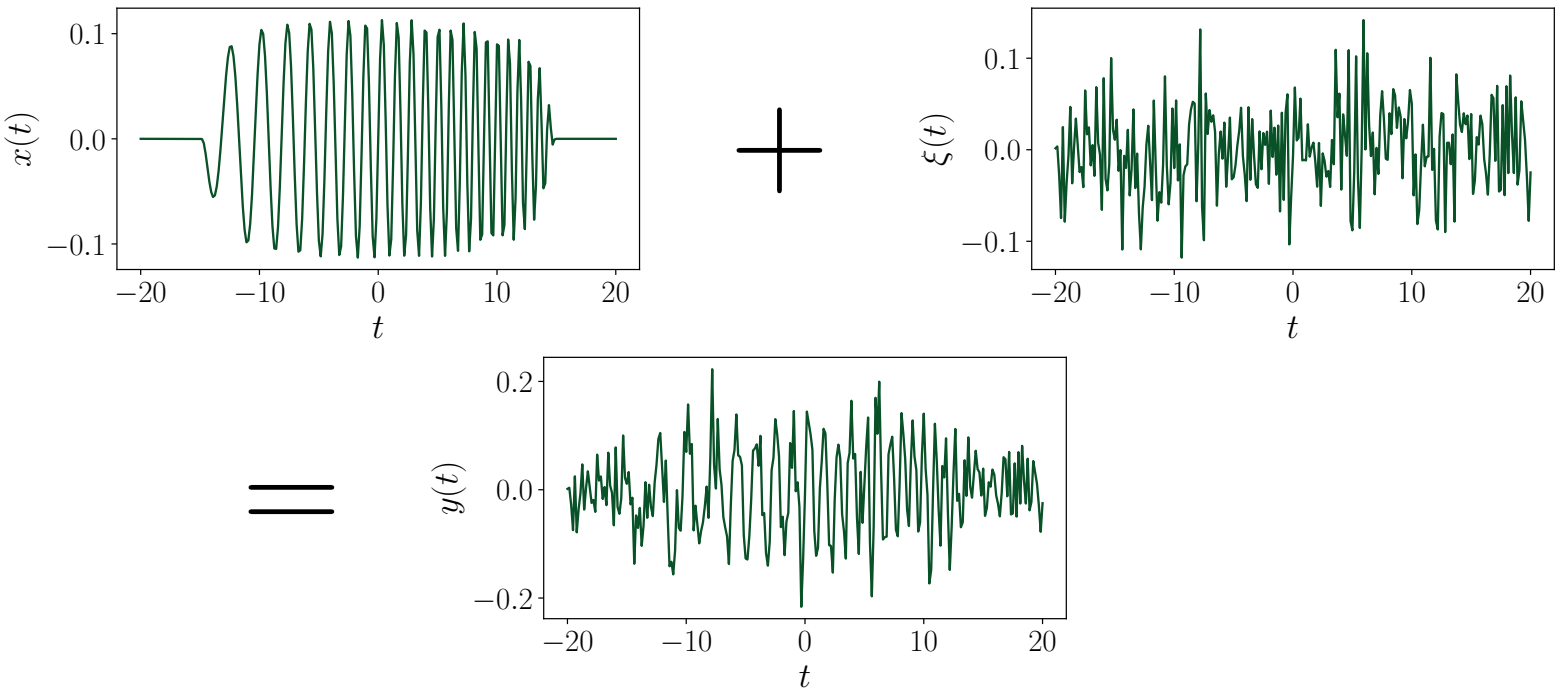
$$\mathbb{E}[\xi(t)] = 0 \quad \text{and} \quad \mathbb{E}[\overline{\xi(t)}\xi(t')] = \delta(t - t')$$



P. Flandrin: 'A signal is characterized by a structured organization.'

Signal-plus-noise observation model

Noisy observations $y(t) = \text{snr} \times x(t) + \xi(t)$

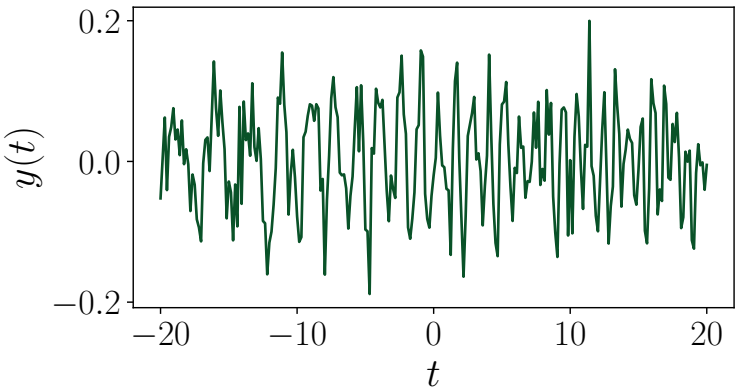


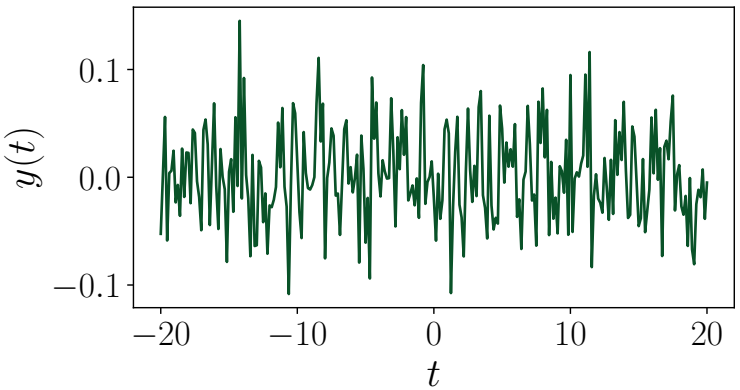
Signal processing task:

Given an observation $y(t)$

detection decides whether there is an underlying signal or only noise.

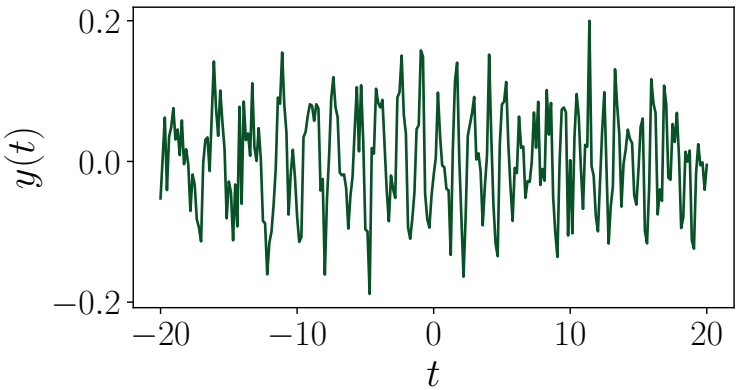
Direct observation

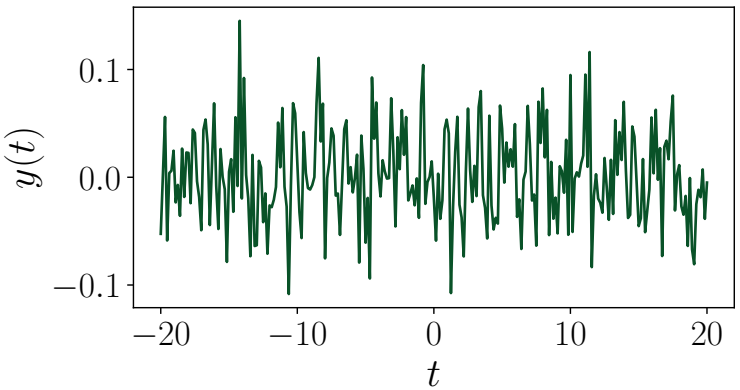




✗ hard to distinguish between oscillations and fluctuations

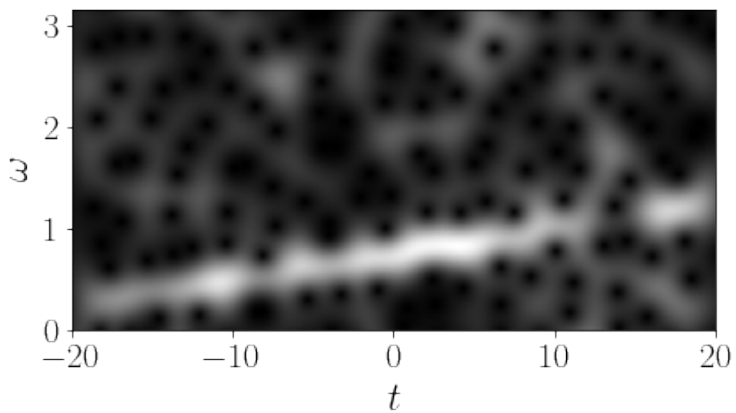
Direct observation

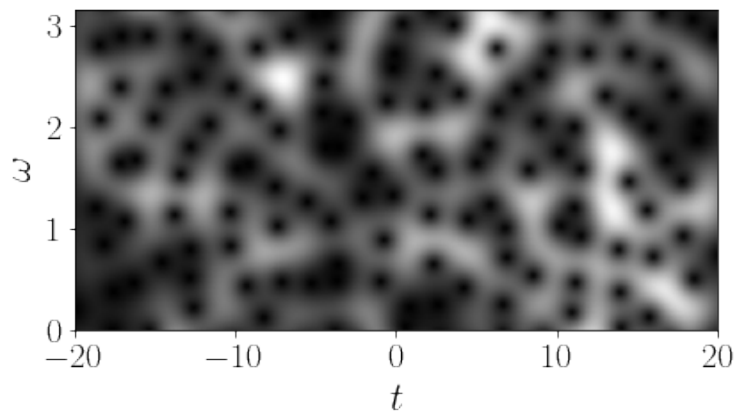




✗ hard to distinguish between oscillations and fluctuations

Time-frequency representation





✓ lines of local maxima: structures are simpler to capture

Outline of the presentation

- Signal detection: the role of representations

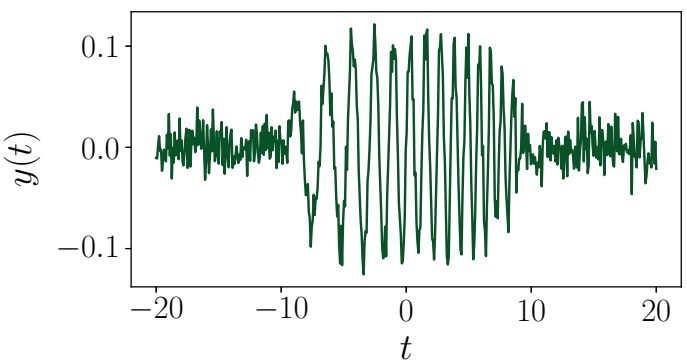
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- Covariance principle and stationary point processes

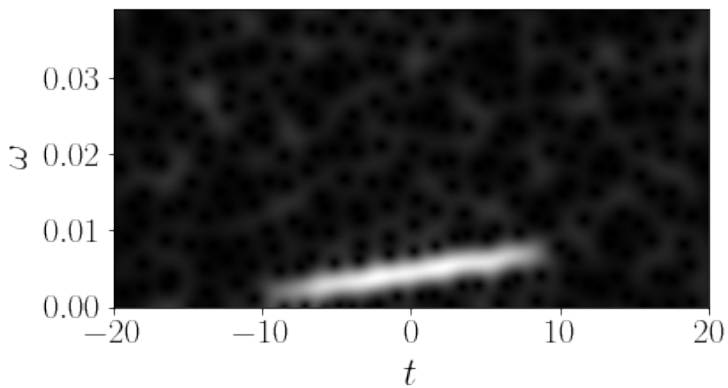
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

- Bridging Kravchuk transform and quantum momentum theory

Time *and* frequency Short-Time Fourier Transform with window h:

$$V_h y(t,\omega) \triangleq \int_{-\infty}^{\infty} \overline{y(u)} h(u-t) \exp(-\mathrm{i}\omega u)\,\mathrm{d}u$$



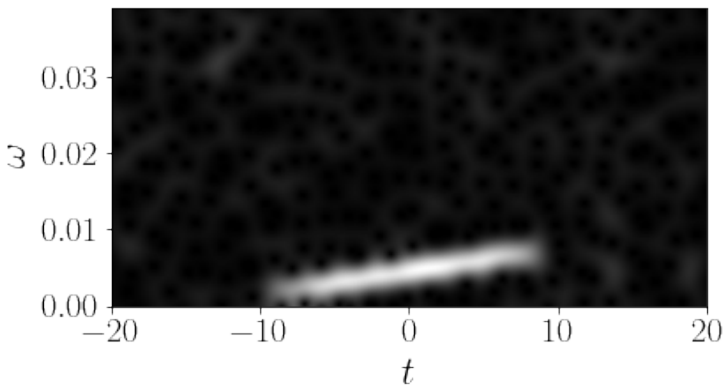


Energy density interpretation $S_h y(t,\omega) = |V_h y(t,\omega)|^2$ the *spectrogram*

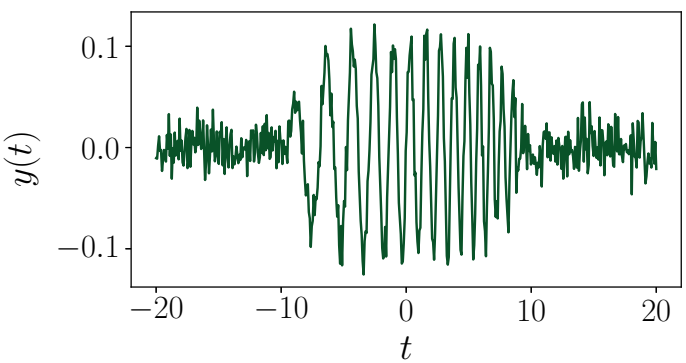
$$\int \int_{-\infty}^{+\infty} S_h y(t,\omega)\,\mathrm{d}t \frac{\mathrm{d}\omega}{2\pi} = \int_{-\infty}^{+\infty} |x(t)|^2\,\mathrm{d}t \quad \text{if} \quad \|h\|_2^2 = 1$$

Signal, i.e., information of interest: regions of **maximal** energy.

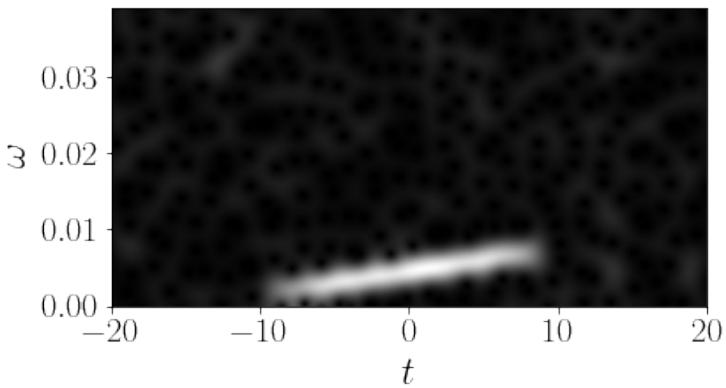
spectrogram



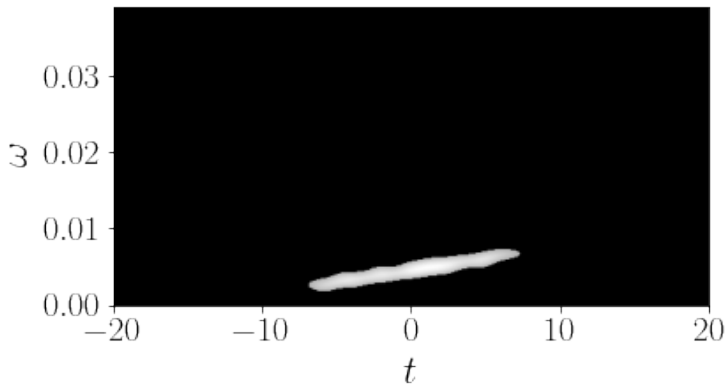
noisy observation



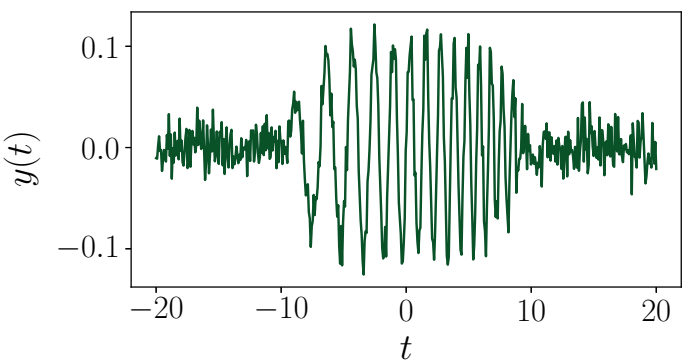
spectrogram



only maxima

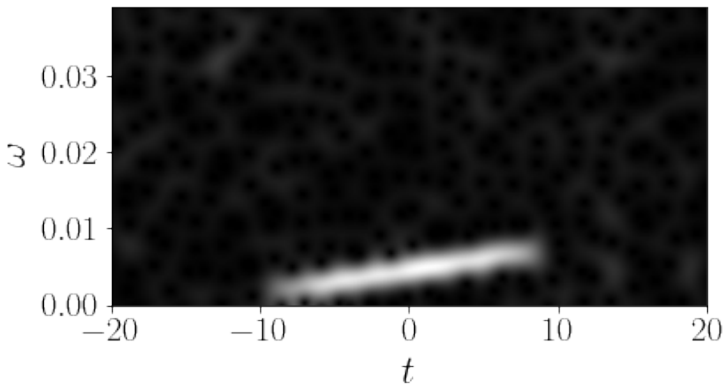


noisy observation

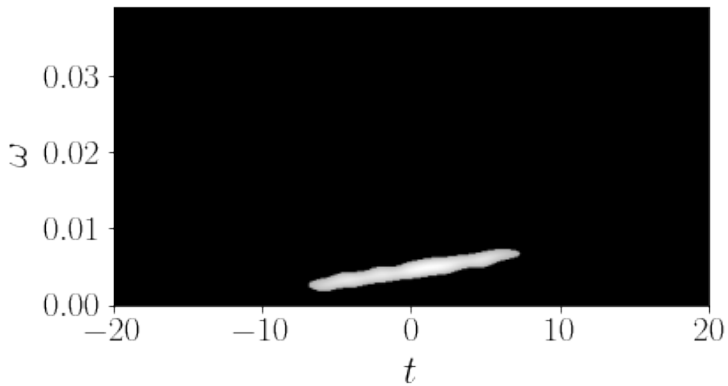


Inversion formula $$y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u,\omega)} h(t-u) \exp(\mathrm{i}\omega u) \, \mathrm{d}u \frac{\mathrm{d}\omega}{2\pi}$$

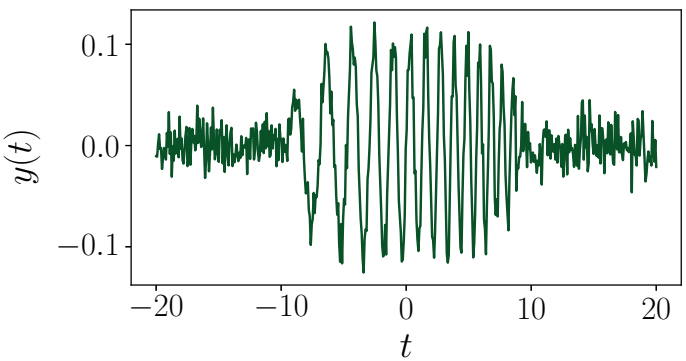
spectrogram



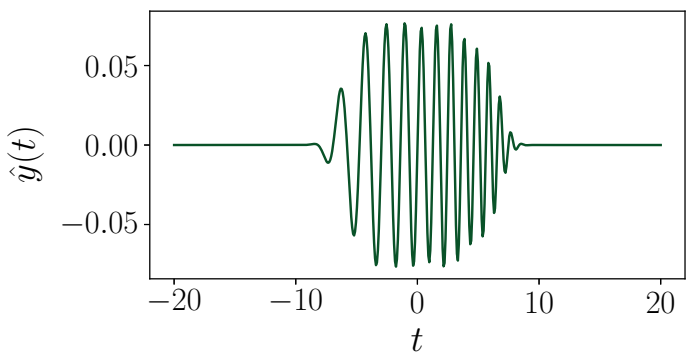
only maxima



noisy observation

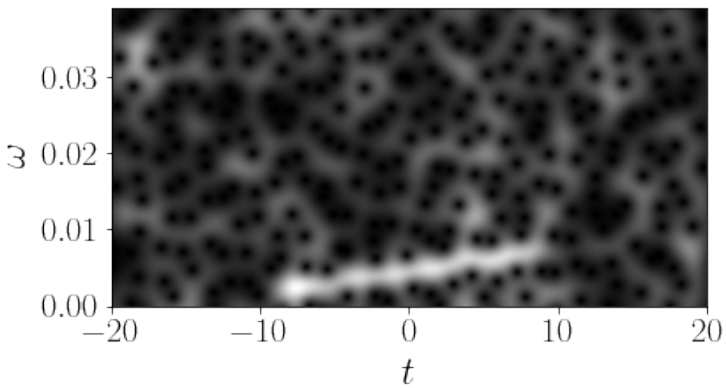


estimate

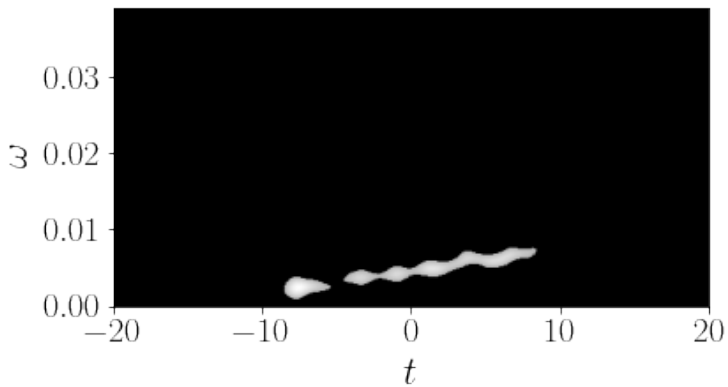


Inversion formula $$y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u,\omega)} h(t-u) \exp(\mathrm{i}\omega u) \, \mathrm{d}u \frac{\mathrm{d}\omega}{2\pi}$$

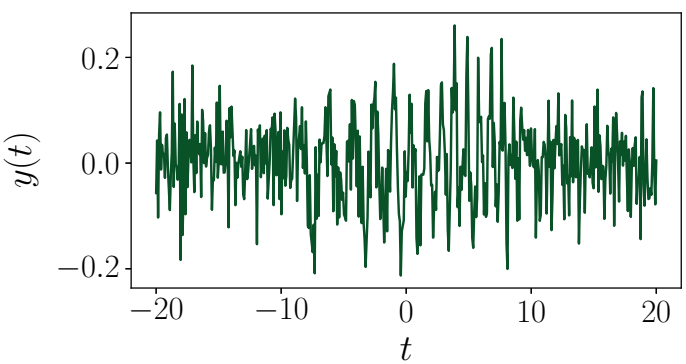
spectrogram



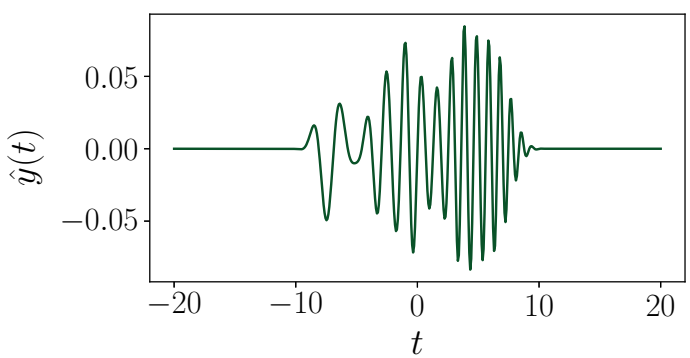
only maxima



noisy observation

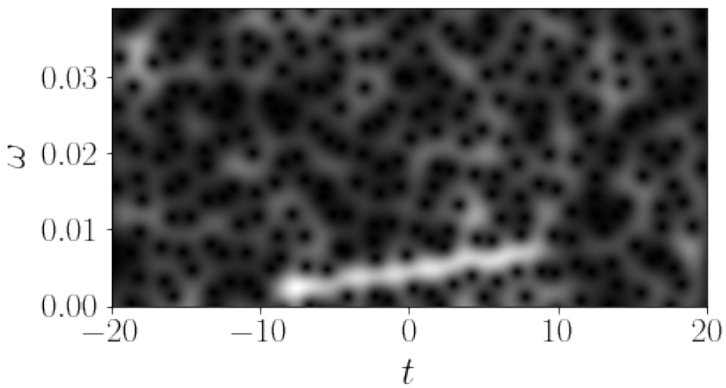


estimate

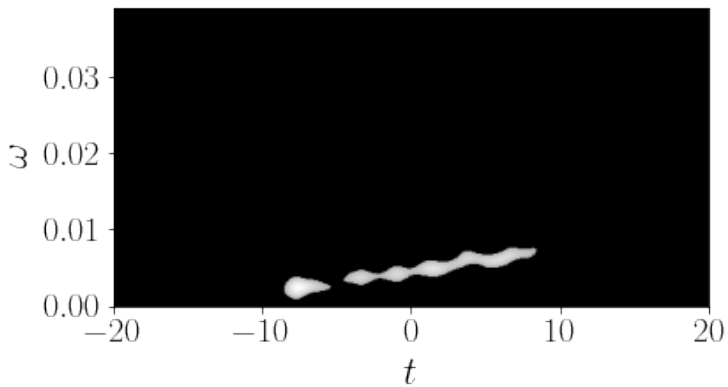


Inversion formula $$y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u,\omega)} h(t-u) \exp(\mathrm{i}\omega u) \, \mathrm{d}u \frac{\mathrm{d}\omega}{2\pi}$$

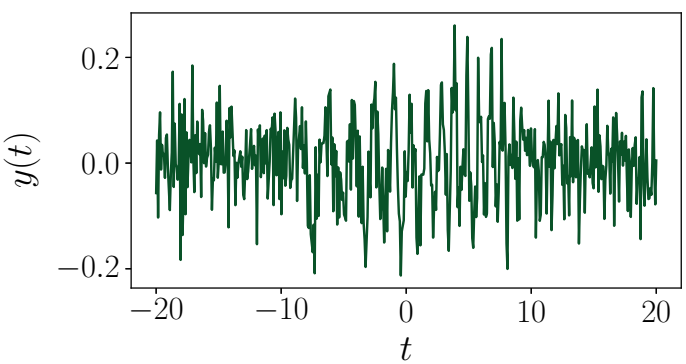
spectrogram



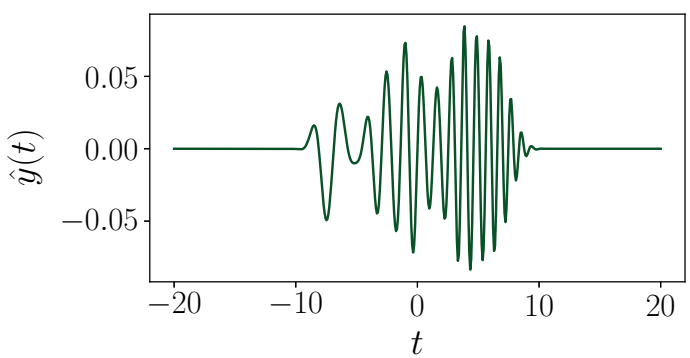
only maxima



noisy observation



estimate



Maxima extraction: *reassignment, synchrosqueezing, ridge extraction*

(Meignen et al., 2017)

Outline of the presentation

- Signal detection: the role of representations

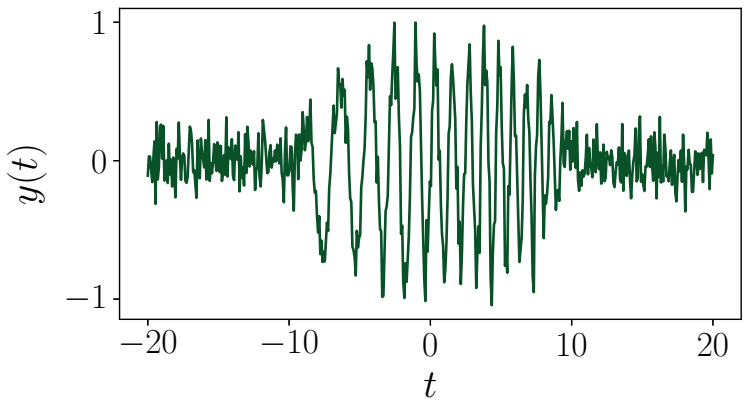
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- Covariance principle and stationary point processes

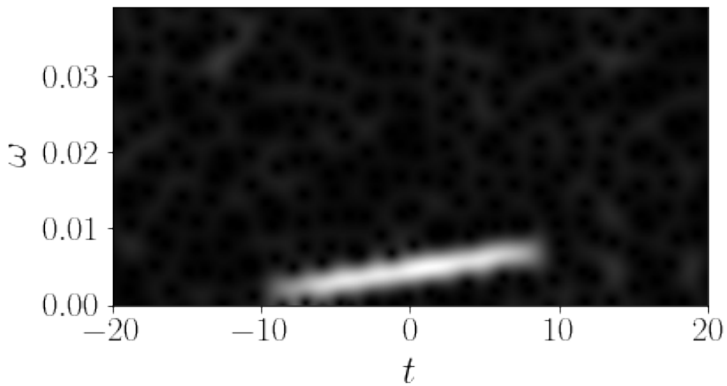
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

- Bridging Kravchuk transform and quantum momentum theory

Restriction to the **circular Gaussian window**: $g(t) = \pi^{-1/4} \mathrm{e}^{-t^2/2}$

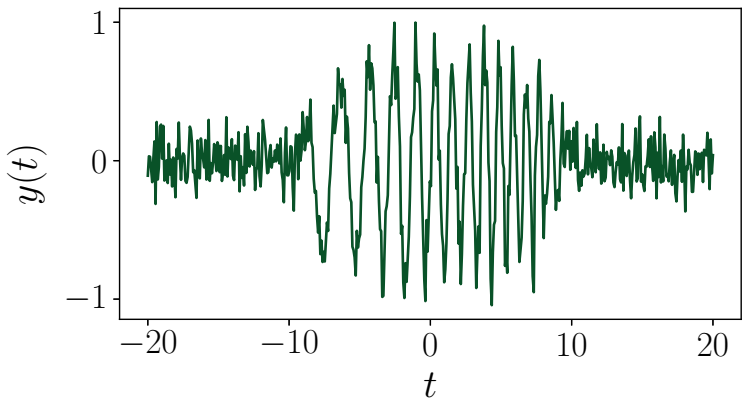
Look for the **zeros**, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.

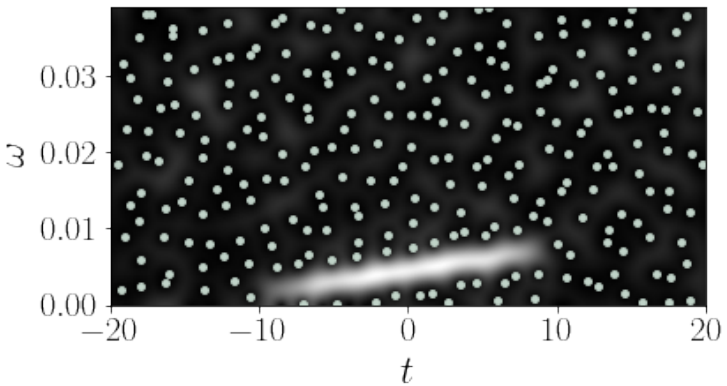




Restriction to the **circular Gaussian window**: $g(t) = \pi^{-1/4} \mathrm{e}^{-t^2/2}$

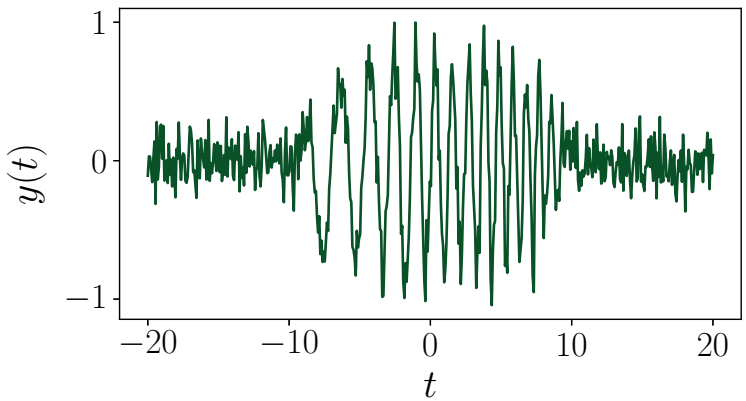
Look for the **zeros**, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.

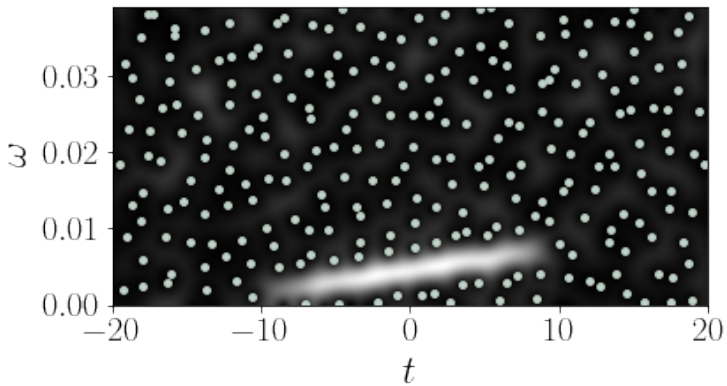




Restriction to the **circular Gaussian window**: $g(t) = \pi^{-1/4} \mathrm{e}^{-t^2/2}$

Look for the **zeros**, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.



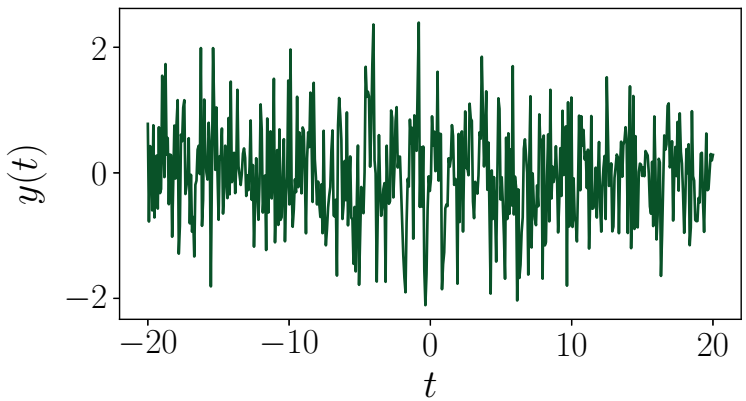


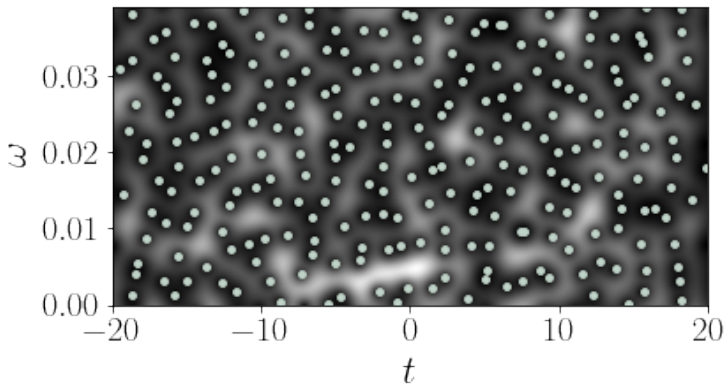
Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

Restriction to the **circular Gaussian window**: $g(t) = \pi^{-1/4} \mathrm{e}^{-t^2/2}$

Look for the **zeros**, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.



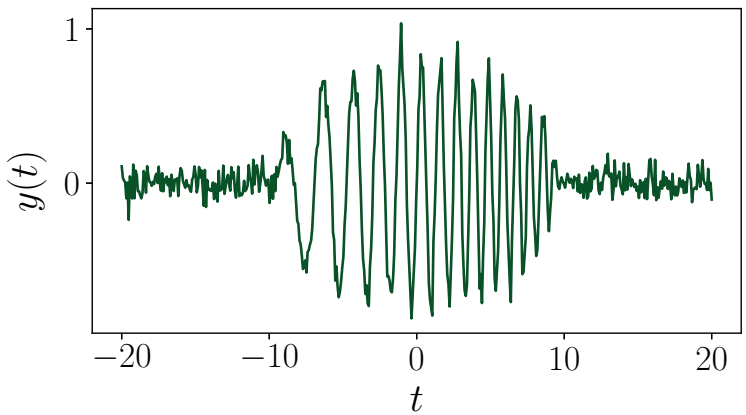


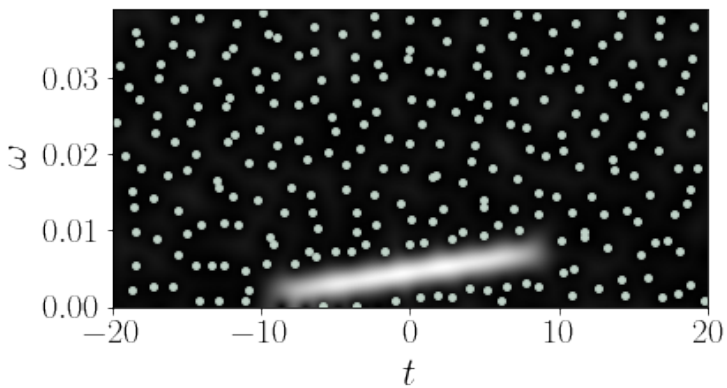
Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

Restriction to the **circular Gaussian window**: $g(t) = \pi^{-1/4} \mathrm{e}^{-t^2/2}$

Look for the **zeros**, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.



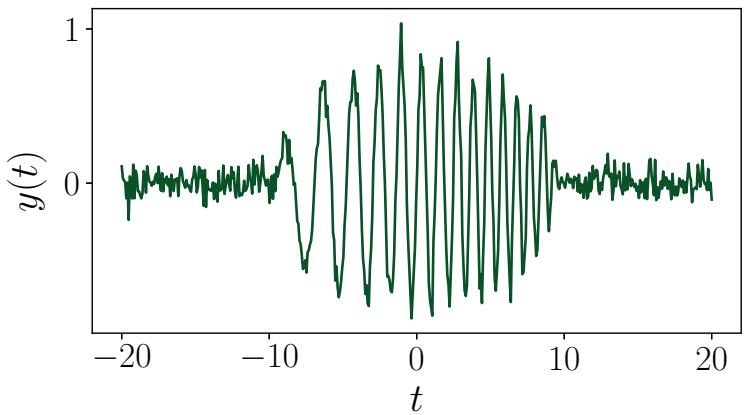


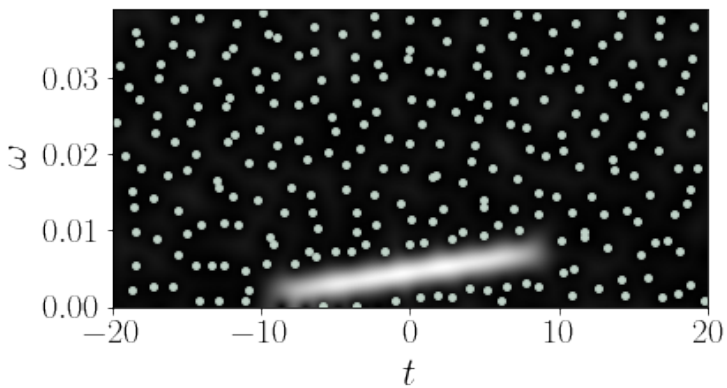
Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

Restriction to the **circular Gaussian window**: $g(t) = \pi^{-1/4} \mathrm{e}^{-t^2/2}$

Look for the **zeros**, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.



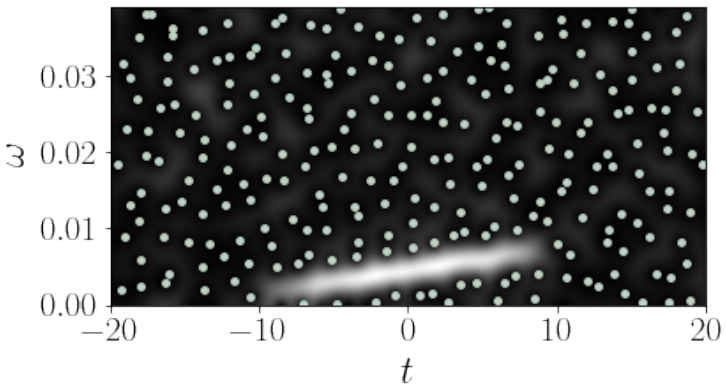


Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

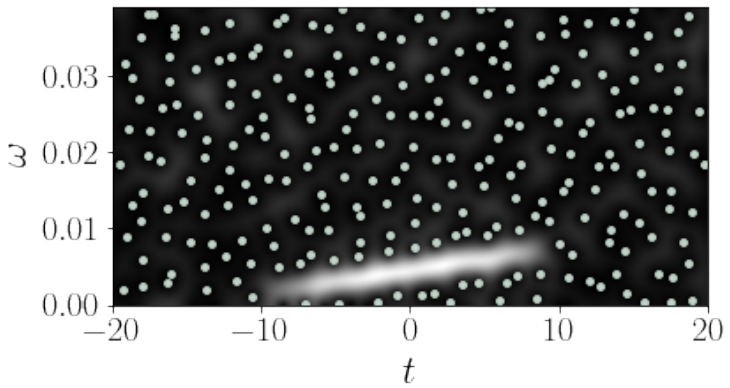
- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

What can be said theoretically about the zeros of the spectrogram?

Idea assimilate the time-frequency plane with $\mathbb{C}$ through $z = (\omega + \mathrm{i}t)/\sqrt{2}$



Idea assimilate the time-frequency plane with $\mathbb{C}$ through $z = (\omega + \mathrm{i}t)/\sqrt{2}$



Bargmann factorization

$$V_g y(t,\omega) = \mathrm{e}^{-|z|^2/2}\mathrm{e}^{-\mathrm{i}\omega t/2} By(z)$$

g the circular Gaussian window

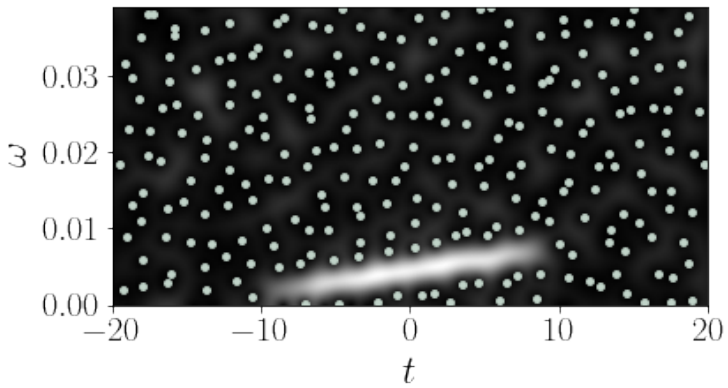
Bargmann transform of the signal y

$$By(z) \triangleq \pi^{-1/4}\mathrm{e}^{-z^2/2} \int_{\mathbb{R}} \overline{y(u)} \exp\left(\sqrt{2}uz - u^2/2\right) \mathrm{d}u,$$

By is an **entire** function, almost characterized by its infinitely many zeros:

$$By(z) = z^m \mathrm{e}^{C_0 + C_1 z + C_2 z^2} \prod_{n\in\mathbb{N}} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2\right).$$

Idea assimilate the time-frequency plane with $\mathbb{C}$ through $z = (\omega + \mathrm{i}t)/\sqrt{2}$



Bargmann factorization

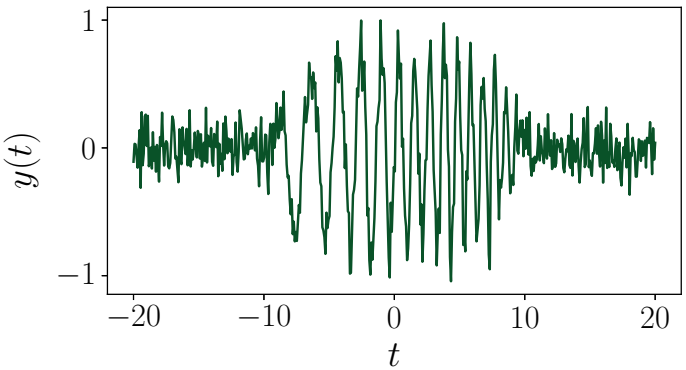
$$V_g y(t,\omega) = \mathrm{e}^{-|z|^2/2}\mathrm{e}^{-\mathrm{i}\omega t/2} By(z)$$

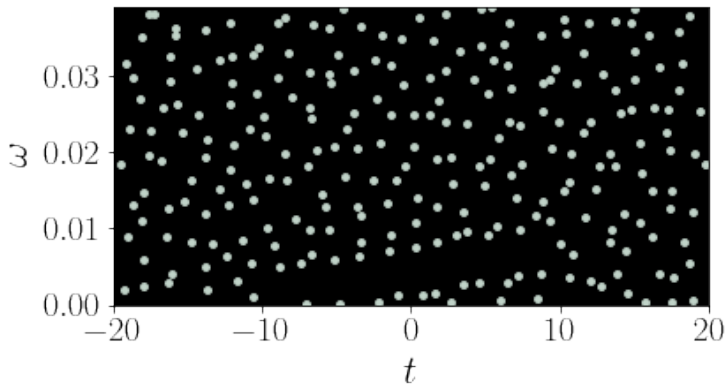
g the circular Gaussian window

Theorem The zeros of the Gaussian spectrogram $V_g y(t,\omega)$

- coincide with the zeros of the **entire** function By,
- hence are **isolated** and constitute a **Point Process**,
- which almost completely **characterizes** the spectrogram.

(Flandrin, 2015)





Advantages of working with the zeros

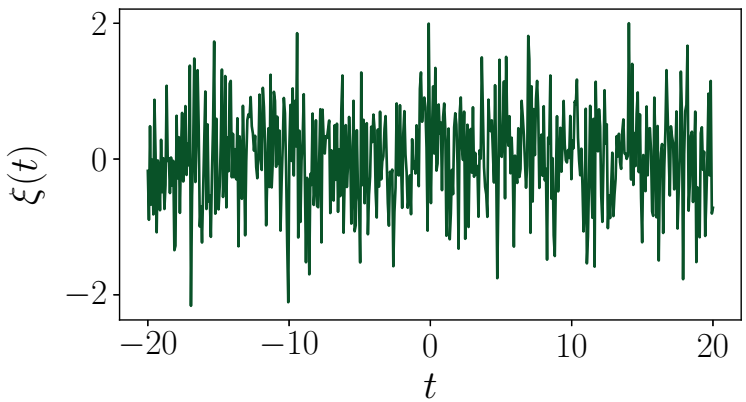
- Easy to find compared to relative maxima.
- Form a robust pattern in the time-frequency plane.
- Require little memory space for storage.
- Efficient tools were recently developed in **stochastic geometry**.

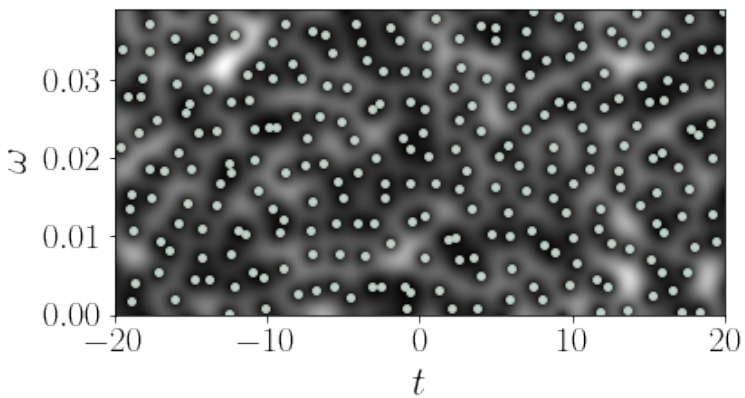
Signal detection based on the spectrogram zeros

(Bardenet, Flamant & Chainais, 2020)

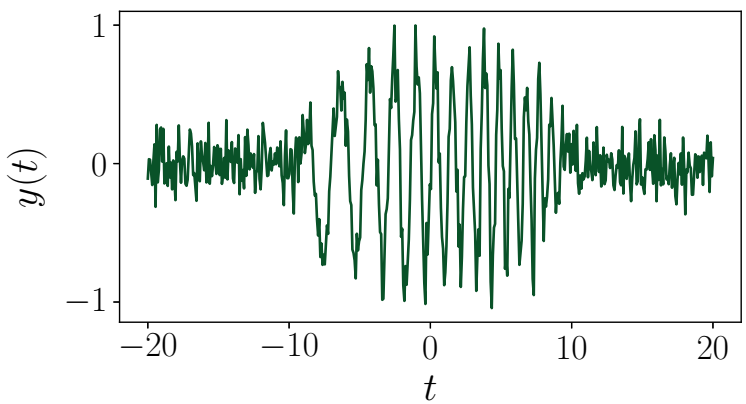
- $\mathbf{H}_0$ white noisy only, i.e., $y(t) = \xi(t)$
- $\mathbf{H}_1$ presence of a signal, i.e., $y(t) = \text{snr} \times x(t) + \xi(t), \quad \text{snr} > 0$

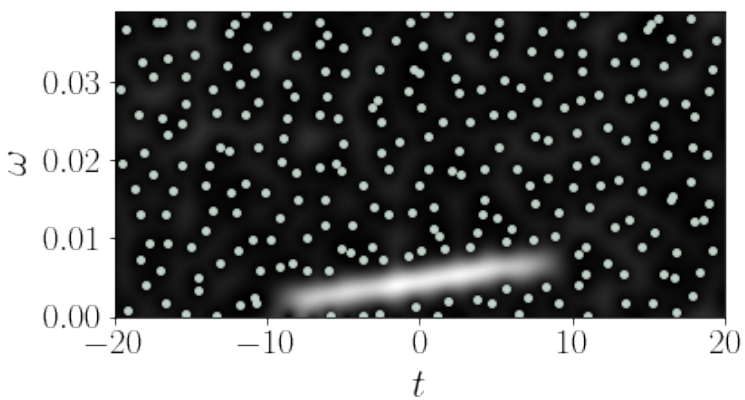
null hypothesis





alternative hypothesis



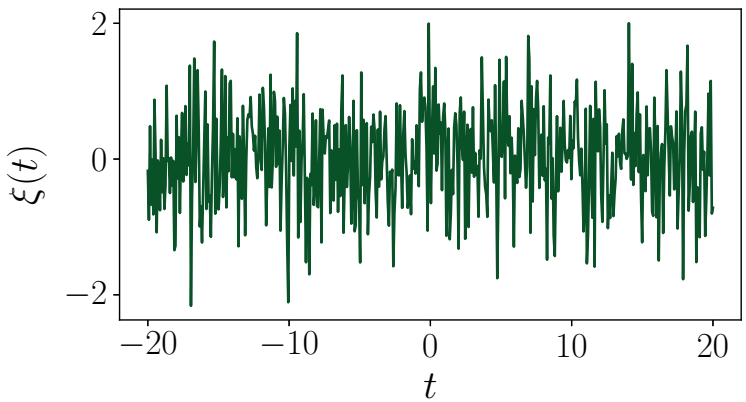


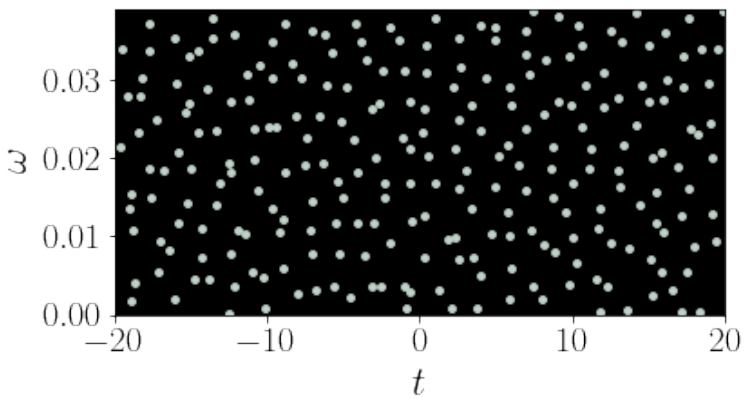
Signal detection based on the spectrogram zeros

(Bardenet, Flamant & Chainais, 2020)

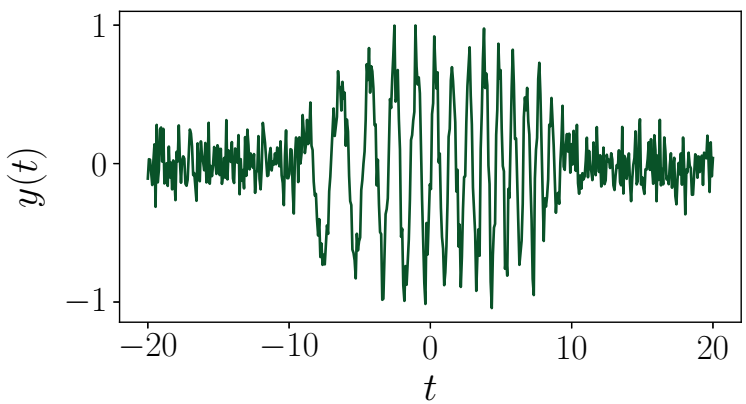
- $\mathbf{H}_0$ white noisy only, i.e., $y(t) = \xi(t)$
- $\mathbf{H}_1$ presence of a signal, i.e., $y(t) = \text{snr} \times x(t) + \xi(t), \quad \text{snr} > 0$

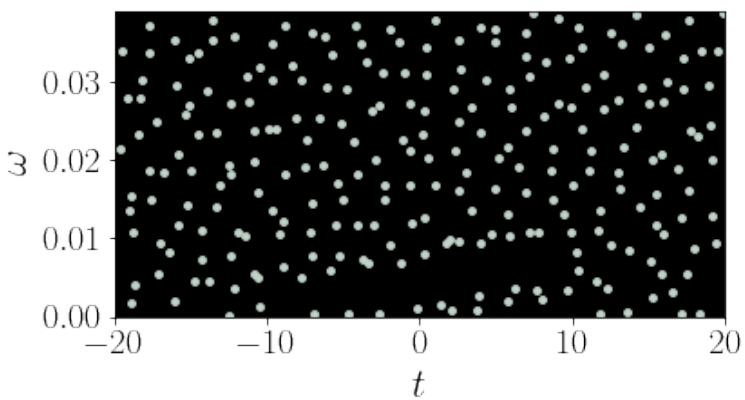
null hypothesis





alternative hypothesis

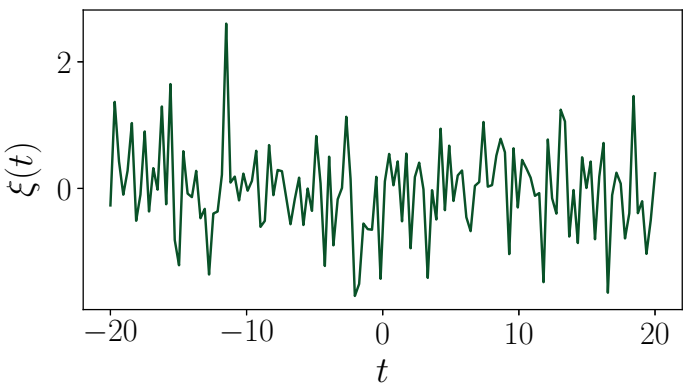


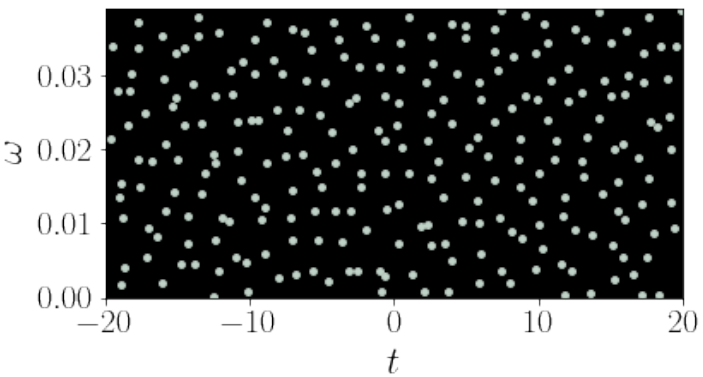


Continuous complex white Gaussian noise

(Bardenet et al., 2020), (Bardenet & Hardy, 2021)

$$\xi(t) = \sum_{n=0}^{\infty} \xi[n] h_n(t), \ \xi[n] \sim \mathcal{N}_{\mathbb{C}}(0,1), \quad \{h_n, k = 0, 1, \ldots\} \text{ Hermite functions}$$

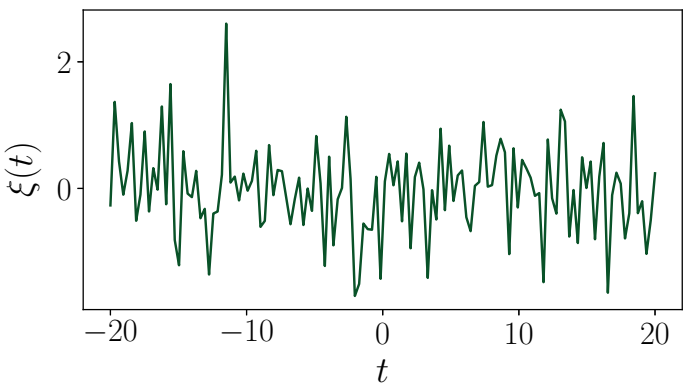


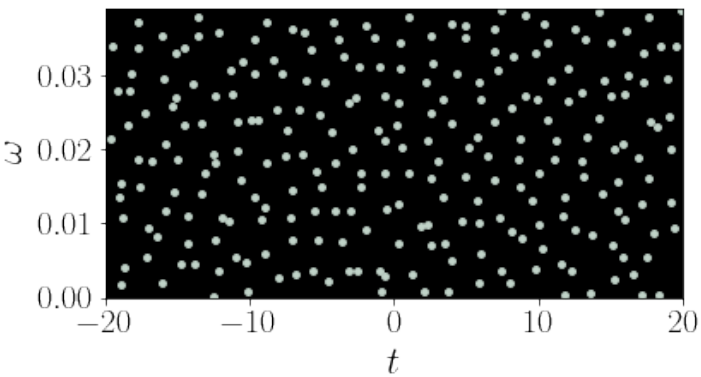


Continuous complex white Gaussian noise

(Bardenet et al., 2020), (Bardenet & Hardy, 2021)

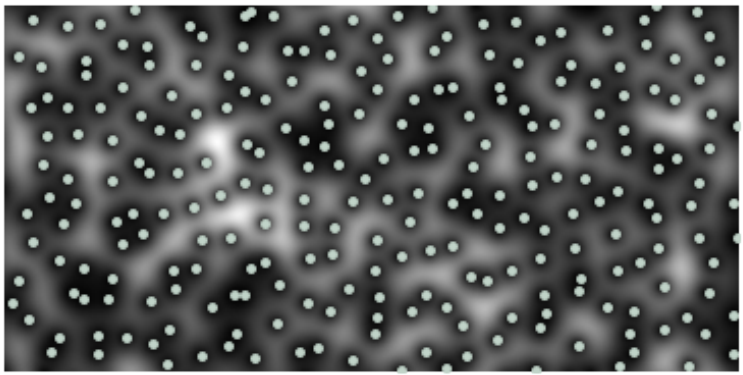
$$\xi(t) = \sum_{n=0}^{\infty} \xi[n] h_n(t), \ \xi[n] \sim \mathcal{N}_{\mathbb{C}}(0,1), \quad \{h_n, k = 0, 1, \ldots\} \text{ Hermite functions}$$





Theorem $V_g\xi(t,\omega) = \mathrm{e}^{-|z|^2/4}\mathrm{e}^{-\mathrm{i}\omega t/2}\,\mathrm{GAF}_{\mathbb{C}}(z)$ (Bardenet & Hardy, 2021)

$\mathrm{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$ the *planar Gaussian Analytic Function* and $z = \frac{\omega + \mathrm{i}t}{\sqrt{2}}$.



$$V_g\xi(t,\omega) \overset{\text{non-vanishing}}{\propto} \mathrm{GAF}_{\mathbb{C}}(z)$$

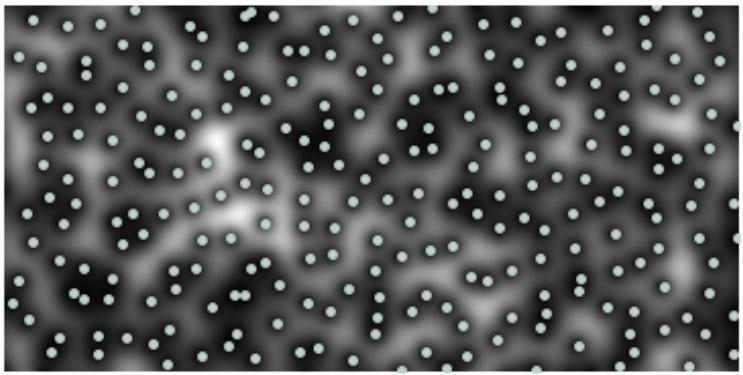
$$z = (\omega + \mathrm{i}t)/\sqrt{2}$$

Zeros of $\mathrm{GAF}_{\mathbb{C}}$: random set of points forming a **Point Process** characterized by a probability distribution on point configurations

Properties of the Point Process of the zeros of $\mathrm{GAF}_{\mathbb{C}}$:

- invariant under the isometries of $\mathbb{C}$, i.e., **stationary**,
- has a uniform density $\rho^{(1)}(z) = \rho^{(1)} = 1/\pi$,
- explicit two-point correlation function $\rho^{(2)}(z, z') = \rho^{(2)}(|z - z'|)$,
- scaling of the *hole probability*: $r^{-4} \log p_r \to -3\mathrm{e}^2/4$, as $r \to \infty$

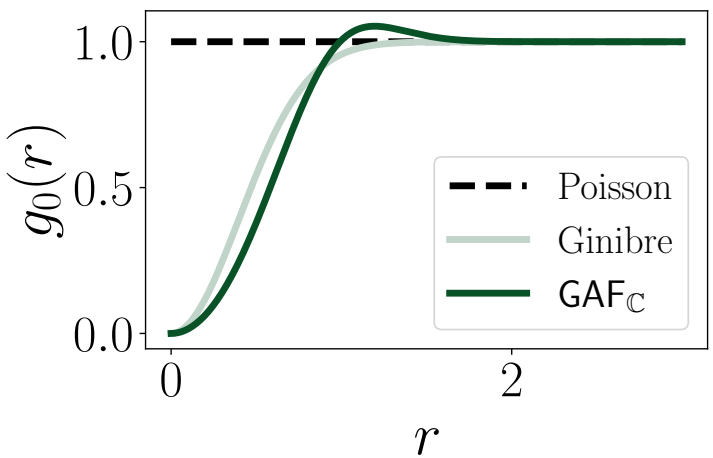
$$p_r = \mathbb{P}(\text{no point in the disk of center 0 and radius } r)$$



$$V_g\xi(t,\omega) \overset{\text{non-vanishing}}{\propto} \mathrm{GAF}_{\mathbb{C}}(z)$$

$$z = (\omega + \mathrm{i}t)/\sqrt{2}$$

Zeros of $\mathrm{GAF}_{\mathbb{C}}$: random set of points forming a **Point Process** characterized by a probability distribution on point configurations



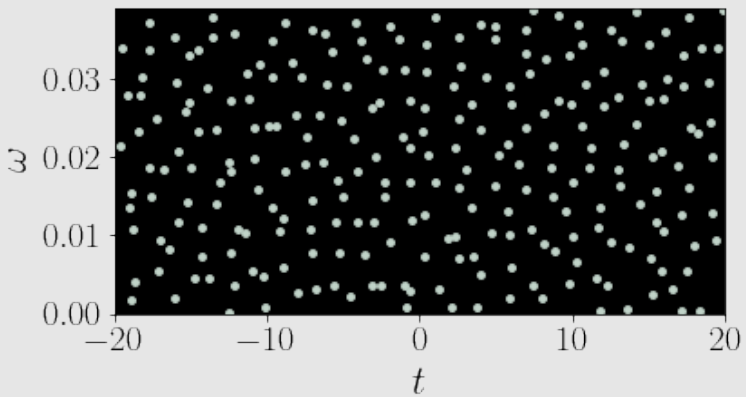
Pair correlation $\rho^{(2)}(z,z')\mathrm{d}z\mathrm{d}z' =$

$$\mathbb{P}\left(1 \text{ point in } B(z,\mathrm{d}z) \text{ and } 1 \text{ in } B(z',\mathrm{d}z')\right)$$

The point process of the zeros of the spectrogram is not **determinantal**.

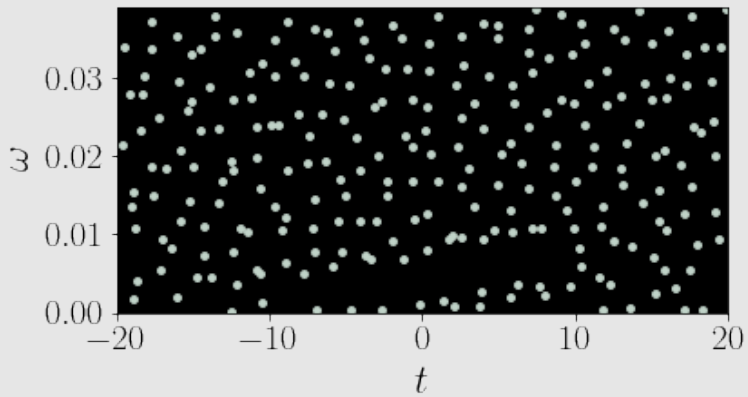
Purpose: summary statistic s, such that for $y = \text{snr} \times x + \xi$

$\mathbb{E}[s(y)|\mathbf{H}_0]$ small



$\text{snr} = 0$

$\mathbb{E}[s(y)|\mathbf{H}_1]$ large

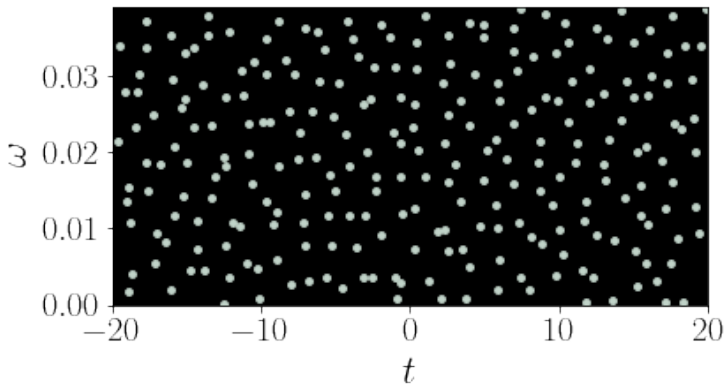


$\text{snr} > 0$

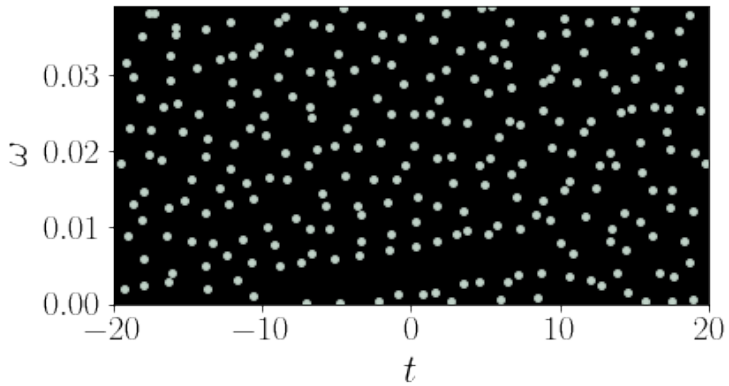
'**Large value of $s(y)$ is a strong indication that there is a signal.**'

Tools from **stochastic geometry** to capture spatial statistics of the zeros.

Unorthodox path: zeros of Gaussian Analytic Functions

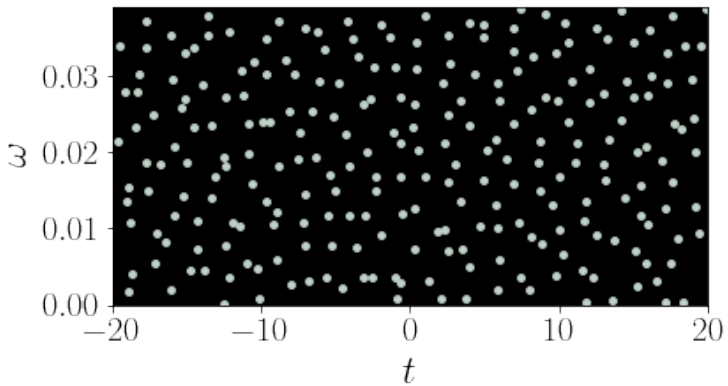


snr $= 0$

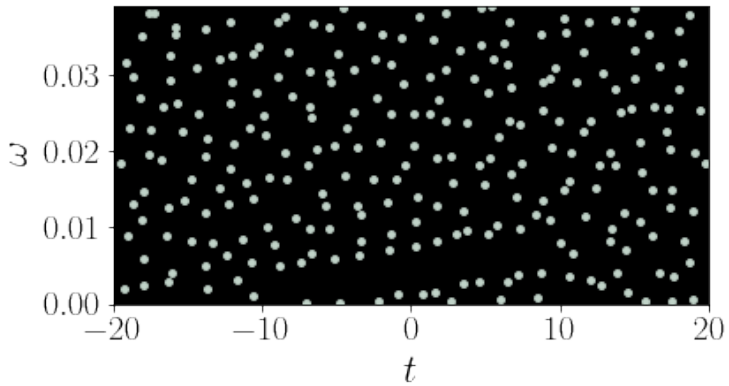


snr > 0

The signal creates **holes** in the zeros pattern: **sedond order** statistics.



snr $= 0$



snr > 0

The signal creates **holes** in the zeros pattern: **sedond order** statistics.

A functional statistic: the empty space function

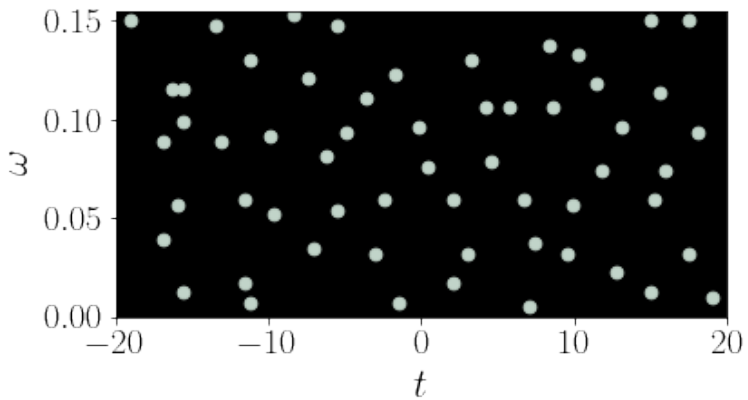
Z a stationary point process, z_0 any reference point

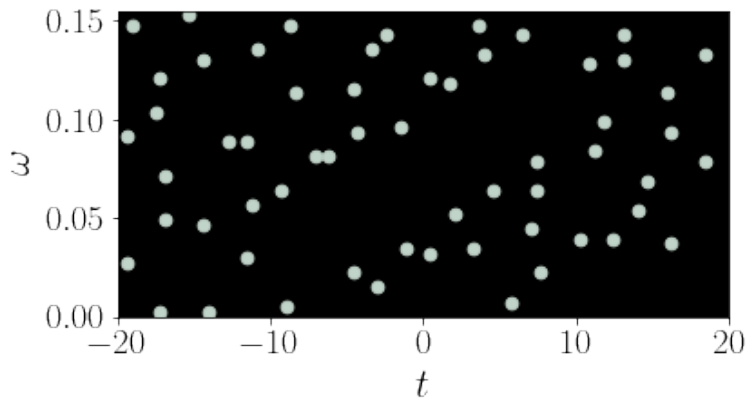
$$F(r) = \mathbb{P}\left(\inf_{z_i \in Z} \mathrm{d}(z_0, z_i) < r\right)$$

$\rightarrow$ probability to find a zero at distance less than r from z_0

Estimation of the F-function of a **stationary** Point Process (Møller, 2007)

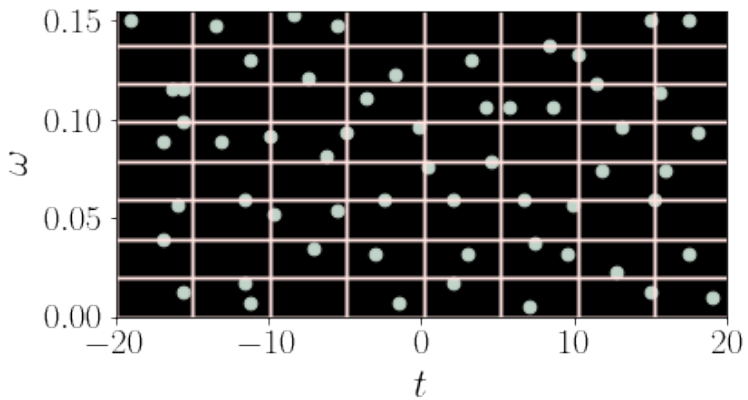
$$F(r) = \mathbb{P}\left(\inf_{z_i \in Z} \mathrm{d}(z_0, z_i) < r\right) : \textbf{ empty space function}$$

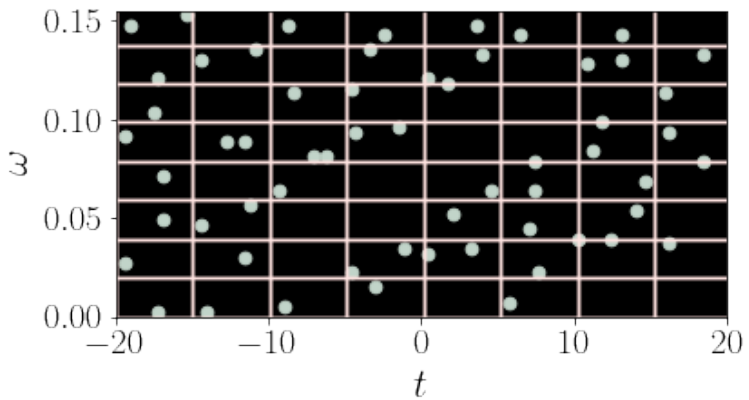




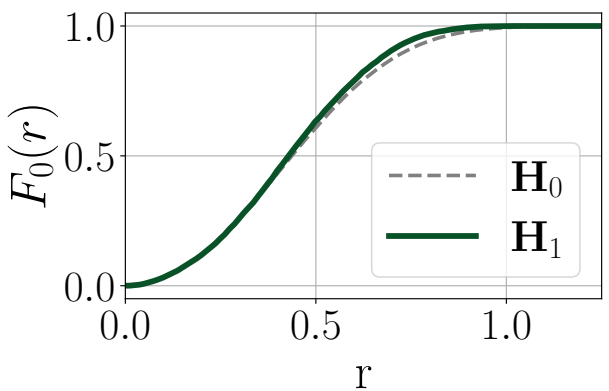
Estimation of the F-function of a **stationary** Point Process (Møller, 2007)

$$F(r) = \mathbb{P}\left(\inf_{z_i \in Z} \mathrm{d}(z_0, z_i) < r\right) : \textbf{ empty space function}$$



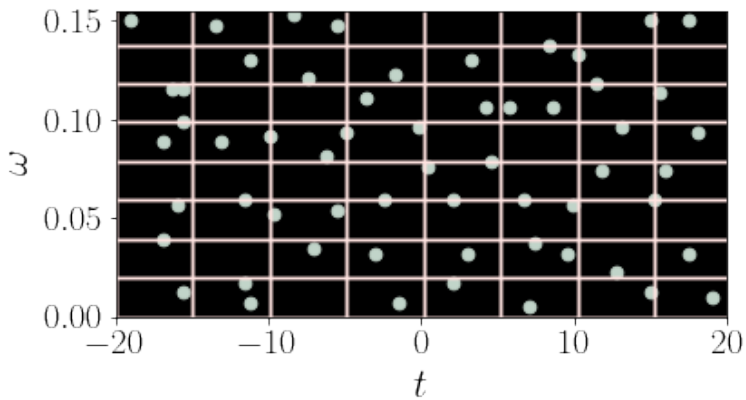


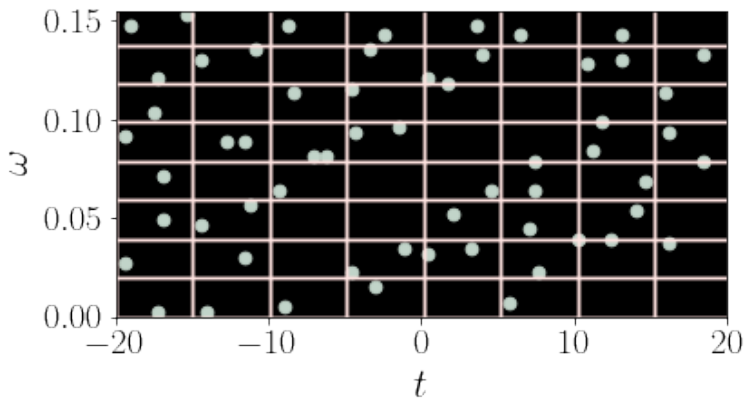
$$\widehat{F}(r) = \frac{1}{N_\#} \sum_{j=1}^{N_\#} \mathbf{1}\left(\inf_{z \in \textbf{Zeros}} \mathrm{d}\left(z_j, z\right) < r\right)$$



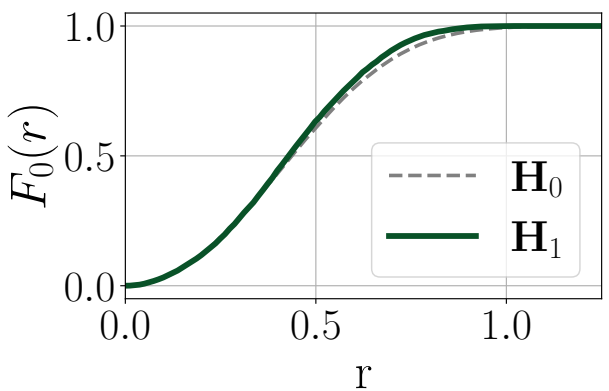
Estimation of the F-function of a **stationary** Point Process (Møller, 2007)

$$F(r) = \mathbb{P}\left(\inf_{z_i \in Z} \mathrm{d}(z_0, z_i) < r\right) : \textbf{ empty space function}$$



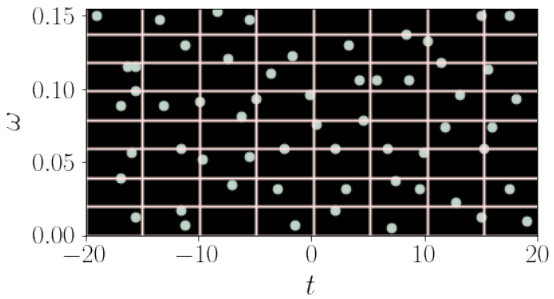


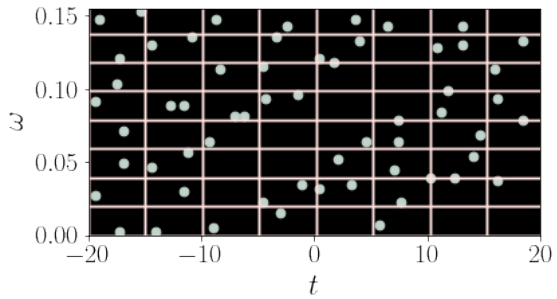
$$\widehat{F}(r) = \frac{1}{N_\#} \sum_{j=1}^{N_\#} \mathbf{1}\left(\inf_{z \in \textbf{Zeros}} \mathrm{d}\left(z_j, z\right) < r\right)$$

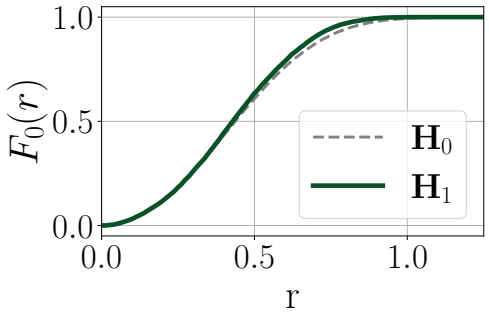


- Monte Carlo envelope test based on the discrepancy between $\widehat{F}$ and F_0

Monte Carlo envelope test







$$s(\boldsymbol{y}) = \sqrt{\int_0^{r_{\max}} \left| \widehat{F}_{\boldsymbol{y}}(r) - F_0(r) \right|^2 \, \mathrm{d}r},$$

Test settings: α level of significance, m number of samples under $\mathbf{H}_0$

Index k, chosen so that $\alpha = k/(m+1)$

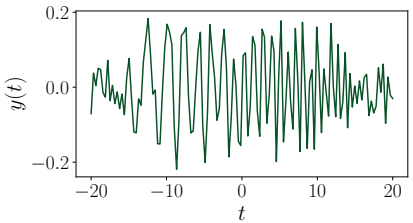
(i) generate m independent samples of complex white Gaussian noise;

(ii) compute their summary statistics $s_1 \geq s_2 \geq \ldots \geq s_m$;

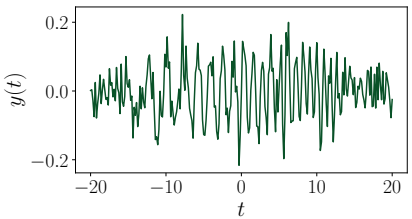
(iii) compute the summary statistic of the observation $\boldsymbol{y}$ under concern;

(iv) if $s(\boldsymbol{y}) \geq s_k$, then reject the null hypothesis with confidence $1 - \alpha$.

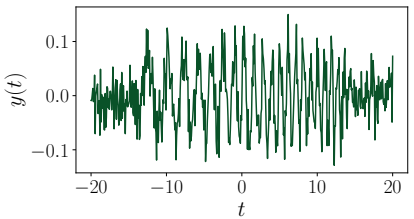
Detection of a noisy chirp of duration $2\nu = 30$ s



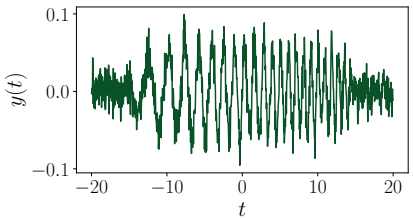
$N = 128$



$N = 256$

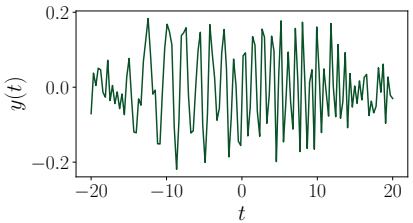


$N = 512$

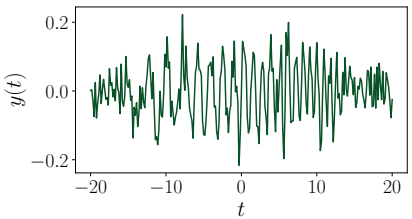


$N = 1024$

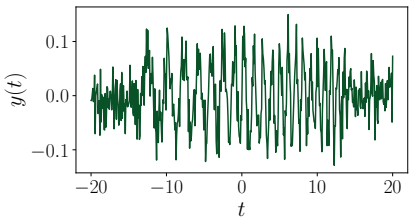
Detection of a noisy chirp of duration $2\nu = 30$ s



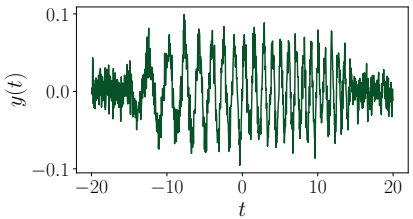
$N = 128$



$N = 256$

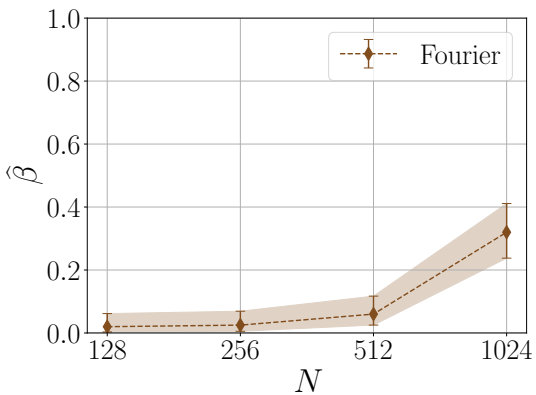


$N = 512$

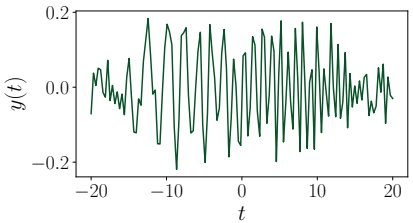


$N = 1024$

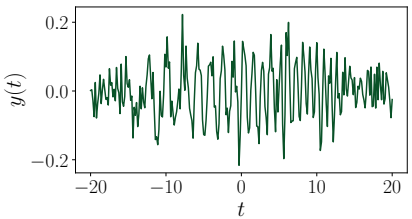
Performance: power of the test computed over 200 samples



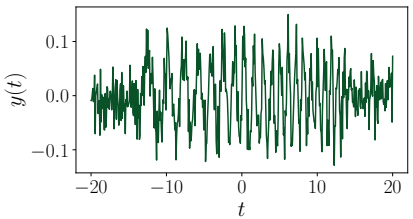
Detection of a noisy chirp of duration $2\nu = 30$ s



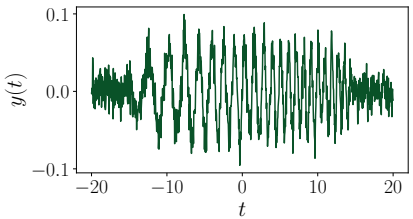
$N = 128$



$N = 256$

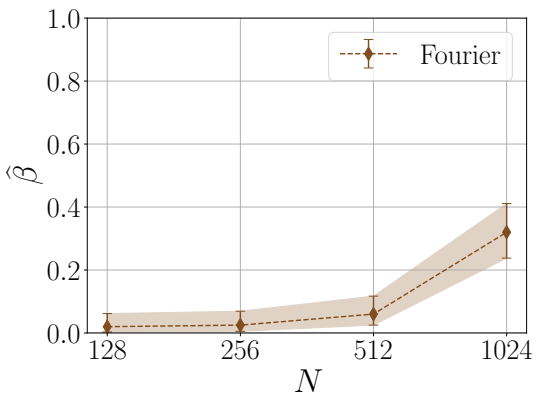


$N = 512$



$N = 1024$

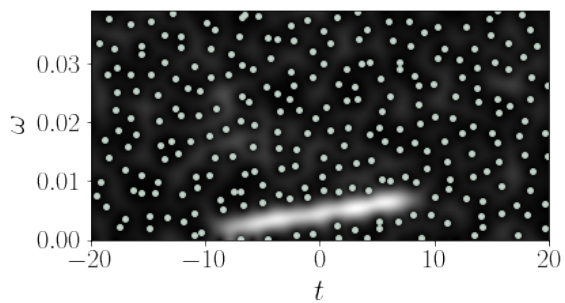
Performance: power of the test computed over 200 samples



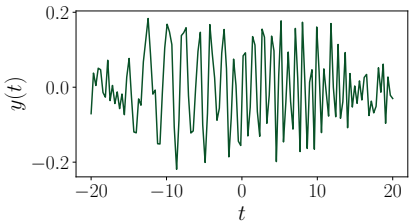
✓ Fast Fourier Transform ;

✗ Low detection power ;

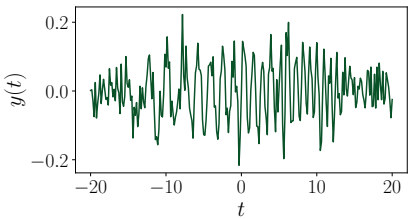
✗ Requires large number of samples



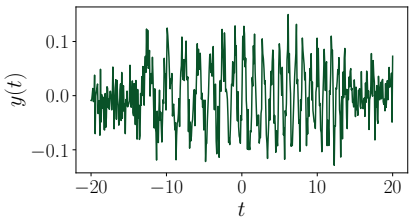
Detection of a noisy chirp of duration $2\nu = 30$ s



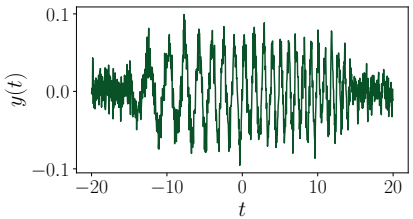
$N = 128$



$N = 256$

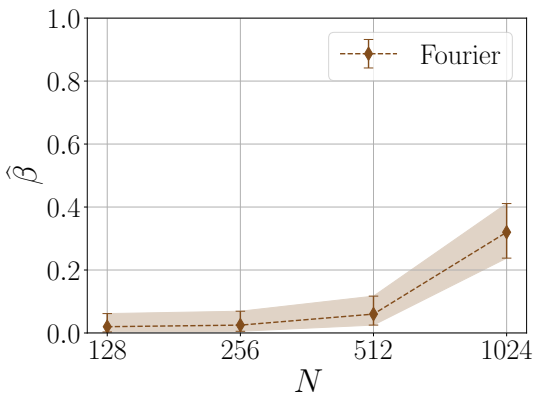


$N = 512$



$N = 1024$

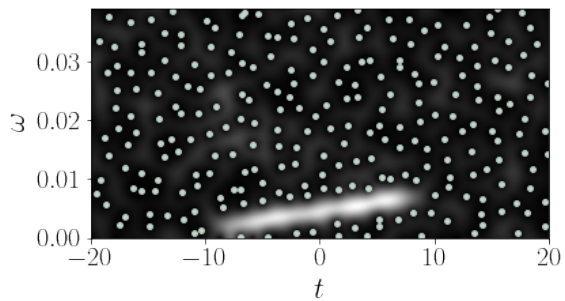
Performance: power of the test computed over 200 samples



✓ Fast Fourier Transform ;

✗ Low detection power ;

✗ Requires large number of samples



Limitations:

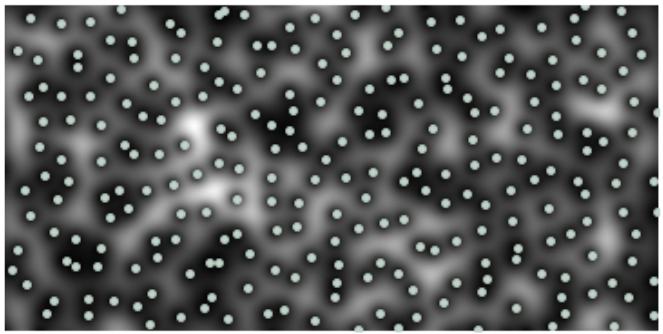
- Necessary discretization of the STFT: arbitrary resolution ;
- Observe only a bounded window: edge corrections to compute $\widehat{F}(r)$.

Other Gaussian Analytic Functions, other transforms?

Short-Time Fourier Transform

$$V_g\xi(t,\omega) \propto \mathrm{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$$

Unbounded phase space $\mathbb{C}$



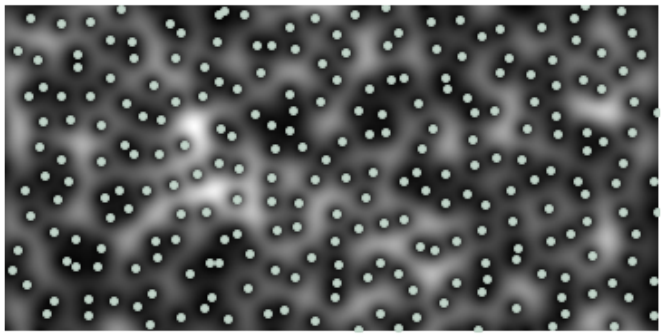
$\rightarrow$ edge corrections

Other Gaussian Analytic Functions, other transforms?

Short-Time Fourier Transform

$$V_g\xi(t,\omega) \propto \mathrm{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$$

Unbounded phase space $\mathbb{C}$



$\rightarrow$ edge corrections

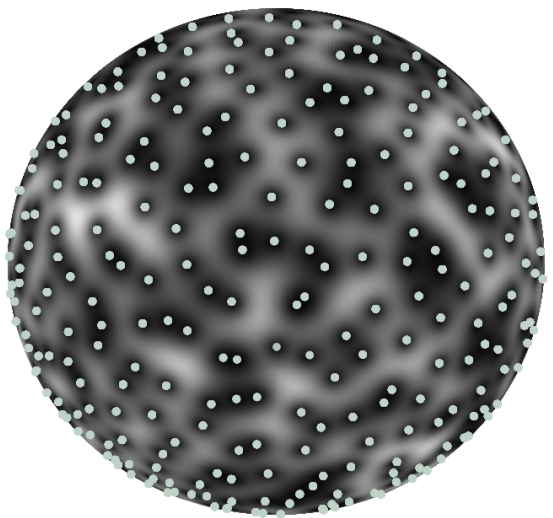
New transform?

$$? \quad \propto \mathrm{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^{N} \xi[n] \sqrt{\binom{N}{n}} z^n$$

stereographic projection $z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$

$\rightarrow$ spherical coordinates $(\vartheta, \varphi) \in S^2$

Compact phase space S^2?



$\rightarrow$ no border!

Outline of the presentation

- Signal detection: the role of representations

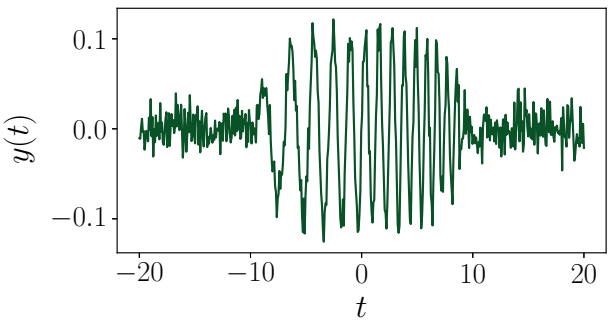
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- Covariance principle and stationary point processes

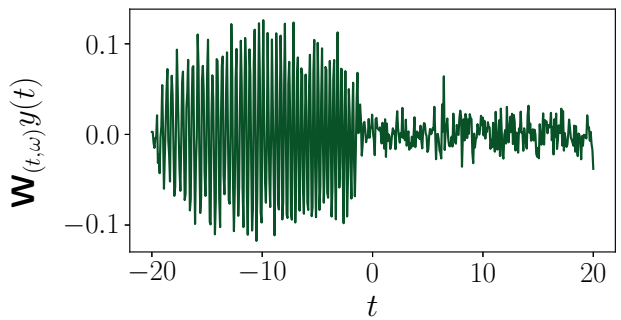
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

- Bridging Kravchuk transform and quantum momentum theory

Time and frequency shifts

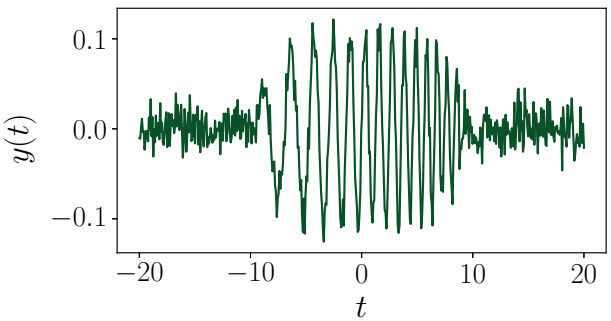
$$\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u - t)$$

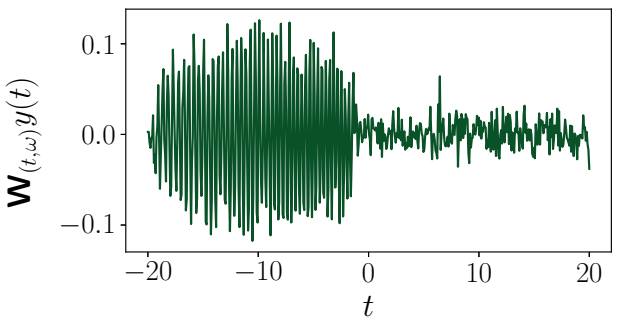




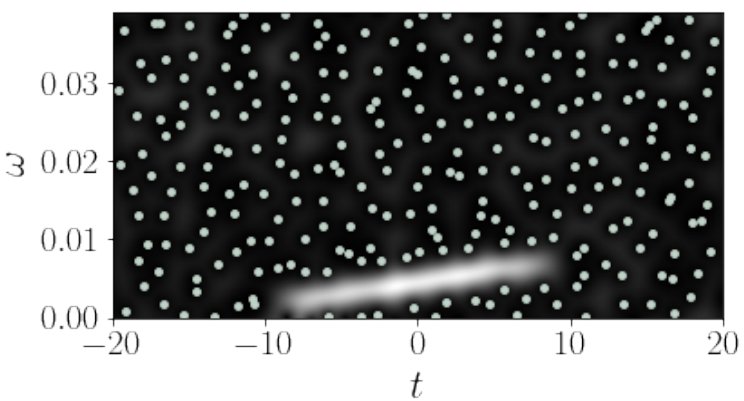
Time and frequency shifts

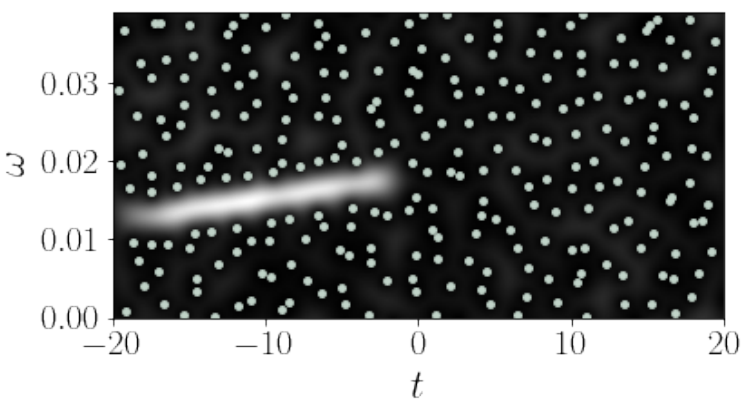
$$\boldsymbol{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$$





$$V_h[\boldsymbol{W}_{(t,\omega)}y](t',\omega') \overset{\text{(covariance)}}{=} \mathrm{e}^{-\mathrm{i}(\omega'-\omega)t}\, V_h y(t'-t,\omega'-\omega),$$





Time and frequency shifts $\boldsymbol{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$

$$\left|V_h[\boldsymbol{W}_{(t,\omega)}y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Algebraic interpretation: covariance under a symmetry group

Time and frequency shifts $\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$

$$\left|V_h[\boldsymbol{W}_{(t,\omega)} y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Complex white Gaussian noise $\widetilde{\xi} = \boldsymbol{W}_{(t,\omega)}\xi$

- $\mathbb{E}[\widetilde{\xi}(u)] \qquad = e^{-\mathrm{i}\omega u}\mathbb{E}[\xi(u-t)] \qquad = 0$

Time and frequency shifts $\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$

$$\left|V_h[\boldsymbol{W}_{(t,\omega)} y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Complex white Gaussian noise $\widetilde{\xi} = \boldsymbol{W}_{(t,\omega)}\xi$

- $\mathbb{E}[\widetilde{\xi}(u)] = e^{-\mathrm{i}\omega u}\mathbb{E}[\xi(u-t)] = 0$
- $\mathbb{E}[\overline{\widetilde{\xi}(u)}\widetilde{\xi}(u')] = e^{\mathrm{i}\omega(u-u')}\mathbb{E}[\overline{\xi(u)}\xi(u')] = \delta(u-u')$

Time and frequency shifts $\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$

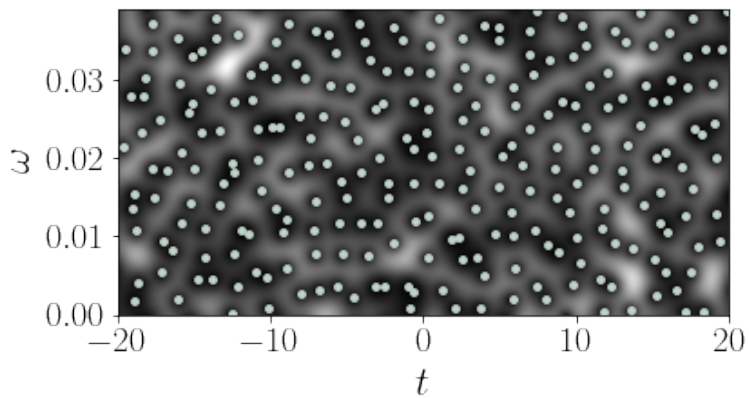
$$\left|V_h[\boldsymbol{W}_{(t,\omega)} y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Complex white Gaussian noise $\widetilde{\xi} = \boldsymbol{W}_{(t,\omega)}\xi$

- $\mathbb{E}[\widetilde{\xi}(u)] = e^{-\mathrm{i}\omega u}\mathbb{E}[\xi(u-t)] = 0$
- $\mathbb{E}[\overline{\widetilde{\xi}(u)}\widetilde{\xi}(u')] = e^{\mathrm{i}\omega(u-u')}\mathbb{E}[\overline{\xi(u)}\xi(u')] = \delta(u-u')$

Invariance under time-frequency shifts:

$$\widetilde{\xi} = \boldsymbol{W}_{(t,\omega)}\xi \overset{\text{(law)}}{=} \xi$$



Time and frequency shifts $\boldsymbol{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$

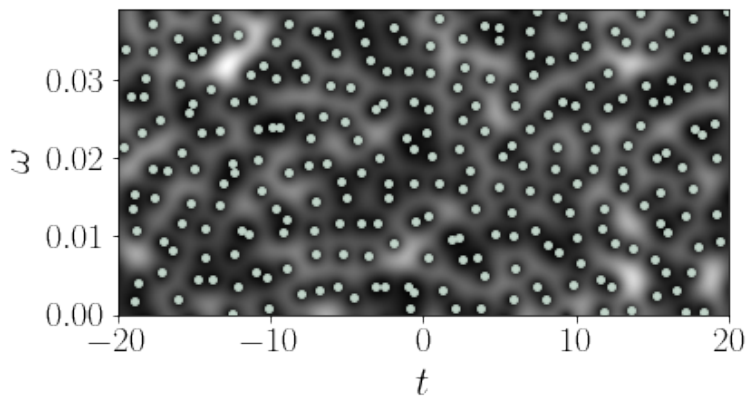
$$\left|V_h[\boldsymbol{W}_{(t,\omega)}y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Complex white Gaussian noise $\widetilde{\xi} = \boldsymbol{W}_{(t,\omega)}\xi$

- $\mathbb{E}[\widetilde{\xi}(u)] = e^{-\mathrm{i}\omega u}\mathbb{E}[\xi(u-t)] = 0$
- $\mathbb{E}[\overline{\widetilde{\xi}(u)}\widetilde{\xi}(u')] = e^{\mathrm{i}\omega(u-u')}\mathbb{E}[\overline{\xi(u)}\xi(u')] = \delta(u-u')$

Invariance under time-frequency shifts:

$$\widetilde{\xi} = \boldsymbol{W}_{(t,\omega)}\xi \overset{\text{(law)}}{=} \xi$$



Covariance is the key to get **stationarity**: how to get covariant transforms?

Time and frequency shifts $\boldsymbol{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$

$$\left|V_h[\boldsymbol{W}_{(t,\omega)}y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Algebraic interpretation: covariance under a symmetry group

Time and frequency shifts $\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$

$$\left|V_h[\boldsymbol{W}_{(t,\omega)} y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Weyl-Heisenberg group $\{\mathrm{e}^{\mathrm{i}\gamma}\boldsymbol{W}_{(t,\omega)},\ (\gamma,t,\omega)\in[0,2\pi]\times\mathbb{R}^2\}$

$$\boldsymbol{W}_{(t',\omega')}\boldsymbol{W}_{(t,\omega)} = \mathrm{e}^{\mathrm{i}\omega t'}\boldsymbol{W}_{(t+t',\omega+\omega')}.$$

Algebraic interpretation: covariance under a symmetry group

Time and frequency shifts $\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$

$$\left|V_h[\boldsymbol{W}_{(t,\omega)} y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Weyl-Heisenberg group $\{\mathrm{e}^{\mathrm{i}\gamma}\boldsymbol{W}_{(t,\omega)},\ (\gamma,t,\omega)\in[0,2\pi]\times\mathbb{R}^2\}$

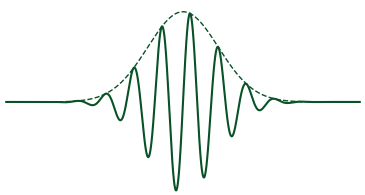
$$\boldsymbol{W}_{(t',\omega')}\boldsymbol{W}_{(t,\omega)} = \mathrm{e}^{\mathrm{i}\omega t'}\boldsymbol{W}_{(t+t',\omega+\omega')}.$$



$g(t) = \pi^{-1/4}\exp\left(-t^2/2\right)$



$\mathbf{T}_u g(t) = g(t-u)$



$\mathbf{M}_\omega g(t) = g(t)\exp\left(-\mathrm{i}\omega t\right)$

Time and frequency shifts $\boldsymbol{W}_{(t,\omega)} y(u) = e^{-\mathrm{i}\omega u} y(u-t)$

$$\left|V_h[\boldsymbol{W}_{(t,\omega)} y](t',\omega')\right|^2 \overset{\text{(covariance)}}{=} \left|V_h y(t'-t,\omega'-\omega)\right|^2,$$

Weyl-Heisenberg group $\{\mathrm{e}^{\mathrm{i}\gamma}\boldsymbol{W}_{(t,\omega)},\ (\gamma,t,\omega)\in[0,2\pi]\times\mathbb{R}^2\}$

$$\boldsymbol{W}_{(t',\omega')}\boldsymbol{W}_{(t,\omega)} = \mathrm{e}^{\mathrm{i}\omega t'}\boldsymbol{W}_{(t+t',\omega+\omega')}.$$

Coherent state interpretation $\{\boldsymbol{W}_{(t,\omega)}h,\ t,\omega\in\mathbb{R}\}$ covariant family

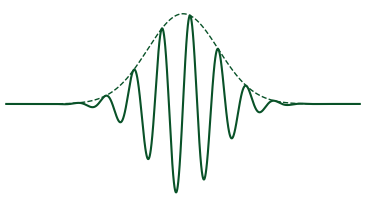
$$V_h y(t,\omega) = \int_{-\infty}^{\infty} \overline{y(u)} h(u-t)\exp(-\mathrm{i}\omega u)\,\mathrm{d}u = \langle y, \boldsymbol{W}_{(t,\omega)}h\rangle$$



$g(t) = \pi^{-1/4}\exp\left(-t^2/2\right)$



$\mathbf{T}_u g(t) = g(t-u)$



$\mathbf{M}_\omega g(t) = g(t)\exp\left(-\mathrm{i}\omega t\right)$

Outline of the presentation

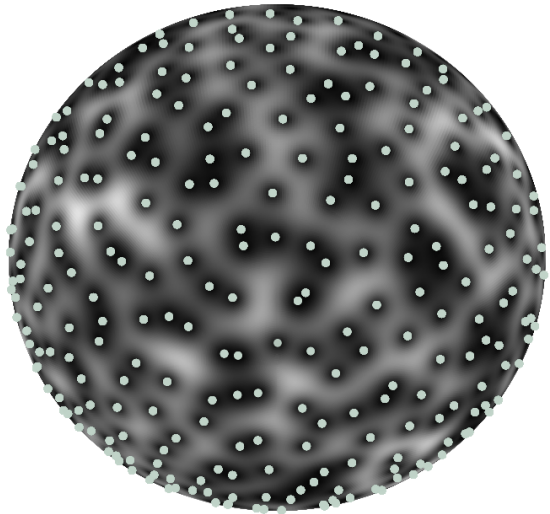
- Signal detection: the role of representations

- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- Covariance principle and stationary point processes

- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

- Bridging Kravchuk transform and quantum momentum theory

The Kravchuk transform: covariance under $\mathrm{SO}(3)$

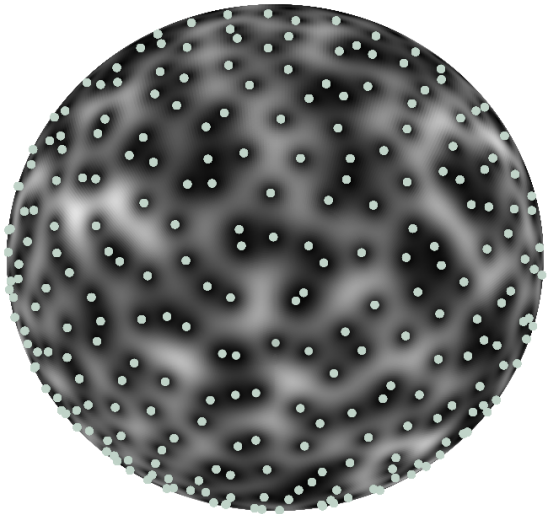


The Kravchuk transform: covariance under $\mathrm{SO}(3)$

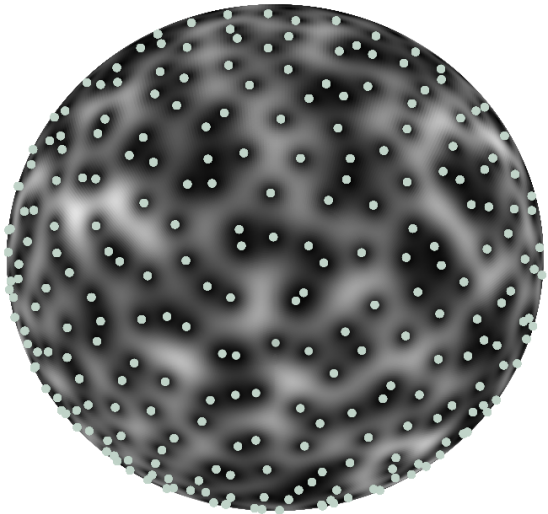
Coherent state interpretation $\quad \boldsymbol{y} \in \mathbb{C}^{N+1}$

$$T\boldsymbol{y}(\vartheta, \varphi) = \langle \boldsymbol{y}, \boldsymbol{\Psi}_{(\vartheta,\varphi)} \rangle$$

$\vartheta \in [0, \pi], \varphi \in [0, 2\pi]$



Coherent state interpretation $\quad \boldsymbol{y} \in \mathbb{C}^{N+1}$



$$T\boldsymbol{y}(\vartheta,\varphi) = \langle \boldsymbol{y}, \boldsymbol{\Psi}_{(\vartheta,\varphi)} \rangle$$

$\vartheta \in [0,\pi], \varphi \in [0,2\pi]$

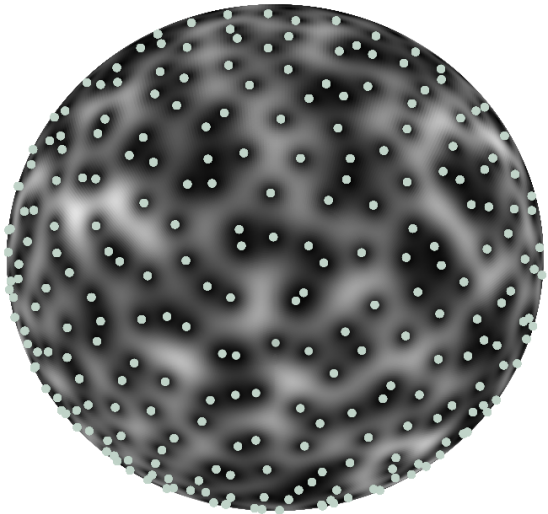
$\mathrm{SO}(3)$ **coherent states** (Gazeau, 2009)

$$\boldsymbol{\Psi}_{\vartheta,\varphi} = \sum_{n=0}^{N} \sqrt{\binom{N}{n}} \left(\cos\frac{\vartheta}{2}\right)^{n} \left(\sin\frac{\vartheta}{2}\right)^{N-n} \mathrm{e}^{\mathrm{i}n\varphi} \boldsymbol{q}_n = \boldsymbol{R}_{\boldsymbol{u}(\vartheta,\varphi)} \boldsymbol{\Psi}_{(0,0)},$$

Coherent state interpretation $\quad \boldsymbol{y} \in \mathbb{C}^{N+1}$

$$T\boldsymbol{y}(\vartheta,\varphi) = \langle \boldsymbol{y}, \boldsymbol{\Psi}_{(\vartheta,\varphi)} \rangle$$

$\vartheta \in [0,\pi], \varphi \in [0,2\pi]$



$\mathrm{SO}(3)$ **coherent states** (Gazeau, 2009)

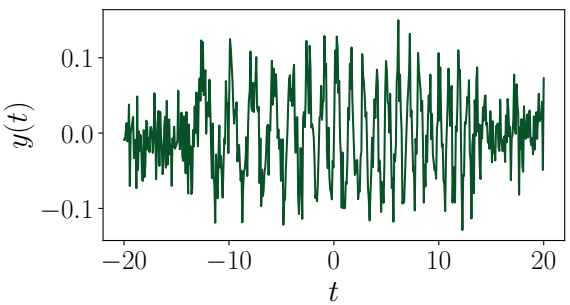
$$\boldsymbol{\Psi}_{\vartheta,\varphi} = \sum_{n=0}^{N} \sqrt{\binom{N}{n}} \left(\cos\frac{\vartheta}{2}\right)^{n} \left(\sin\frac{\vartheta}{2}\right)^{N-n} \mathrm{e}^{\mathrm{i}n\varphi} \boldsymbol{q}_n = \boldsymbol{R}_{\boldsymbol{u}(\vartheta,\varphi)} \boldsymbol{\Psi}_{(0,0)},$$

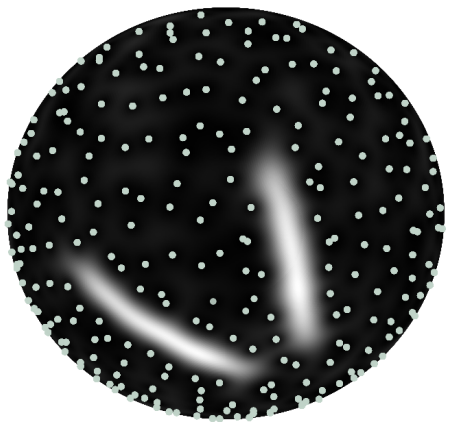
Kravchuk transform $\quad \{\boldsymbol{q}_n, n = 0,1,...,N\}$ the *Kravchuk functions*

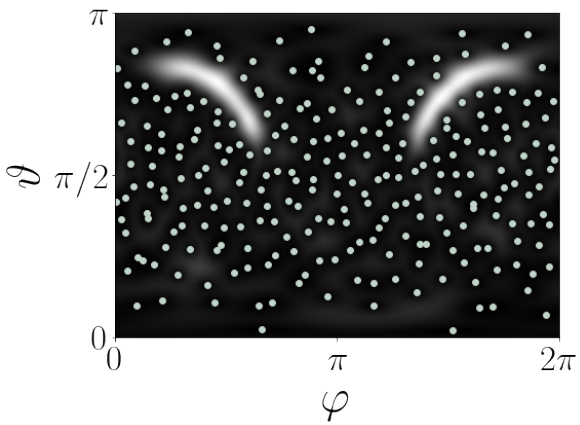
$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^{N} \langle \boldsymbol{y}, \boldsymbol{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$$

Kravchuk transform $\{\boldsymbol{q}_n, n = 0, 1, ..., N\}$ the *Kravchuk functions*

$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^{N} \langle \boldsymbol{y}, \boldsymbol{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$$

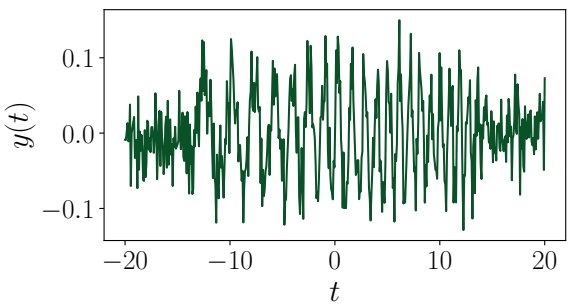


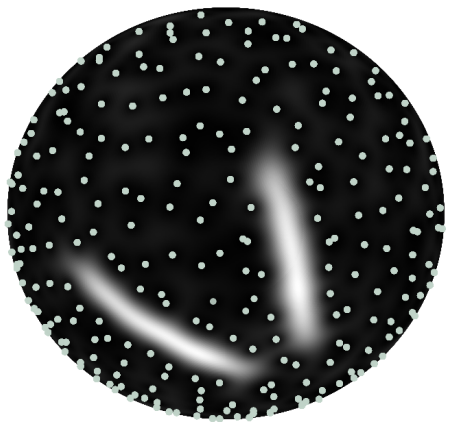


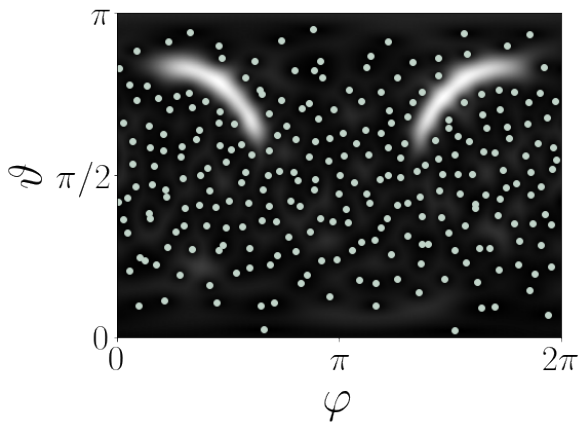


Kravchuk transform $\{\boldsymbol{q}_n, n = 0, 1, ..., N\}$ the *Kravchuk functions*

$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^{N} \langle \boldsymbol{y}, \boldsymbol{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$$







Theorem $T\boldsymbol{\xi}(\vartheta, \varphi) = \sqrt{(1+|z|^2)}^{-N} \, \mathrm{GAF}_{\mathbb{S}}(z), \qquad z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$

$\mathrm{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^{N} \boldsymbol{\xi}[n] \sqrt{\binom{N}{n}} z^n$ the *spherical Gaussian Analytic Function*

(Pascal & Bardenet, 2022)

Outline of the presentation

- Signal detection: the role of representations

- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- Covariance principle and stationary point processes

- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

- Bridging Kravchuk transform and quantum momentum theory

Kravchuk transform $\{\boldsymbol{q}_n, n = 0, 1, ..., N\}$ the Kravchuk basis

$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^{N} \langle \boldsymbol{y}, \boldsymbol{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$$

$\rightarrow$ first: basis change, i.e., computation of $\langle \boldsymbol{y}, \boldsymbol{q}_n \rangle = \sum_{\ell=0}^{N} \overline{\boldsymbol{y}[\ell]} q_n(\ell; N)$

Kravchuk transform $\{\boldsymbol{q}_n, n = 0, 1, ..., N\}$ the Kravchuk basis

$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^{N} \langle \boldsymbol{y}, \boldsymbol{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$$

$\rightarrow$ first: basis change, i.e., computation of $\langle \boldsymbol{y}, \boldsymbol{q}_n \rangle = \sum_{\ell=0}^{N} \overline{\boldsymbol{y}[\ell]} q_n(\ell; N)$

Evaluation of Kravchuk functions $q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$

$$(N-n)Q_{n+1}(t; N) = (N-2t)Q_n(t; N) - nQ_{n-1}(t; N),$$

$\{Q_n(t; N), n = 0, 1, \ldots, N\}$ orthogonal family of Kravchuk polynomials

$$\sum_{\ell=0}^{N} \binom{N}{\ell} Q_n(\ell; N) Q_{n'}(\ell; N) = 2^N \binom{N}{n}^{-1} \delta_{n,n'}$$

Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell;N) = \frac{1}{\sqrt{2^N}}\sqrt{\binom{N}{n}}Q_n(\ell;N)\sqrt{\binom{N}{\ell}}$$

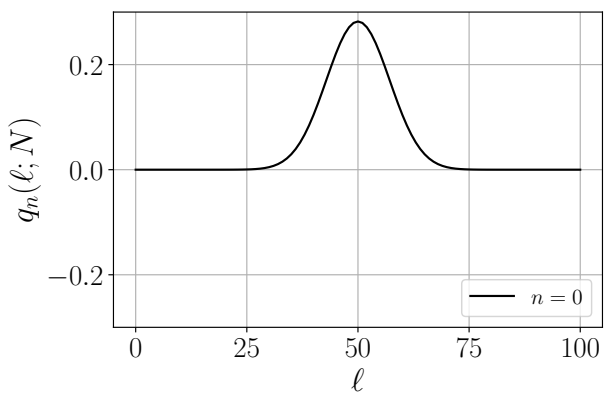
Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell;N) = \frac{1}{\sqrt{2^N}}\sqrt{\binom{N}{n}}\,Q_n(\ell;N)\sqrt{\binom{N}{\ell}}$$



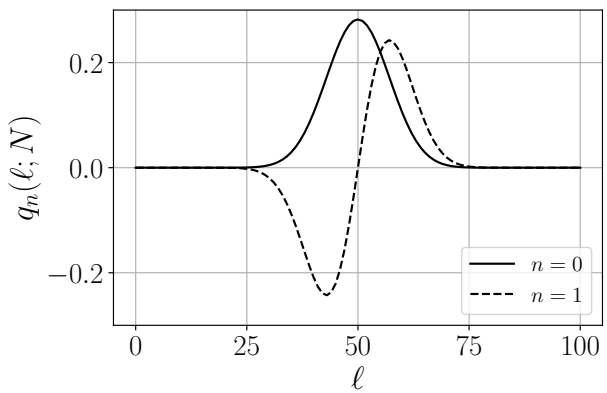
Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell;N) = \frac{1}{\sqrt{2^N}}\sqrt{\binom{N}{n}}\,Q_n(\ell;N)\sqrt{\binom{N}{\ell}}$$



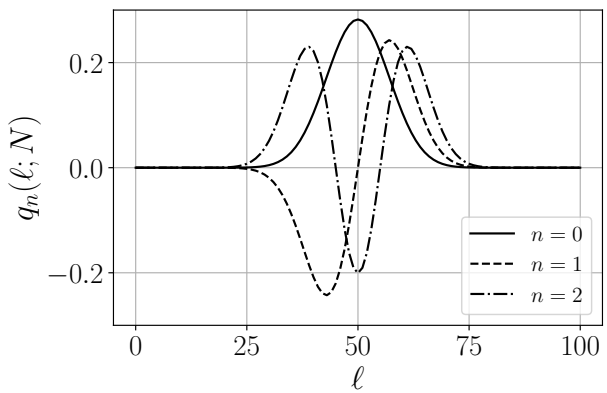
Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell;N) = \frac{1}{\sqrt{2^N}}\sqrt{\binom{N}{n}}\,Q_n(\ell;N)\sqrt{\binom{N}{\ell}}$$



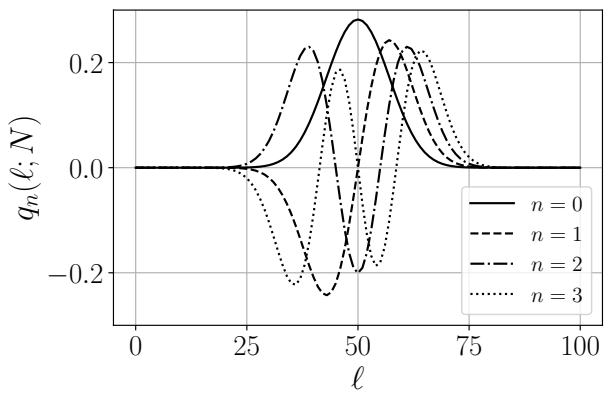
Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell;N) = \frac{1}{\sqrt{2^N}}\sqrt{\binom{N}{n}}\,Q_n(\ell;N)\sqrt{\binom{N}{\ell}}$$



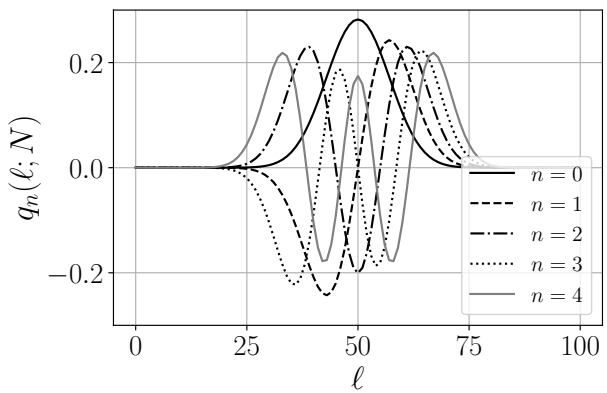
Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell;N) = \frac{1}{\sqrt{2^N}}\sqrt{\binom{N}{n}}\,Q_n(\ell;N)\sqrt{\binom{N}{\ell}}$$



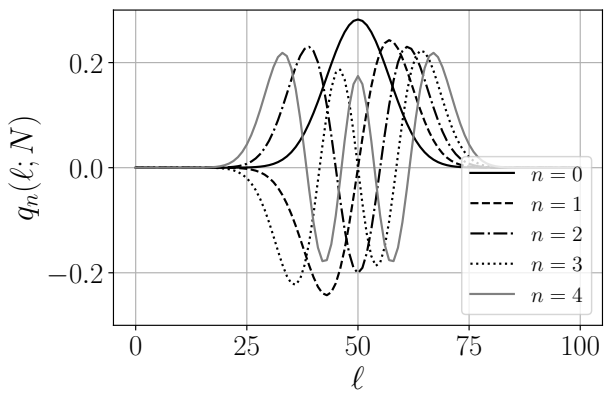
Evaluation of Kravchuk functions

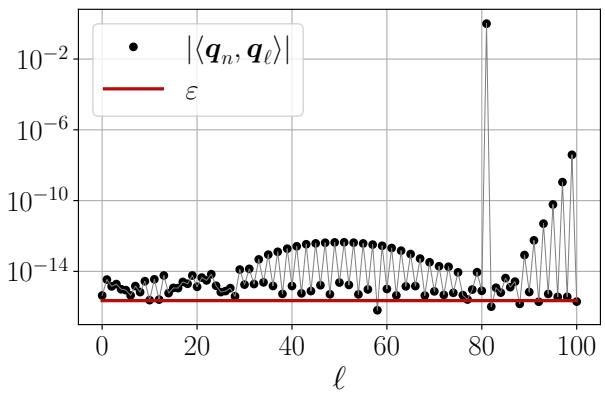
(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell;N) = \frac{1}{\sqrt{2^N}}\sqrt{\binom{N}{n}}\,Q_n(\ell;N)\sqrt{\binom{N}{\ell}}$$





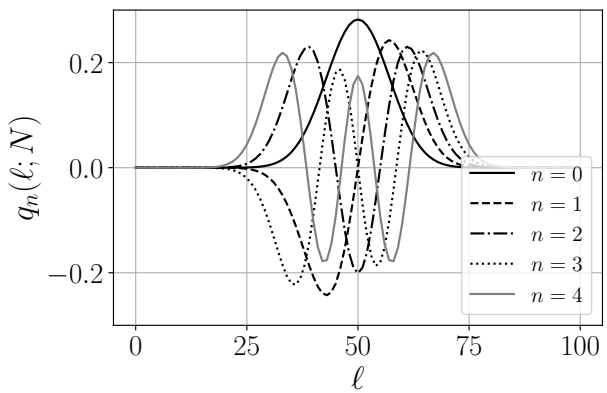
Evaluation of Kravchuk functions

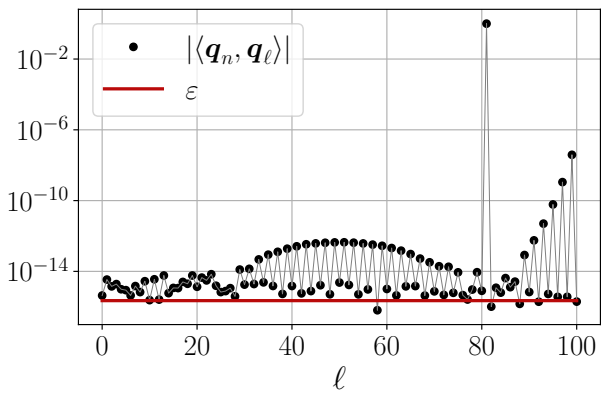
(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell;N) = \frac{1}{\sqrt{2^N}}\sqrt{\binom{N}{n}}Q_n(\ell;N)\sqrt{\binom{N}{\ell}}$$





$\rightarrow$ estimated basis is **not orthogonal**! Not possible to compute $\langle \boldsymbol{y}, \boldsymbol{q}_n \rangle$.

Kravchuk transform $\{\boldsymbol{q}_n, n = 0, 1, ..., N\}$ the Kravchuk basis

$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^{N} \left(\sum_{\ell=0}^{N} \overline{\boldsymbol{y}[\ell]} q_n(\ell; N) \right) \sqrt{\binom{N}{n}} z^n \rightarrow \text{intractable}$$

A generative function for Kravchuk polynomials

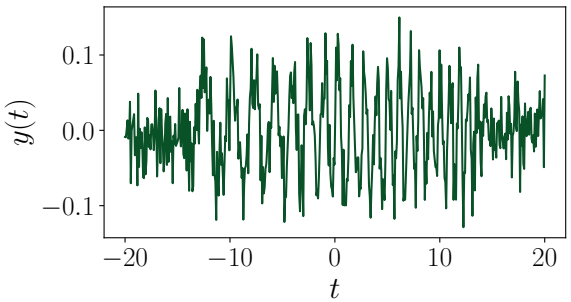
$$\sum_{n=0}^{N} \binom{N}{n} Q_n(\ell; N) z^n = (1-z)^\ell (1+z)^{N-\ell}$$

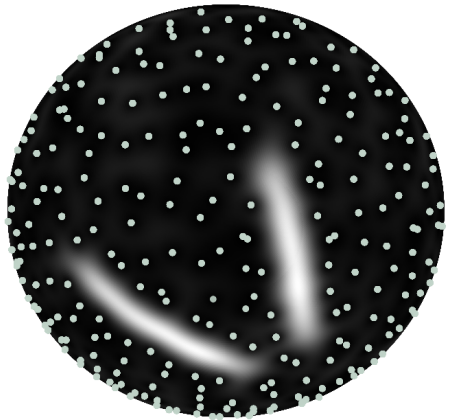
$$\Longrightarrow \quad \sum_{n=0}^{N} \sqrt{\binom{N}{n}} q_n(\ell; N) z^n = \sqrt{\binom{N}{\ell}} \frac{(1-z)^\ell (1+z)^{N-\ell}}{\sqrt{2^N}}$$

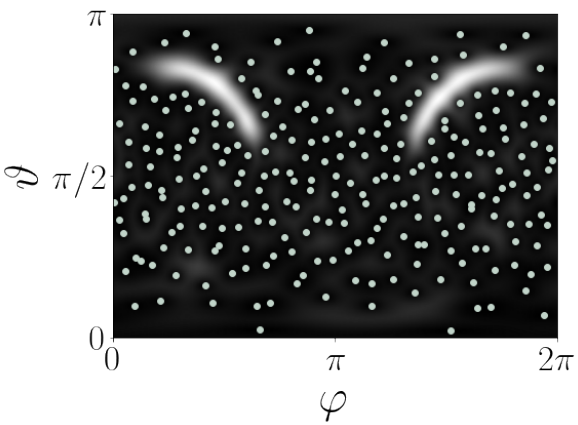
$$T\boldsymbol{y}(z) = \frac{1}{\sqrt{(1+|z^2|)^N}} \sum_{\ell=0}^{N} \sqrt{\binom{N}{\ell}} \overline{\boldsymbol{y}[\ell]} \frac{(1-z)^\ell (1+z)^{N-\ell}}{\sqrt{2^N}}$$

✗ no more Fast Fourier Transform algorithm using $z^n = \cot(\vartheta/2)^n \mathrm{e}^{\mathrm{i}n\varphi}$

Detection of the zeros of the Kravchuk spectrogram $|T\boldsymbol{y}(z_i)|^2 = 0$

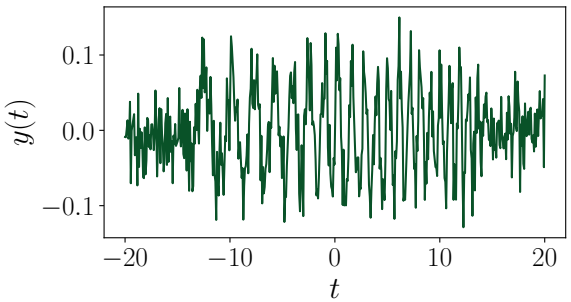


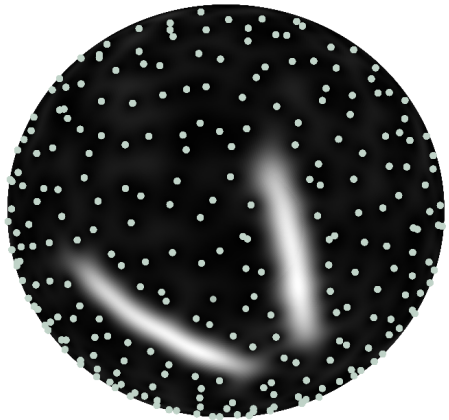


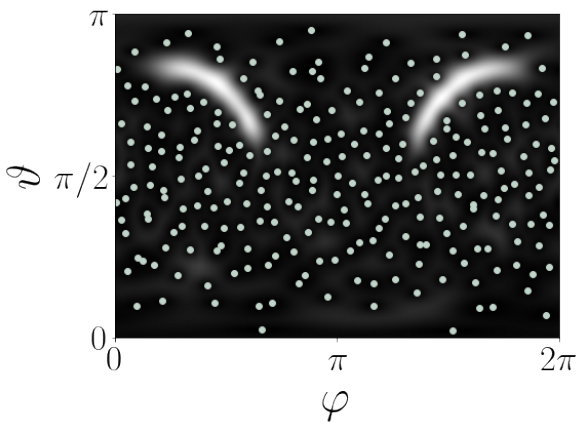


Advantage compared to Fourier: can tune the resolution of phase space.

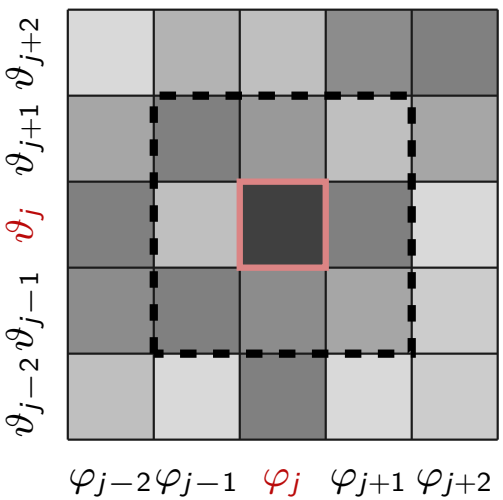
Detection of the zeros of the Kravchuk spectrogram $|T\boldsymbol{y}(z_i)|^2 = 0$





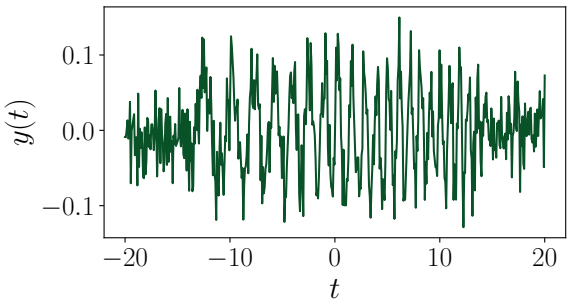


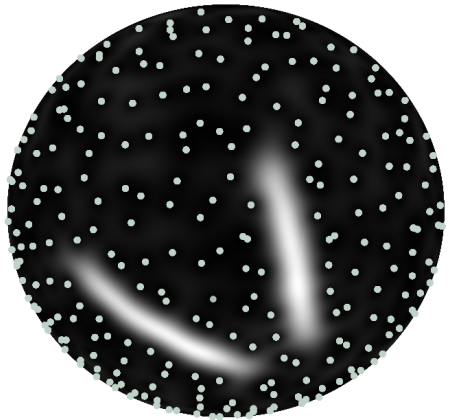
Advantage compared to Fourier: can tune the resolution of phase space.

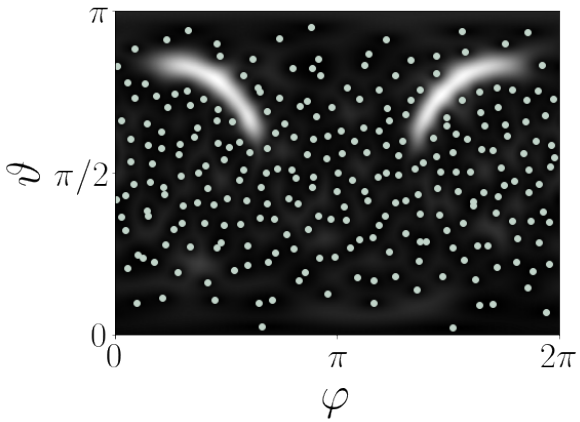


Minimal Grid Neighbors

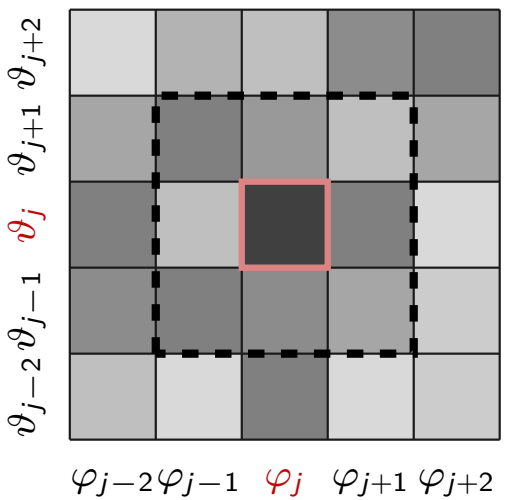
Detection of the zeros of the Kravchuk spectrogram $|T\boldsymbol{y}(z_i)|^2 = 0$



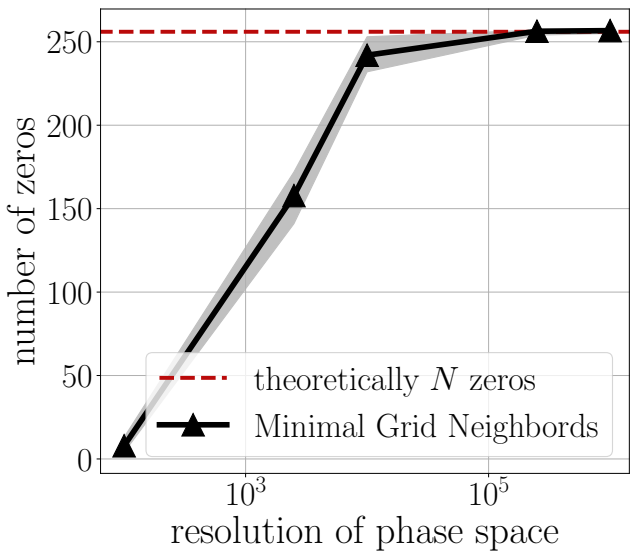




Advantage compared to Fourier: can tune the resolution of phase space.



Minimal Grid Neighbors



Proposition: all local minima of $|T\boldsymbol{y}(z)|^2$ are zeros.

Outline of the presentation

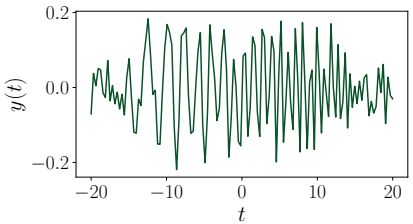
- Signal detection: the role of representations

- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- Covariance principle and stationary point processes

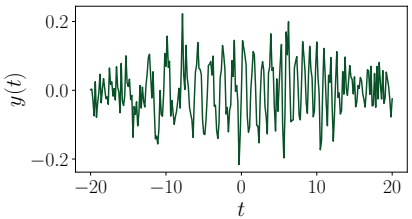
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

- Bridging Kravchuk transform and quantum momentum theory

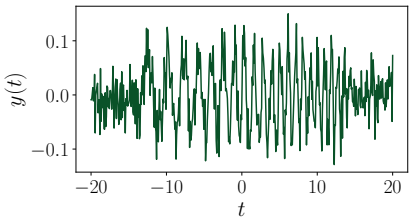
Detection of a noisy chirp of duration $2\nu = 30$ s



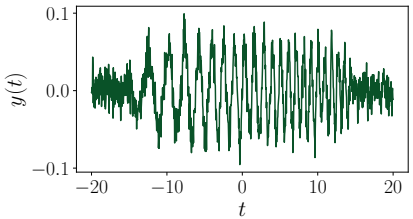
$N = 128$



$N = 256$

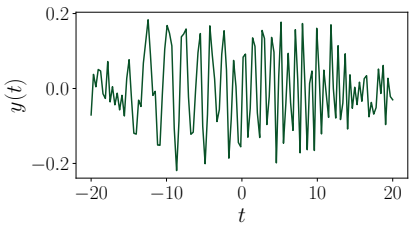


$N = 512$

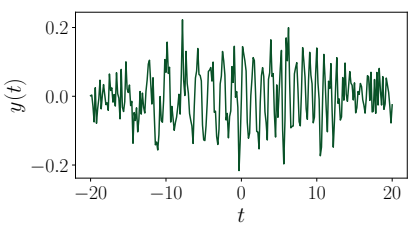


$N = 1024$

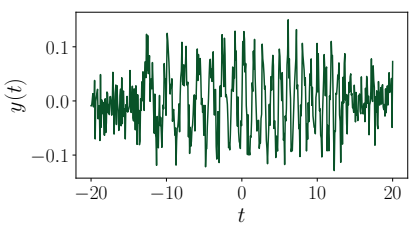
Detection of a noisy chirp of duration $2\nu = 30$ s



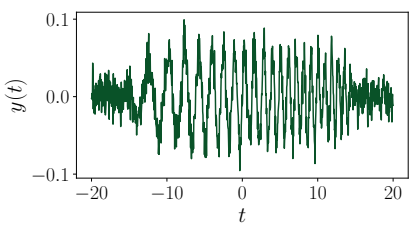
$N = 128$



$N = 256$

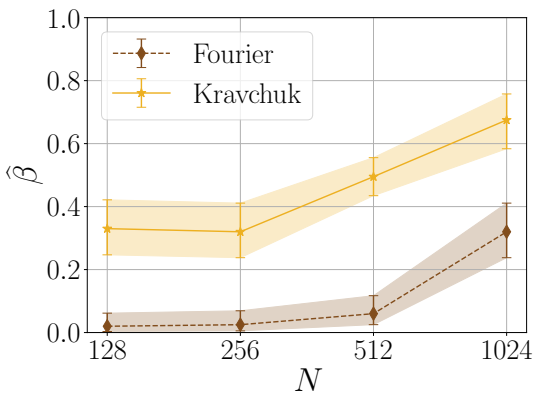


$N = 512$

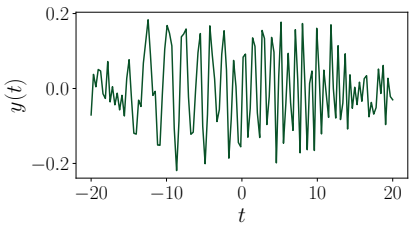


$N = 1024$

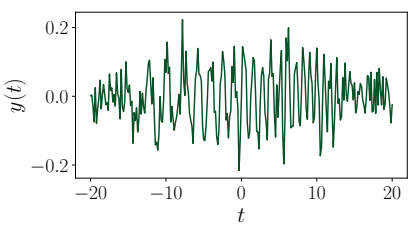
Performance: power of the test computed over 200 samples



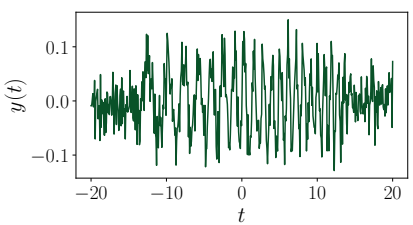
Detection of a noisy chirp of duration $2\nu = 30$ s



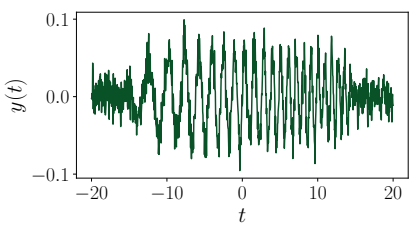
$N = 128$



$N = 256$

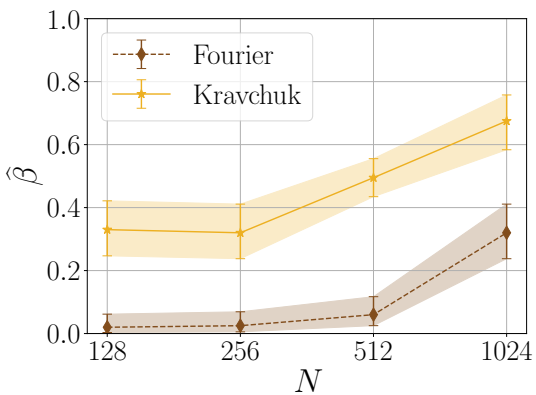


$N = 512$



$N = 1024$

Performance: power of the test computed over 200 samples

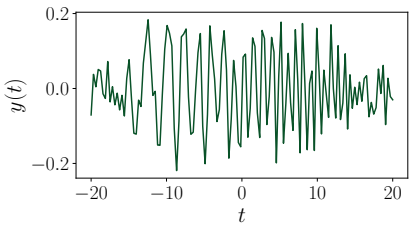


✓ higher detection power

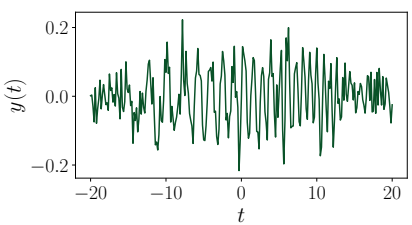
✓ more robust to small N

✗ no fast algorithm yet

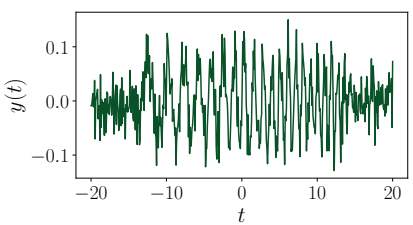
Detection of a noisy chirp of duration $2\nu = 30$ s



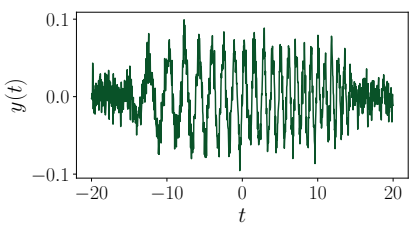
$N = 128$



$N = 256$

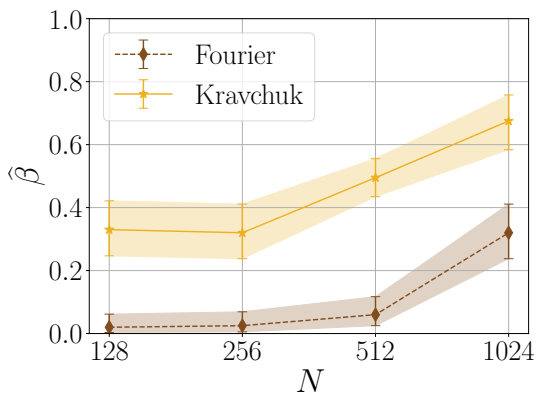


$N = 512$



$N = 1024$

Performance: power of the test computed over 200 samples



✓ higher detection power

✓ more robust to small N

✗ no fast algorithm yet

Advantages of using Kravchuk vs. Fourier spectrogram

- intrinsically encoded resolution: no need for prior knowledge
- compact phase space: no edge correction

Outline of the presentation

- Signal detection: the role of representations

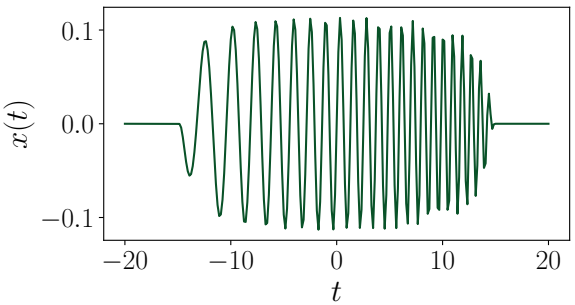
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros - I
- Covariance principle and stationary point processes

- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

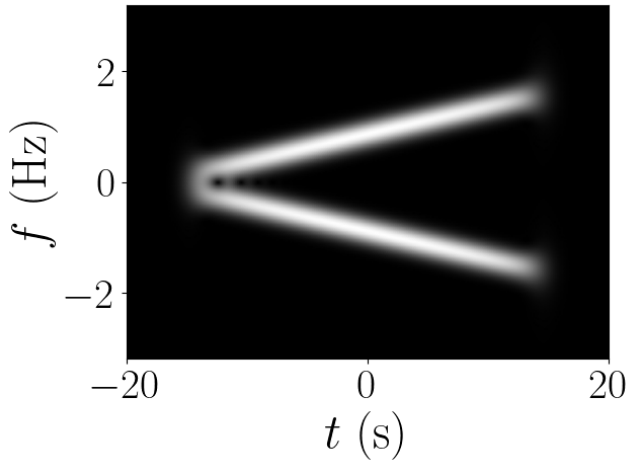
- Bridging Kravchuk transform and quantum momentum theory

In search of the North Pole, a.k.a., time–frequency interpretation

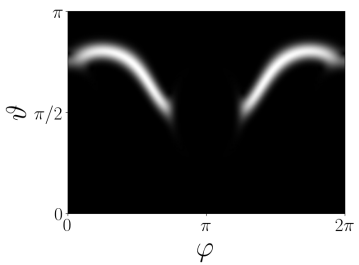
Chirp signal



Gaussian spectrogram



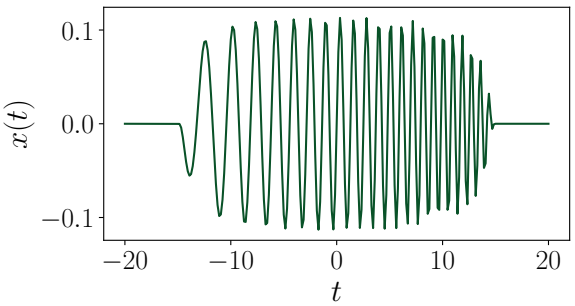
Kravchuk spectrogram



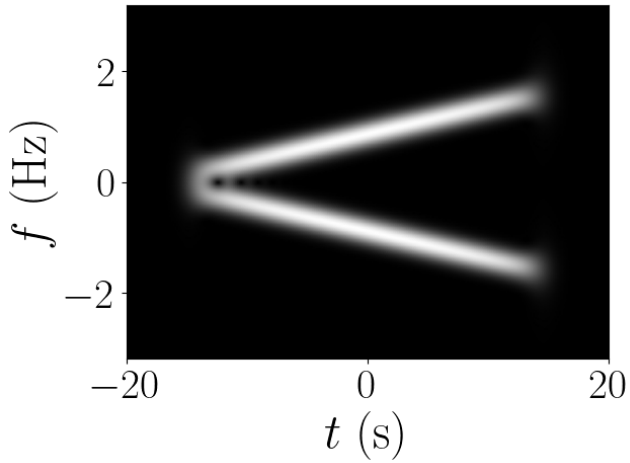


In search of the North Pole, a.k.a., time–frequency interpretation

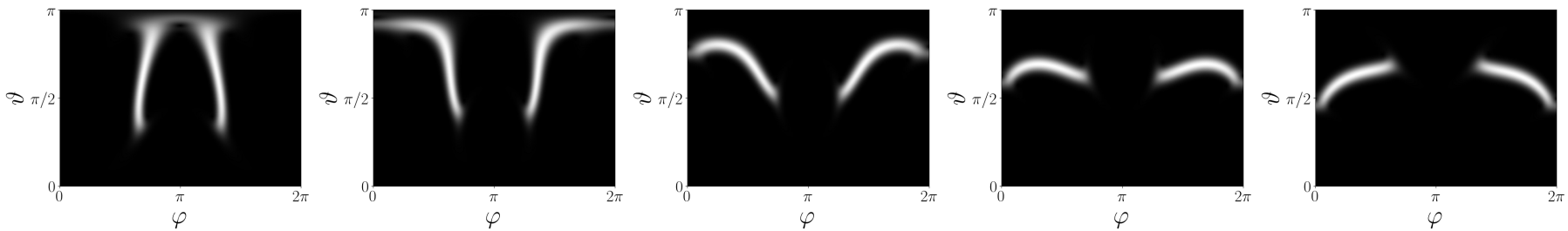
Chirp signal



Gaussian spectrogram



Kravchuk spectrogram





Covariance property: $T[\boldsymbol{R_u y}](\vartheta, \varphi) = T\boldsymbol{y}(R_u(\vartheta, \varphi))$

Energy preservation:

$$\forall \boldsymbol{y} \in \mathbb{C}^{N+1}, \quad \int_{\mathbb{S}^2} |T\boldsymbol{y}(\vartheta,\varphi)|^2 \sin(\vartheta)\,\mathrm{d}\vartheta\mathrm{d}\varphi = \frac{4\pi}{N+1}\|\boldsymbol{y}\|^2$$

$$\Longrightarrow (\vartheta,\varphi) \mapsto T\boldsymbol{y}(\vartheta,\varphi) \textbf{ square integrable} \text{ on the sphere}$$

(Pascal et al., 2022, *IEEE Trans. Sig. Process.*)



Energy preservation:

$$\forall \boldsymbol{y} \in \mathbb{C}^{N+1}, \quad \int_{\mathbb{S}^2} |T\boldsymbol{y}(\vartheta,\varphi)|^2 \sin(\vartheta)\,\mathrm{d}\vartheta\mathrm{d}\varphi = \frac{4\pi}{N+1}\|\boldsymbol{y}\|^2$$

$$\Longrightarrow (\vartheta,\varphi) \mapsto T\boldsymbol{y}(\vartheta,\varphi) \textbf{ square integrable} \text{ on the sphere}$$



(Pascal et al., 2022, *IEEE Trans. Sig. Process.*)

Any function of $L^2(\mathbb{S}^2)$ *decomposes onto the orthonormal basis* of spin spherical harmonics: $\left\{ {}_s Y_m^\ell \,:\, s,m \in \mathbb{Z}, \ell \in \mathbb{N} \text{ such that } |s| \leq \ell, |m| \leq \ell \right\}$.

(Newman et coll., 1966, *J. Math. Phys.* ; McEwen et coll., 2011, *IEEE Trans. Sig. Process.*)

Energy preservation:

$$\forall \boldsymbol{y} \in \mathbb{C}^{N+1}, \quad \int_{\mathbb{S}^2} |T\boldsymbol{y}(\vartheta,\varphi)|^2 \sin(\vartheta)\,\mathrm{d}\vartheta\mathrm{d}\varphi = \frac{4\pi}{N+1}\|\boldsymbol{y}\|^2$$

$$\Longrightarrow (\vartheta,\varphi) \mapsto T\boldsymbol{y}(\vartheta,\varphi) \textbf{ square integrable} \text{ on the sphere}$$



(Pascal et al., 2022, *IEEE Trans. Sig. Process.*)

Any function of $L^2(\mathbb{S}^2)$ decomposes onto the orthonormal basis of spin spherical harmonics: $\left\{ {}_s Y_m^\ell \,:\, s,m \in \mathbb{Z}, \ell \in \mathbb{N} \text{ such that } |s| \leq \ell, |m| \leq \ell \right\}$.

(Newman et coll., 1966, *J. Math. Phys.* ; McEwen et coll., 2011, *IEEE Trans. Sig. Process.*)

Theorem. Let $p \in (0,1)$, and $N \in \mathbb{N}$ even.

For any $\boldsymbol{y} \in \mathbb{C}^{N+1}$, let $\widetilde{\boldsymbol{y}} = \mathbf{r}_{\pi/2} * \boldsymbol{y}$, with $\mathbf{r}_{\pi/2}$ the rotation of Euler angles $(0,\pi/2,0)$

$$T\boldsymbol{y}(\vartheta,\,\varphi) = \mathcal{C}_N \mathrm{e}^{\mathrm{i}\frac{N\varphi}{2}} \sum_{n=0}^{N} {}_{\frac{N}{2}}Y^{\frac{N}{2}}_{n-\frac{N}{2}}(\pi-\vartheta,\varphi)\overline{\widetilde{y}[n]}, \quad \forall(\vartheta,\,\varphi) \in [0,\pi]\times[0,2\pi]$$

where $\mathcal{C}_N$ is an explicit normalization factor.

$T\boldsymbol{y}$ decomposes fully onto the $\boldsymbol{N/2}$-spin spherical harmonics only.

(Pascal, Flamant, Bardenet 2025, *GRETSI*)

Time–frequency interpretaion of spherical coordinates

Aligned Kravchuk transform: let $N \in \mathbb{N}$ even, and $\boldsymbol{y} \in \mathbb{C}^{N+1}$

$$\widetilde{T}\boldsymbol{y}(\vartheta, \varphi) = \sum_{n=0}^{N} {}_{\frac{N}{2}}Y^{\frac{N}{2}}_{n-\frac{N}{2}}(\vartheta, \varphi)\overline{y[n]}.$$

(Pascal, Flamant, Bardenet 2025, *GRETSI*)

Time–frequency interpretaion of spherical coordinates

Aligned Kravchuk transform: let $N \in \mathbb{N}$ even, and $\boldsymbol{y} \in \mathbb{C}^{N+1}$

$$\widetilde{T}\boldsymbol{y}(\vartheta, \varphi) = \sum_{n=0}^{N} {}_{\frac{N}{2}}Y^{\frac{N}{2}}_{n-\frac{N}{2}}(\vartheta, \varphi)\overline{y[n]}.$$

(Pascal, Flamant, Bardenet 2025, *GRETSI*)

- fast and stable algorithms to compute ${}_{\frac{N}{2}}Y^{\frac{N}{2}}_{n-\frac{N}{2}}(\vartheta, \varphi)$: pyssht

Time–frequency interpretaion of spherical coordinates

Aligned Kravchuk transform: let $N \in \mathbb{N}$ even, and $\boldsymbol{y} \in \mathbb{C}^{N+1}$

$$\widetilde{T}\boldsymbol{y}(\vartheta, \varphi) = \sum_{n=0}^{N} {}_{\frac{N}{2}}Y^{\frac{N}{2}}_{n-\frac{N}{2}}(\vartheta, \varphi)\overline{y[n]}.$$

(Pascal, Flamant, Bardenet 2025, *GRETSI*)

- fast and stable algorithms to compute ${}_{\frac{N}{2}}Y^{\frac{N}{2}}_{n-\frac{N}{2}}(\vartheta, \varphi)$: pyssht
- directly expressed in terms of $\boldsymbol{y}$: no costly "rotation" $\mathbf{r}_\beta$ needed

Time–frequency interpretaion of spherical coordinates

Aligned Kravchuk transform: let $N \in \mathbb{N}$ even, and $\boldsymbol{y} \in \mathbb{C}^{N+1}$

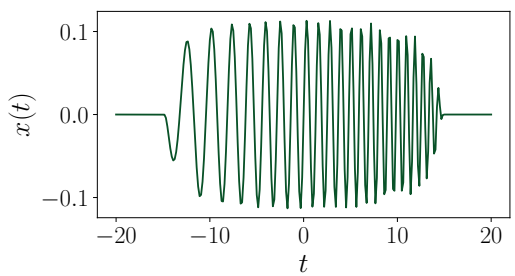
$$\widetilde{T}\boldsymbol{y}(\vartheta, \varphi) = \sum_{n=0}^{N} {}_{\frac{N}{2}}Y^{\frac{N}{2}}_{n-\frac{N}{2}}(\vartheta, \varphi)\overline{y[n]}.$$

(Pascal, Flamant, Bardenet 2025, *GRETSI*)

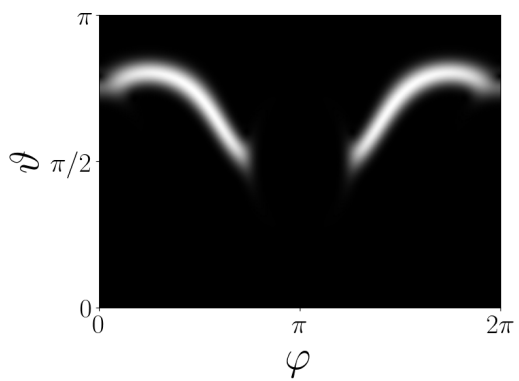
- fast and stable algorithms to compute ${}_{\frac{N}{2}}Y^{\frac{N}{2}}_{n-\frac{N}{2}}(\vartheta, \varphi)$: pyssht
- directly expressed in terms of $\boldsymbol{y}$: no costly "rotation" $\mathbf{r}_\beta$ needed
- variables (ϑ, φ) aligned with the time–frequency axes

  ϑ : time variable

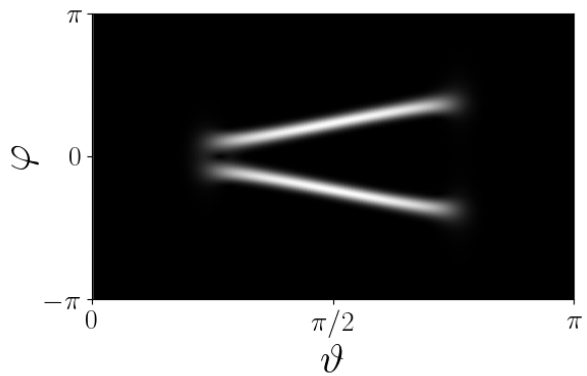
  φ : frequency variable



signal $\boldsymbol{y}$

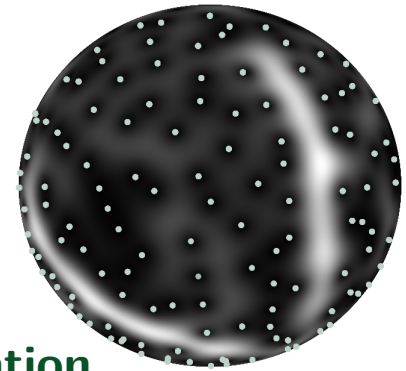


original transform $|T\boldsymbol{y}|^2$



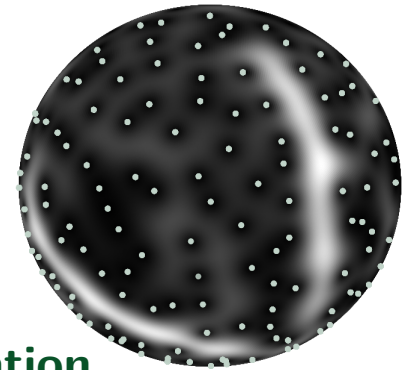
aligned transform $|\widetilde{T}\boldsymbol{y}|^2$

Take home messages



- **Novel covariant discrete Kravchuk transform** $T\boldsymbol{y}(\vartheta, \varphi)$
  - Interpreted as a coherent state decomposition,
  - Representation on a compact phase space,
  - **Aligned version with proper time–frequency interpretation**,
  - Fast, **scalable to long signals**, and robust implementation,
  - Zeros of the Kravchuk spectrogram of white noise fully characterized.

Take home messages



- **Novel covariant discrete Kravchuk transform $T\boldsymbol{y}(\vartheta, \varphi)$**
  - Interpreted as a coherent state decomposition,
  - Representation on a compact phase space,
  - **Aligned version with proper time–frequency interpretation**,
  - Fast, **scalable to long signals**, and robust implementation,
  - Zeros of the Kravchuk spectrogram of white noise fully characterized.

- **Signal detection based on spectrogram zeros**
  - Preliminary work using the zeros of the Fourier spectrogram,
  - Significant improvement using the Kravchuk spectrogram.

(Pascal & Bardenet, 2022, *IEEE Trans. Signal Process.* `arxiv:2202.03835`;
Pascal & Bardenet, 2022, *GRETSI*;
Pascal, Flamant & Bardenet, 2025, *GRETSI*;
Pascal & Bardenet, 2025, "Point processes and spatial statistics in time-frequency analysis")

`https://github.com/bpascal-fr/kravchuk-transform-and-its-zeros`
`https://github.com/bpascal-fr/from-kravchuk-to-ssht`
`https://github.com/bpascal-fr/GeoSto-PP-for-TF`

Work in progress and perspectives

- **real-world problems** in signal processing from $N \geq 10^3$ to 10^4

  signal **detection** from aligned Kravchuk transform zeros

  denoising by filtering using the fast implementation of the inverse transform

- surveillance of **bird populations** and migration monitoring

  annotated dataset of $70 \cdot 10^3$ recordings: bird call or not **BirdVox** (Lostanlen et al., 2018)

- **asymptotic** behvaior of the aligned Kravchuk transform as $N \to \infty$

  covariant discretization of the Short Time Fourier Transform?

(Pascal & Bardenet, 2022, *IEEE Trans. Signal Process.* `arxiv:2202.03835`;
Pascal & Bardenet, 2022, *GRETSI*;
Pascal, Flamant & Bardenet, 2025, *GRETSI*;
Pascal & Bardenet, 2025, "Point processes and spatial statistics in time-frequency analysis")

`https://github.com/bpascal-fr/kravchuk-transform-and-its-zeros`
`https://github.com/bpascal-fr/from-kravchuk-to-ssht`
`https://github.com/bpascal-fr/GeoSto-PP-for-TF`