Texture segmentation based on fractal attributes using convex functional minimization with generalized Stein formalism for automated regularization parameter selection.

Barbara Pascal

October 12$^{\text{th}}$ 2021

Under the supervision of **Patrice Abry** and **Nelly Pustelnik**

Collaboration with **Valérie Vidal** and **Samuel Vaiter**

Image segmentation

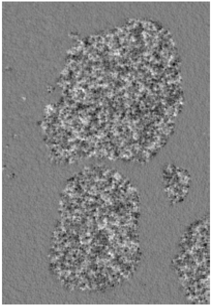
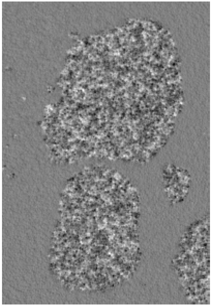
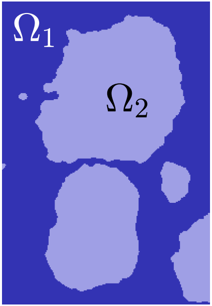


Image segmentation

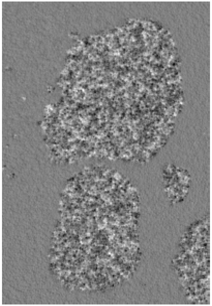


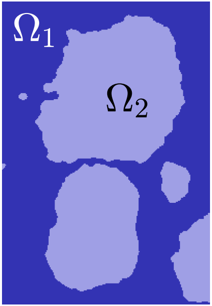


Goal: obtain a partition of the image into K homogeneous regions

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_K$$

Image segmentation





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Textures

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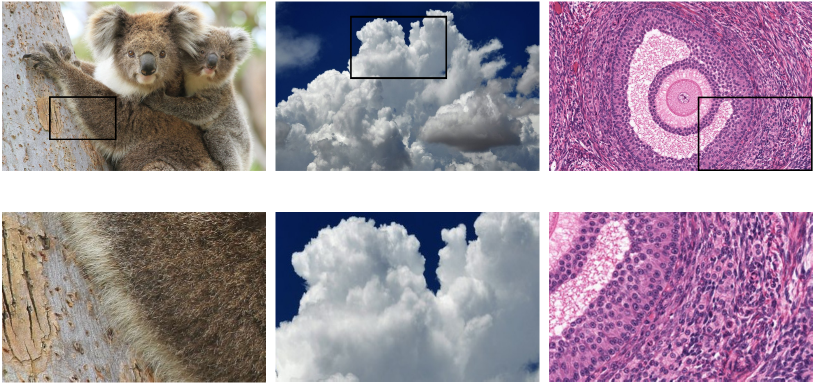

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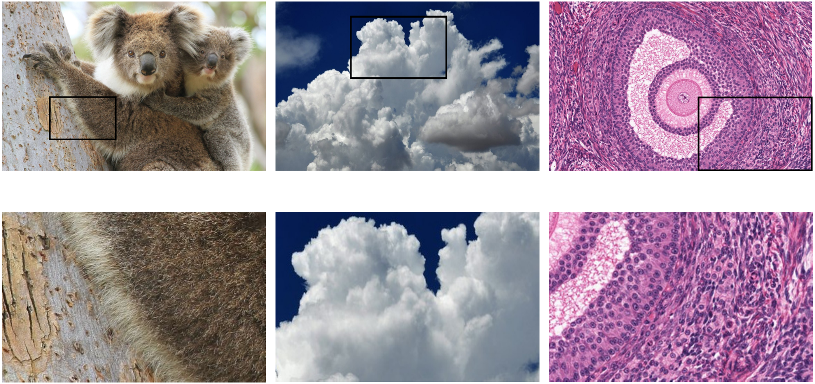
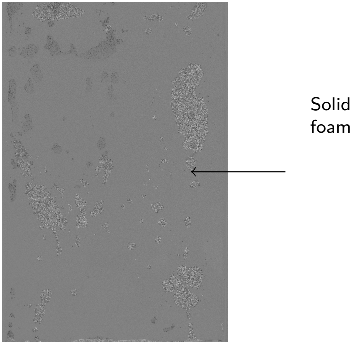


Crucial to describe real-world images

Multiphase flow through porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



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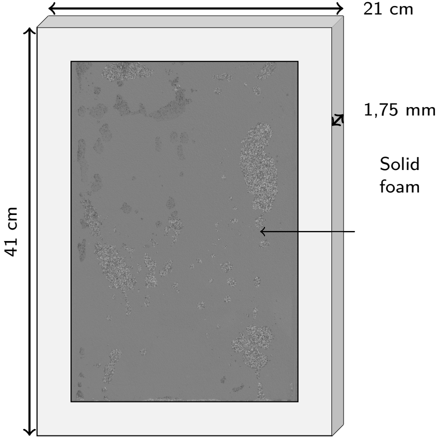

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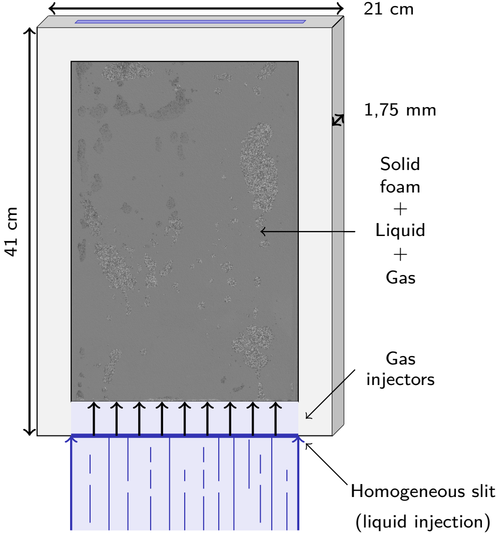


→ 1600 × 1100 pixels
→ video: ∼ 1000 images
→ phase diagram: ∼ 10 flow rates

Presentation outline

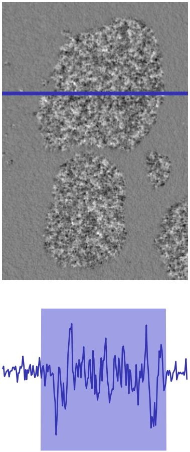
1. Texture characterization: *fractals attributes*
   - local variance σ^2
   - local regularity h

2. Design of functionals: *penalized least-squares*
   - free contours
   - co-localized contours

3. Accelerated minimization algorithms: *splitting proximal algorithms*
   - computation of proximal operators
   - acceleration based on strong-convexity

4. Hyperparameters tuning: *SURE under Gaussian correlated noise*
   - projected estimation error
   - quasi-Newton minimization

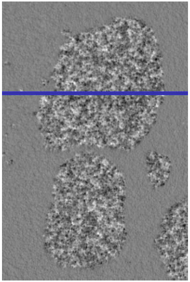
Piecewise monofractal model

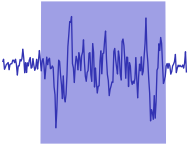


Piecewise monofractal model

Fractals attributes

- variance σ^2 *amplitude of variations*

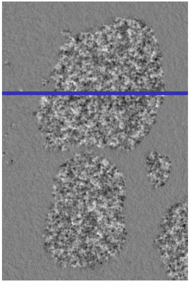


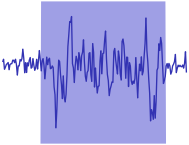


Piecewise monofractal model

Fractals attributes

- variance σ^2 — *amplitude of variations*
- local regularity h — *scale invariance*





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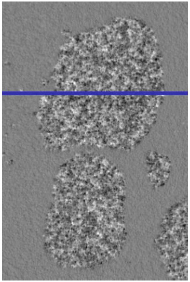
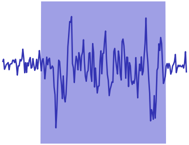
$$|f(x) - f(y)| \leq \sigma(x)|x - y|^{h(x)}$$





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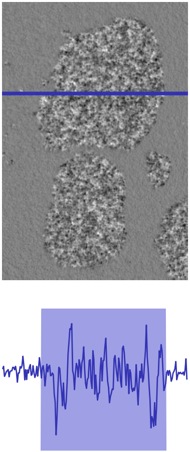

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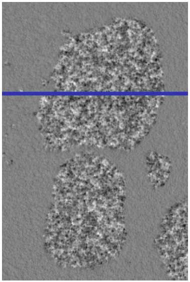
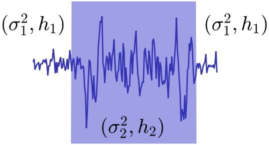


Segmentation

- σ^2 and h piecewise constant



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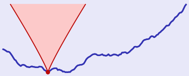

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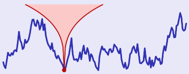

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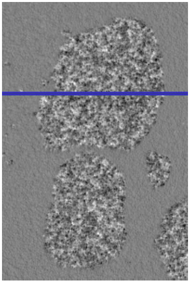
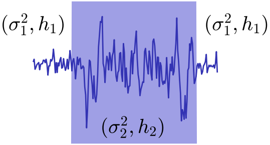
Segmentation

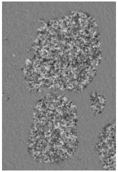
- σ^2 and h piecewise constant
- region Ω_k characterized by (σ_k^2, h_k)





Multiscale analysis

Textured image



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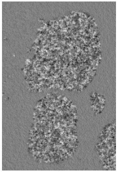

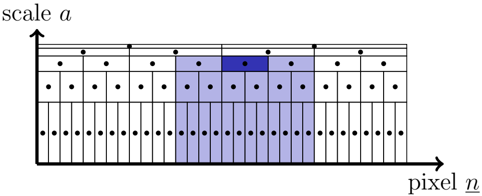

Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$

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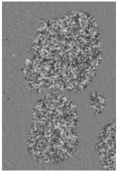

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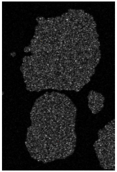

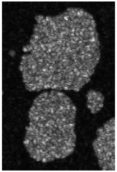

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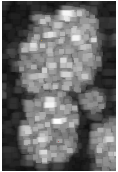

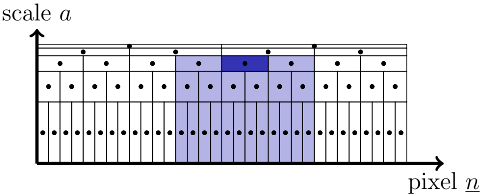

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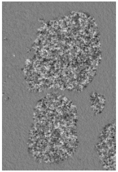

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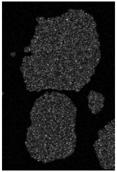

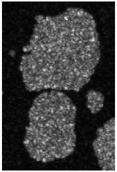

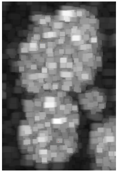

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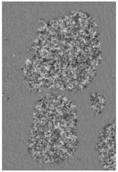

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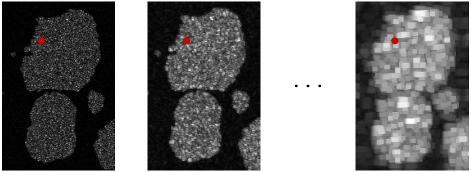

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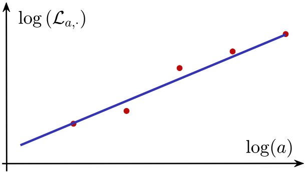

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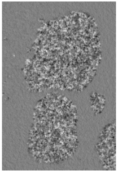

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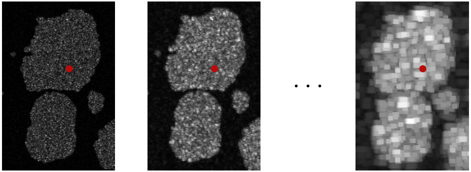

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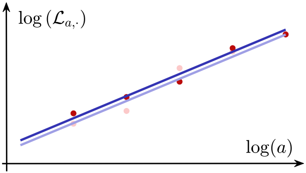

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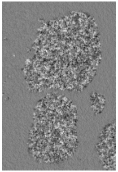

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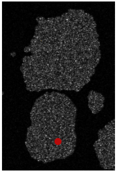

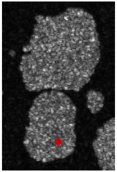

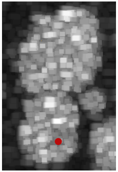

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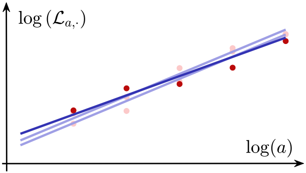

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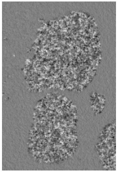

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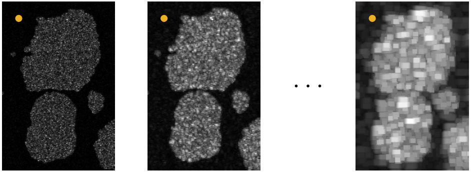

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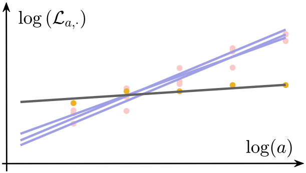

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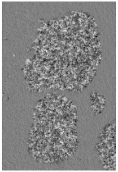

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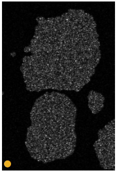
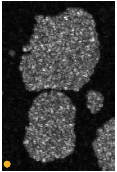
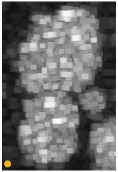


$a = 2^2$



...

$a = 2^5$



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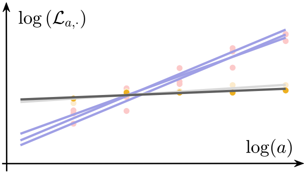

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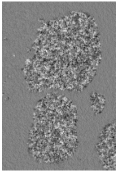

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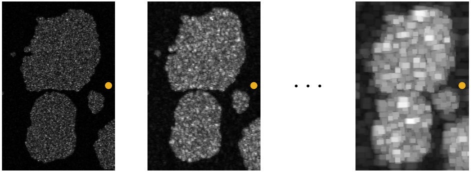

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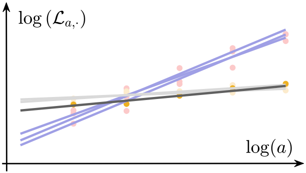
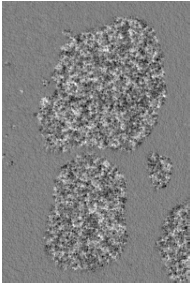


Direct punctual estimation

Linear regression $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

Textured image

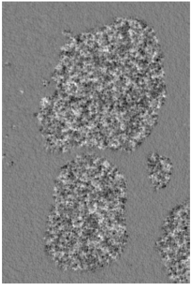


Direct punctual estimation

Linear regression $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Textured image

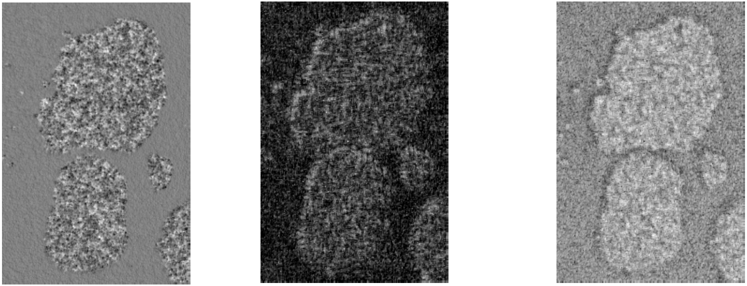


Direct punctual estimation

Linear regression $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Textured image | Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$

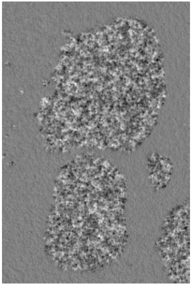


Direct punctual estimation

Linear regression $\underbrace{\mathbb{E}\log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{expected value}} = \log(a)\underbrace{\bar{\boldsymbol{h}}}_{\text{regularity}} + \underbrace{\bar{\boldsymbol{v}}}_{\propto\log(\boldsymbol{\sigma}^2)}$

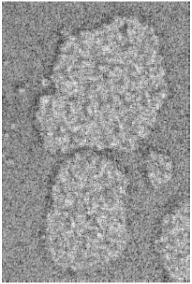
$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Textured image



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$



$\longrightarrow$ large estimation variance

A posteriori regularization

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

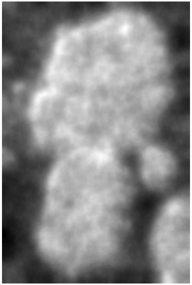
A posteriori regularization

Filter smoothing (linear)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Lissage



A posteriori regularization

Filter smoothing (linear)

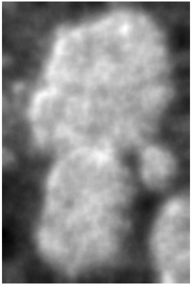
$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

ROF denoising (nonlinear)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \; \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Lissage



ROF

A posteriori regularization

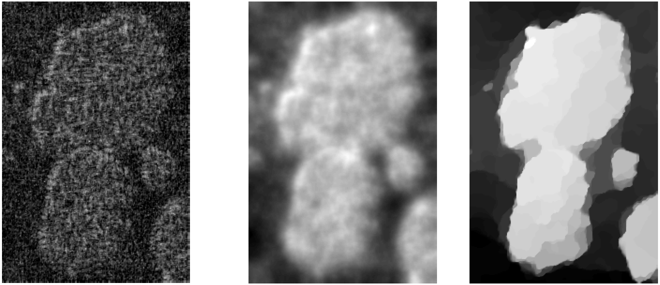
Filter smoothing (linear)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{LR}}$$

ROF denoising (nonlinear)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \ \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Linear regression $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Lissage | ROF

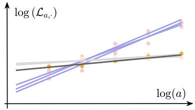


$\longrightarrow$ cumulative estimation variance and regularization bias

Functionals with either free or co-localized contours

$$\sum_a \underbrace{\|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}}$$

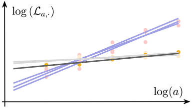
$\rightarrow$ fidelity to the log-linear model



Functionals with either free or co-localized contours

$$\underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

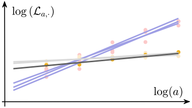


→ favors piecewise constancy

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model



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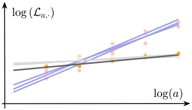
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

→ favors piecewise constancy



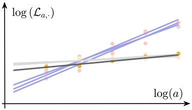
Finite differences $\mathbf{D}_1\boldsymbol{x}$ (horizontal), $\mathbf{D}_2\boldsymbol{x}$ (vertical) in each pixel

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

→ favors piecewise constancy



Finite differences $\mathbf{D}\boldsymbol{x} = [\mathrm{D}_1\boldsymbol{x}, \mathrm{D}_2\boldsymbol{x}]$

Free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

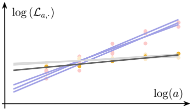
$$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

→ favors piecewise constancy



Finite differences $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x}, \mathbf{D}_2\boldsymbol{x}]$

Free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

$$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

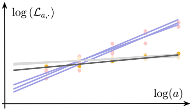
Co-localized: $\boldsymbol{h}$, $\boldsymbol{v}$ are **concomitantly** piecewise constant

$$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha) = \|[\alpha\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}]\|_{2,1}$$

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \; \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Least-Squares → fidelity to the log-linear model



Total Variation → favors piecewise constancy

Disjoint contours

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

Common contours

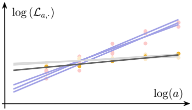
$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

→ favors piecewise constancy



Disjoint contours

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_{\mathsf{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

Common contours

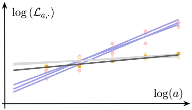
$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_{\mathsf{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Least-Squares → fidelity to the log-linear model



Total Variation → favors piecewise constancy

Disjoint contours

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad \boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 2 + \sqrt{2} \simeq 3.4$

Common contours

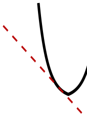
$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad \boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 2\sqrt{2} \simeq 2.8$

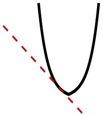
Functionals minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



Functionals minimization

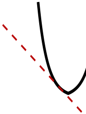
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \ + \ \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



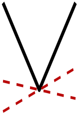
- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$

Functionals minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



nonsmooth

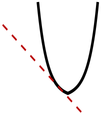


- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau\nabla\varphi(\boldsymbol{x}^n)$
- implicit subgradient descent: proximal point algorithm

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau\boldsymbol{u}^n, \ \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \ \Leftrightarrow \ \boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

Functionals minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \ + \ \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



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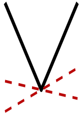

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- splitting proximal algorithm

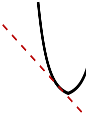
$$\boldsymbol{y}^{n+1} = \mathrm{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

$$\boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau \mathbf{D}^\top \boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

Functionals minimization

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



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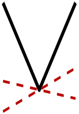


- gradient descent $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$
- implicit subgradient descent: proximal point algorithm

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \boldsymbol{u}^n, \;\; \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \;\Leftrightarrow\; \boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

- splitting proximal algorithm $\quad \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}) = \underset{\boldsymbol{u}}{\mathrm{argmin}} \, \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{u}\|^2 + \tau\varphi(\boldsymbol{u})$

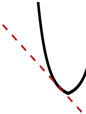
$$\boldsymbol{y}^{n+1} = \mathrm{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma\mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

$$\boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}}-\boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau\mathbf{D}^\top\boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



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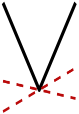
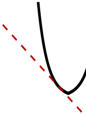


Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



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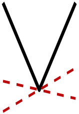
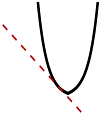


Ex. **Mixed norm:** for $\boldsymbol{z} = [\boldsymbol{z}_1; \ldots, ; \boldsymbol{z}_I]$

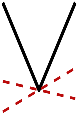
$$\mathcal{Q}(\boldsymbol{z}) = \|\boldsymbol{z}\|_{2,1} = \sum_{\underline{n}\in\Omega} \sqrt{\sum_{i=1}^{I} z_i^2(\underline{n})} = \sum_{\underline{n}\in\Omega} \|\boldsymbol{z}(\underline{n})\|_2$$

Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



nonsmooth



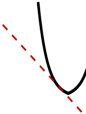
Ex. Mixed norm: for $\boldsymbol{z} = [\boldsymbol{z}_1; \dots; \boldsymbol{z}_I]$

$$\mathcal{Q}(\boldsymbol{z}) = \|\boldsymbol{z}\|_{2,1} = \sum_{\underline{n}\in\Omega} \sqrt{\sum_{i=1}^{I} z_i^2(\underline{n})} = \sum_{\underline{n}\in\Omega} \|\boldsymbol{z}(\underline{n})\|_2$$

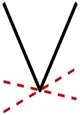
$$\boldsymbol{p} = \mathrm{prox}_{\lambda\|\cdot\|_{2,1}}(\boldsymbol{z}) \quad \Leftrightarrow \quad p_i(\underline{n}) = \max\left(0, 1 - \frac{\lambda}{\|\boldsymbol{z}(\underline{n})\|_2}\right) z_i(\underline{n})$$

Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



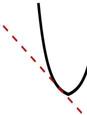
nonsmooth



Least-Squares: $\|\log \boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}(\boldsymbol{h}, \boldsymbol{v})\|^2, \quad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



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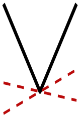


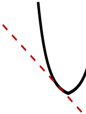
Least-Squares: $\|\log \mathcal{L} - \Phi(\boldsymbol{h}, \boldsymbol{v})\|^2, \quad \Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Proposition *(Pascal, 2019)*

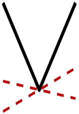
$$(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \mathrm{prox}_{\tau\|\mathcal{L} - \Phi\cdot\|^2}(\boldsymbol{h}, \boldsymbol{v}) \Longleftrightarrow (\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \left(\mathbf{I} + \tau\Phi^\top\Phi\right)^{-1}\left((\boldsymbol{h}, \boldsymbol{v}) + \tau\Phi^\top \log \mathcal{L}\right)$$

Computation of proximal operators

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



nonsmooth



Least-Squares: $\|\log \boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}(\boldsymbol{h}, \boldsymbol{v})\|^2, \quad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Proposition *(Pascal, 2019)*

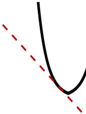
Let $S_m = \sum_a \log^m(a)$, $\mathcal{D} = (1 + \tau S_2)(1 + \tau S_0) - \tau^2 S_1^2$,
$\boldsymbol{\mathcal{T}} = \sum_a \log \boldsymbol{\mathcal{L}}_a$ and $\boldsymbol{\mathcal{G}} = \sum_a \log(a) \log \boldsymbol{\mathcal{L}}_a$, alors

$$(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \text{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|^2}(\boldsymbol{h}, \boldsymbol{v}) \iff (\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}) = \left(\mathbf{I} + \tau \boldsymbol{\Phi}^\top \boldsymbol{\Phi}\right)^{-1} \left((\boldsymbol{h}, \boldsymbol{v}) + \tau \boldsymbol{\Phi}^\top \log \boldsymbol{\mathcal{L}}\right)$$

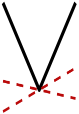
$$\iff \begin{cases} \widetilde{\boldsymbol{h}} = \mathcal{D}^{-1}\left((1 + \tau S_0)(\tau \boldsymbol{\mathcal{G}} + \boldsymbol{h}) - \tau S_1(\tau \boldsymbol{\mathcal{T}} + \boldsymbol{v})\right) \\ \widetilde{\boldsymbol{v}} = \mathcal{D}^{-1}\left((1 + \tau S_2)(\tau \boldsymbol{\mathcal{T}} + \boldsymbol{v}) - \tau S_1(\tau \boldsymbol{\mathcal{G}} + \boldsymbol{h})\right) \end{cases}$$

Accelerated algorithm based on strong-convexity

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \;\; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;+\; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



nonsmooth



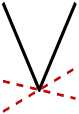
Primal-dual algorithm *(Chambolle, 2011)*

$$\delta\text{: duality gap, } \delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$$

Accelerated algorithm based on strong-convexity

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

nonsmooth



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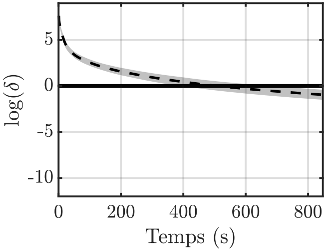
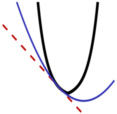


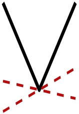
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



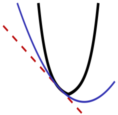
μ-strongly convex

nonsmooth



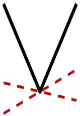
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



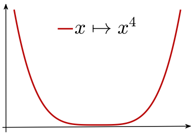
$\boldsymbol{\mu}$-strongly convex

nonsmooth

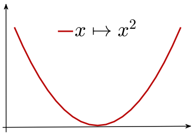


Strong-convexity

- φ $\boldsymbol{\mu}$-strongly convex iff $\varphi - \dfrac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convex



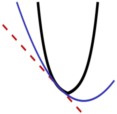
✓ strictly convex
✗ non strongly convex



✓ strictly convex
✓ 1-strongly convex

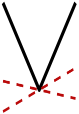
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



μ-strongly convex

nonsmooth

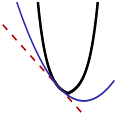


Strong-convexity

- φ μ-strongly convex iff $\varphi - \frac{\mu}{2}\|\cdot\|^2$ convex
- φ $\mathcal{C}^2$ with Hessian matrix $\boldsymbol{H}\varphi \succeq 0 \implies \mu = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

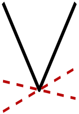
Convexity properties

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



μ-strongly convex

nonsmooth



Strong-convexity

- φ μ-strongly convex iff $\varphi - \dfrac{\mu}{2}\|\cdot\|^2$ convex
- φ $\mathcal{C}^2$ with Hessian matrix $\boldsymbol{H}\varphi \succeq 0 \implies \mu = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

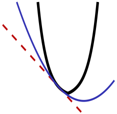
Proposition *(Pascal, 2019)*

$\sum_a \|\log \boldsymbol{\mathcal{L}} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2$ est μ-strongly convex.

$a_{\min} = 2^1$, $a_{\max}$	2^2	2^3	2^4	2^5	2^6
$\mu = \min \mathrm{Sp}\left(2\boldsymbol{\Phi}^\top \boldsymbol{\Phi}\right)$	0.29	**0.72**	1.20	1.69	2.20

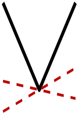
Accelerated algorithm based on strong-convexity

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



μ-strongly convex

nonsmooth



Accelerated Primal-dual algorithm *(Chambolle, 2011)*

for $n = 0, 1, \dots$ $\qquad\qquad \boldsymbol{x} = (\boldsymbol{h}, \boldsymbol{v})$

$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma_n(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma_n \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

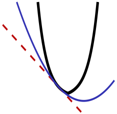
$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau_n\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau_n \mathbf{D}^\top \boldsymbol{y}^{n+1}\right)$$

$$\theta_n = \sqrt{1 + 2\mu\tau_n}, \quad \tau_{n+1} = \tau_n/\theta_n, \quad \sigma_{n+1} = \theta_n\sigma_n$$

$$\bar{\boldsymbol{x}}^{n+1} = \boldsymbol{x}^{n+1} + \theta_n^{-1}\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

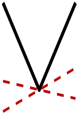
Algorithme accéléré par forte-convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$



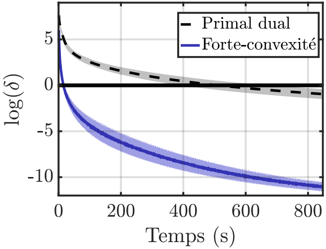
μ-strongly convex

nonsmooth



Accelerated Primal-dual algorithm *(Chambolle, 2011)*

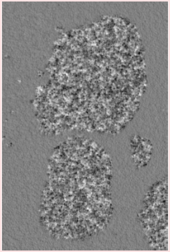
δ: duality gap, $\delta(\boldsymbol{x}^n, \boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$



Segmentation *via* iterated thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;+\; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image

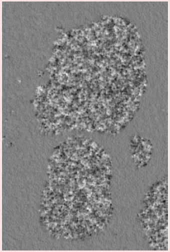


Segmentation *via* iterated thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image

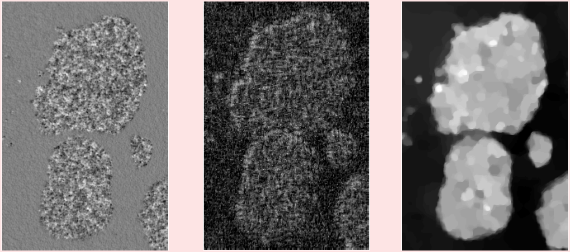
Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$



Segmentation *via* iterated thresholding

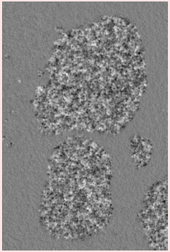
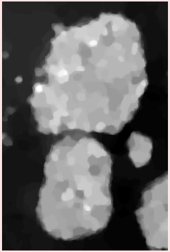
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image | Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Co-localized contours $\widehat{\boldsymbol{h}}^{\mathrm{C}}$



Segmentation *via* iterated thresholding

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image	Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$	Co-localized contours $\widehat{\boldsymbol{h}}^{\mathrm{C}}$	Threshold estimate† $T\widehat{\boldsymbol{h}}^{\mathrm{C}}$
			

† (Cai, 2013)

State-of-the-art methods for texture segmentation

Threshold-ROF on $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

(Nafornita, 2014), (Pustelnik, 2016)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Lin. reg.

ROF

Threshold

Only based on regularity $\boldsymbol{h}$.

State-of-the-art methods for texture segmentation

Threshold-ROF on $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

(Nafornita, 2014), (Pustelnik, 2016)

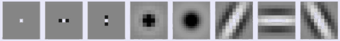
$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \; \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Lin. reg. | ROF | Threshold

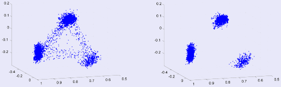
Only based on regularity $\boldsymbol{h}$.

Factorization based segmentation[†] *(Yuan, 2015)*

(i) local histograms



(ii) matrix factorization



(a) (b)

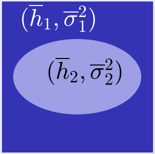
Fig. 2. Scatterplot of features in subspace. (a) Scatterplot of features projected onto the 3-d subspace. (b) Scatterplot after removing features with high edgeness.

[†] `https://sites.google.com/site/factorizationsegmentation/`

Compared segmentation performance on synthetic textures

Piecewise monofractal texture synthesis *(Pascal, 2019)*

- mask: $\Omega = \Omega_1 \sqcup \Omega_2$,
- attributes: $(\overline{h}_k, \overline{\sigma}_k^2)_{k=1,2}$

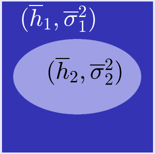


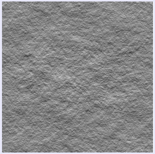
Compared segmentation performance on synthetic textures

Piecewise monofractal texture synthesis *(Pascal, 2019)*

- mask: $\Omega = \Omega_1 \sqcup \Omega_2$,
- attributes: $(\overline{h}_k, \overline{\sigma}_k^2)_{k=1,2}$

Ex. $\overline{h}_1 = 0.5$, $\overline{\sigma}_1^2 = 0.6$
$\overline{h}_2 = 0.6$, $\overline{\sigma}_2^2 = 0.7$





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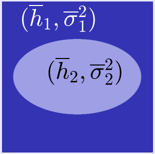
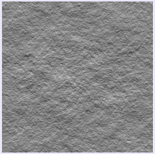
Ex. $\bar{h}_1 = 0.5$, $\bar{\sigma}_1^2 = 0.6$
$\bar{h}_2 = 0.6$, $\bar{\sigma}_2^2 = 0.7$





Averaged segmentation performances over 5 realizations

Yuan

$71.1 \pm 1.3\%$

T-ROF

$78.5 \pm 1.1\%$

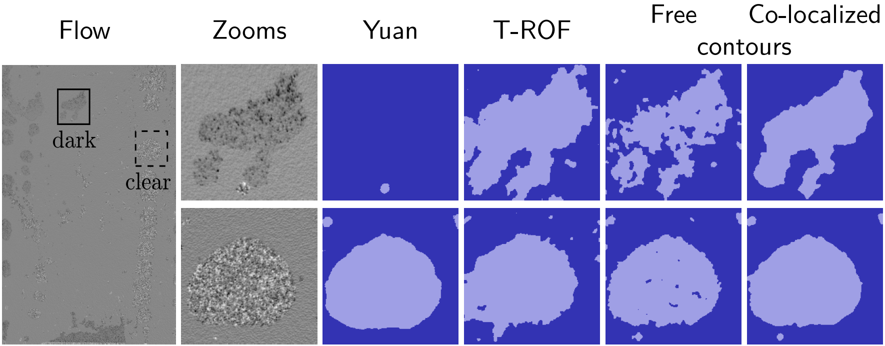
free contours

$90.2 \pm 1.9\%$

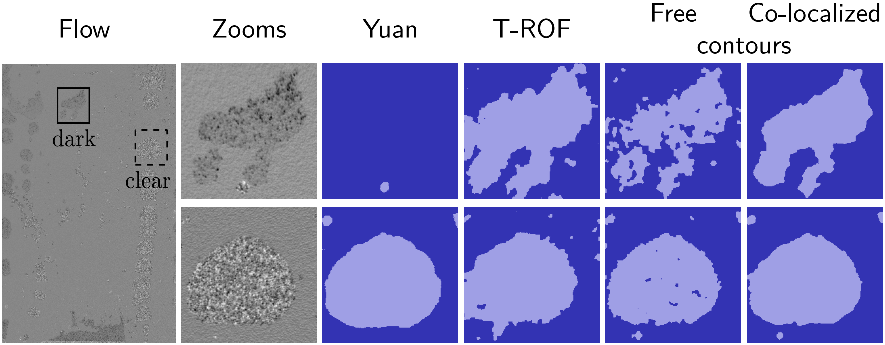
co-localized contours

$91.1 \pm 1.5\%$

Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$



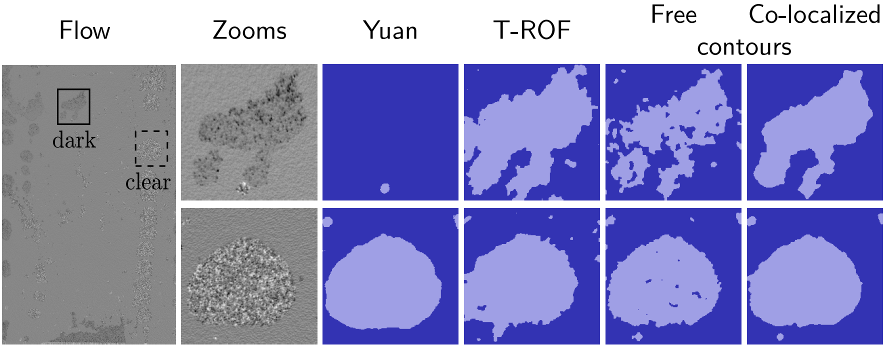
Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$



Liquid: $h_{\mathrm{L}} = 0.4$

Gas: $h_{\mathrm{G}} = 0.9$

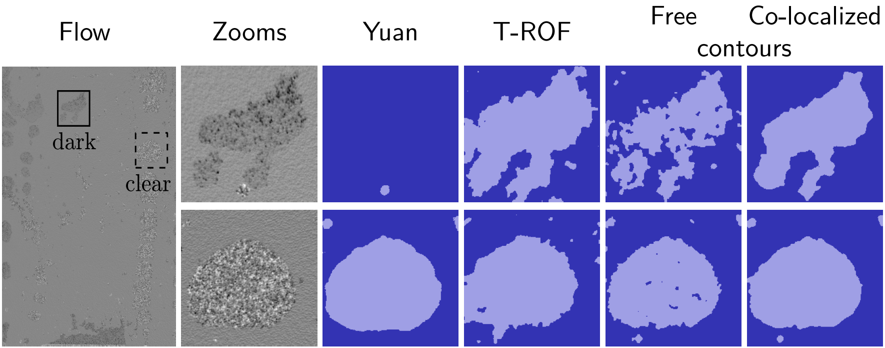
Low activity: $Q_{\mathrm{G}} = 300$mL/min - $Q_{\mathrm{L}} = 300$mL/min



Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$

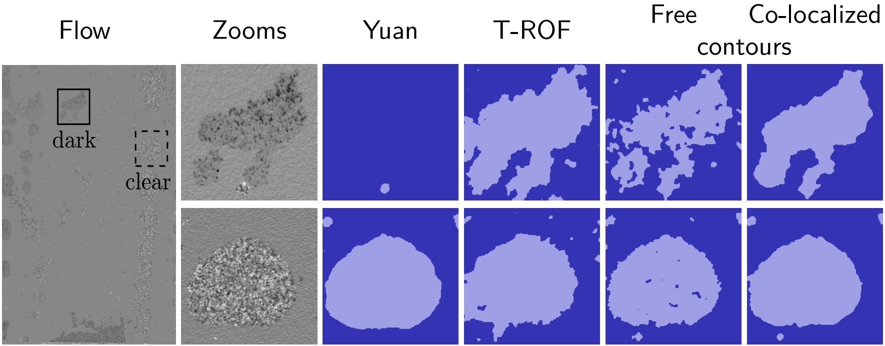
Low activity: $Q_{\mathrm{G}} = 300\text{mL}/\text{min}$ - $Q_{\mathrm{L}} = 300\text{mL}/\text{min}$



Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$

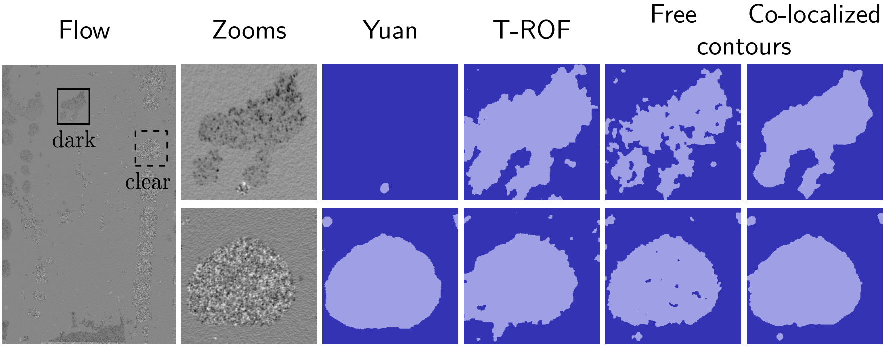
Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$



Liquid: $h_{\mathrm{L}} = 0.4$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$ (dark bubbles)

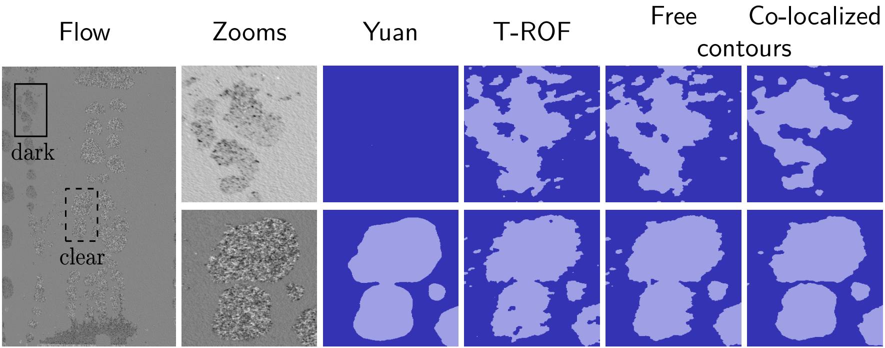
Low activity: $Q_{\mathrm{G}} = 300\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$



Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ | $\sigma^2_{\mathrm{dark}} = 10^{-2}$ (dark bubbles)
$\sigma^2_{\mathrm{clear}} = 10^{-1}$ (clear bubbles)

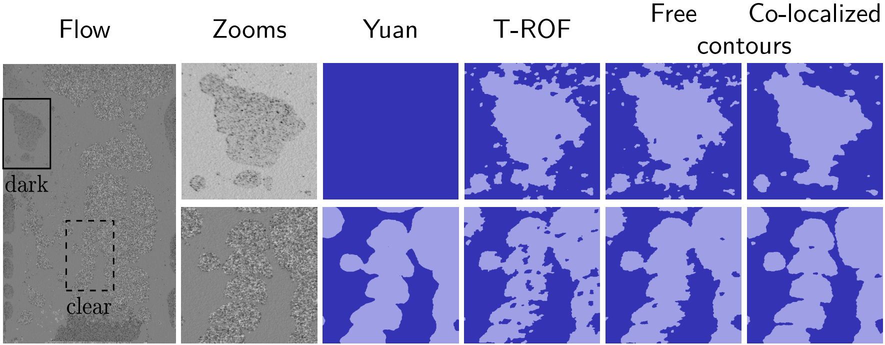
Transition: $Q_{\mathrm{G}} = 400\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 700\mathrm{mL}/\mathrm{min}$



Liquid: $h_{\mathrm{L}} = 0.4$ $\sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

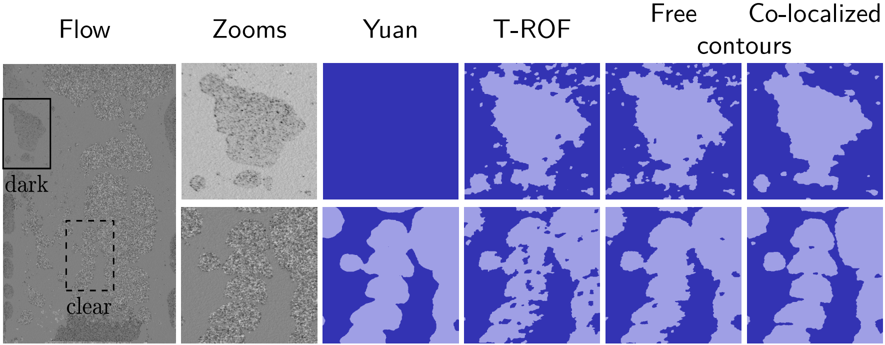
High activity: $Q_{\mathrm{G}} = 1200\mathrm{mL/min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL/min}$



Liquid: $h_{\mathrm{L}} = 0.4 \qquad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

High activity: $Q_{\mathrm{G}} = 1200\text{mL}/\text{min}$ - $Q_{\mathrm{L}} = 300\text{mL}/\text{min}$



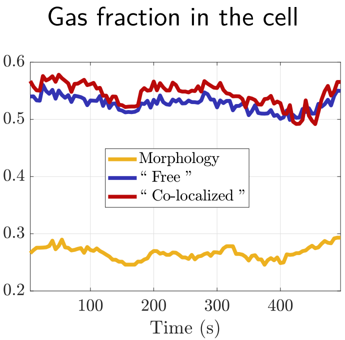
Computational load	1s	12s	700s	**2100s**

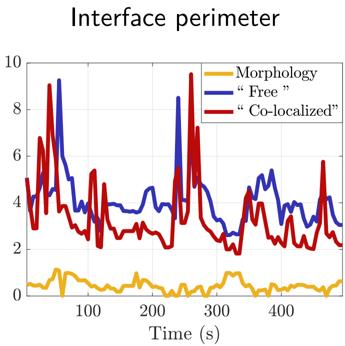
Liquid: $h_{\mathrm{L}} = 0.4$ $\quad \sigma^2_{\mathrm{dark}} = 10^{-2}$

Gas: $h_{\mathrm{G}} = 0.9$ $\quad \left| \begin{array}{ll} \sigma^2_{\mathrm{dark}} = 10^{-2} & \text{(dark bubbles)} \\ \sigma^2_{\mathrm{clear}} = 10^{-1} & \text{(clear bubbles).} \end{array} \right.$

Multiphase flow through porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)





Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

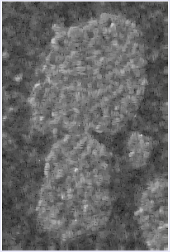
$(\lambda, \alpha) = (0, 0)$

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$ $(\lambda, \alpha) = (0.5, 0.5)$



too small

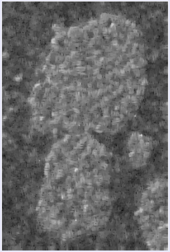
Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

$(\lambda, \alpha) = (500, 500)$

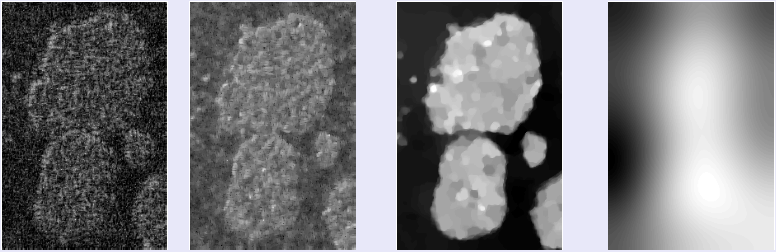
too large

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$ | $(\lambda, \alpha) = (0.5, 0.5)$ | $(\lambda^{\dagger}, \alpha^{\dagger}) = (11.5, 0.8)$ | $(\lambda, \alpha) = (500, 500)$



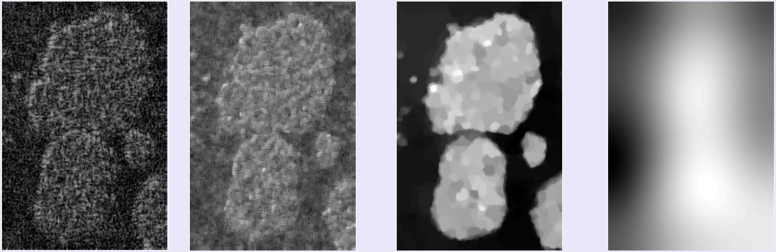
too small | optimal | too large

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\mathcal{L}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ — Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$ — $(\lambda, \alpha) = (0.5, 0.5)$ — $(\lambda^\dagger, \alpha^\dagger) = (11.5, 0.8)$ — $(\lambda, \alpha) = (500, 500)$



too small — optimal — too large

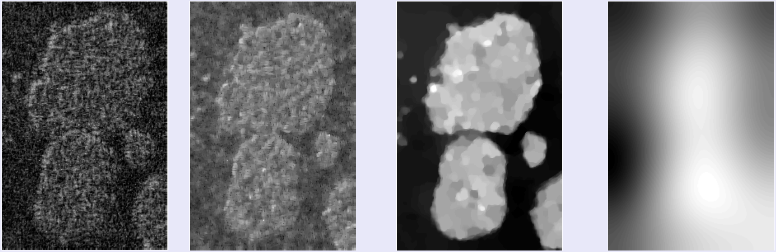
What *optimal* means?

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\mathcal{L}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ | Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$ | $(\lambda, \alpha) = (0.5, 0.5)$ | $(\lambda^\dagger, \alpha^\dagger) = (11.5, 0.8)$ | $(\lambda, \alpha) = (500, 500)$



too small | optimal | too large

What *optimal* means? How to determine $\lambda^\dagger$ and $\alpha^\dagger$?

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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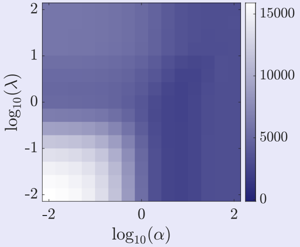
$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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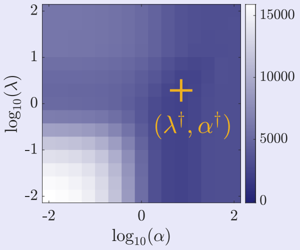

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

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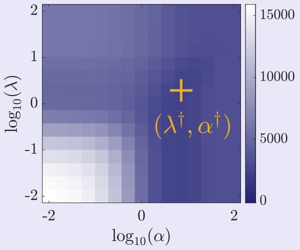
$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

?

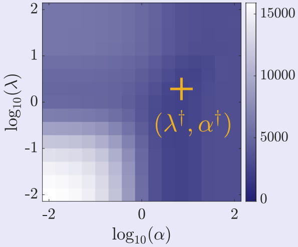
Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliary*

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

?

Stein Unbiased Risk Estimate
(SURE)

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \rho^2 \mathbf{I})$

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex.
$$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \operatorname*{argmin}_{\boldsymbol{x}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Quadratic error $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \overset{?}{=} \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ unknown

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

Quadratic error $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \overset{?}{=} \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ unknown

Theorem *(Stein, 1981)*

Let $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\zeta \mapsto \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \zeta; \lambda), \zeta \rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$.

$$\widehat{R}(\boldsymbol{y}; \lambda) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\rho^2 \operatorname{tr}\left(\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)\right) - \rho^2 P$$
$$\Longrightarrow R(\lambda) = \mathbb{E}_\zeta[\widehat{R}(\boldsymbol{y}; \lambda)].$$

Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \Phi \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\Phi : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \mathcal{S})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \mathcal{L} = \Phi(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$

$\Phi : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

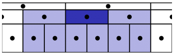
Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \mathcal{S})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\quad\quad$ $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \mathcal{S})$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



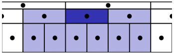
Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\quad$ $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$ $\quad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

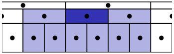
Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\qquad$ $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$ $\qquad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

$R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N,\ \boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\| \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}}) \right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top
\end{aligned}
$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

$$\begin{aligned} R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\ &= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\ &= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \end{aligned}$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2
\end{aligned}
$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right]
\end{aligned}$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \mathcal{S})$

$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right]
\end{aligned}$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\bar{\boldsymbol{x}}+\boldsymbol{\zeta}), \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right]
\end{aligned}
$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A} \qquad \boldsymbol{\zeta}\ , \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right]
\end{aligned}
$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

$$\mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top \boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top$$

$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A} \quad \boldsymbol{\zeta}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathbb{E}_{\boldsymbol{\zeta}}\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - \mathbb{E}_{\boldsymbol{\zeta}}\|\mathbf{A}\boldsymbol{\zeta}\|^2
\end{aligned}$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}\qquad\boldsymbol{\zeta}\,, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathbb{E}_{\boldsymbol{\zeta}}\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)
\end{aligned}
$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \; \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A} \qquad \boldsymbol{\zeta}\;, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathbb{E}_{\boldsymbol{\zeta}} \left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - \underbrace{\operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \zeta)\|^2 \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\zeta\right\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\zeta\right\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\right\rangle - 2\left\langle \mathbf{A} \qquad \zeta\ , \mathbf{A}\zeta\right\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \underbrace{\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2}_{\text{accessible}} + 2\mathbb{E}_\zeta \left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\right\rangle - \underbrace{\operatorname{tr}\left(\mathbf{A}\mathcal{S}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - 2\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - 2\langle \mathbf{A} \qquad \boldsymbol{\zeta}\ , \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \underbrace{\mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2}_{\text{accessible}} + 2\mathbb{E}_{\boldsymbol{\zeta}} \langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - \underbrace{\operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}
$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - 2\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - 2\langle \mathbf{A} \qquad \boldsymbol{\zeta}\ , \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \underbrace{\mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2}_{\text{accessible}} + 2\mathbb{E}_\zeta \langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - \underbrace{\operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}
$$

$$
\mathbb{E}_\zeta \langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle = \mathbb{E}_\zeta \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle
$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\zeta}\|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\zeta}\left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\zeta}\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\zeta}\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\zeta}\left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\zeta}\left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle\mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - 2\langle\mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\zeta}\left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle\mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - 2\langle\mathbf{A}\qquad\boldsymbol{\zeta}, \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \underbrace{\mathbb{E}_{\zeta}\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2}_{\text{accessible}} + 2\mathbb{E}_{\zeta}\langle\mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - \underbrace{\operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_{\zeta}\langle\mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle &= \mathbb{E}_{\zeta}\langle\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle \\
&= \int \langle\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta};\lambda), \mathbf{A}\boldsymbol{\zeta}\rangle \exp\left(-\frac{\boldsymbol{\zeta}^\top\boldsymbol{\mathcal{S}}^{-1}\boldsymbol{\zeta}}{2}\right)\mathrm{d}\boldsymbol{\zeta}
\end{aligned}
$$

Generalized *Stein Unbiased Risk Estimate* (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - 2\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - 2\langle \mathbf{A}\quad\boldsymbol{\zeta}, \mathbf{A}\boldsymbol{\zeta}\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \underbrace{\mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2}_{\text{accessible}} + 2\mathbb{E}_{\boldsymbol{\zeta}}\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle - \underbrace{\mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\zeta}}\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle &= \mathbb{E}_{\boldsymbol{\zeta}}\langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\rangle \\
&= \int \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta};\lambda), \mathbf{A}\boldsymbol{\zeta}\rangle \exp\left(-\frac{\boldsymbol{\zeta}^\top\boldsymbol{\mathcal{S}}^{-1}\boldsymbol{\zeta}}{2}\right) \mathrm{d}\boldsymbol{\zeta} \\
\text{(Gen. I.b.P.)} &= \mathbb{E}_{\boldsymbol{\zeta}}\mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)
\end{aligned}$$

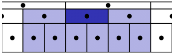
Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\qquad \boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$ $\qquad \mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Theorem *(Pascal, 2020)*

Let $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\boldsymbol{\zeta} \mapsto \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \mathbf{A}\boldsymbol{\zeta}\rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$.

$$\widehat{R}(\boldsymbol{\Lambda}) \triangleq \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right) - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

$$\Longrightarrow R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{\Lambda})].$$

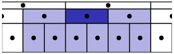
Generalized *Stein Unbiased Risk Estimate*

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\zeta \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\mathcal{L}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \zeta$ $\qquad \zeta \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$ $\qquad \mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

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Let $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
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$$\widehat{R}(\boldsymbol{\Lambda}) \triangleq \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right) - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

$$\Longrightarrow R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \mathbb{E}_\zeta[\widehat{R}(\boldsymbol{\Lambda})].$$

Computation of the degrees of freedom

Degrees of freedom $\mathrm{dof} \triangleq \mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)$

Computation of the degrees of freedom

Degrees of freedom $\quad \mathrm{dof} \triangleq \mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi} \partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right)$

- **Monte Carlo strategy** (MC) **Large size matrix M** $\in \mathbb{R}^{P \times P}$

$$\mathrm{tr}(\mathbf{M}) = \mathbb{E}_{\boldsymbol{\varepsilon}} \langle \mathbf{M}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} \rangle, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_P)$$

Computation of the degrees of freedom

Degrees of freedom $\mathrm{dof} \triangleq \mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \mathbf{\Pi} \partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right)$

- **Monte Carlo strategy** (MC) **Large size matrix $\mathbf{M} \in \mathbb{R}^{P \times P}$**

$$\mathrm{tr}(\mathbf{M}) = \mathbb{E}_{\boldsymbol{\varepsilon}} \langle \mathbf{M}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} \rangle, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_P)$$

- **Finite Differences** (FD) **Inaccessible Jacobian matrix**

$$\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}[\boldsymbol{\varepsilon}] \underset{\nu \to 0}{\simeq} \frac{1}{\nu} \left(\widehat{\boldsymbol{x}}(\boldsymbol{y} + \nu \boldsymbol{\varepsilon}; \boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) \right)$$

Computation of the degrees of freedom

Degrees of freedom $\mathrm{dof} \triangleq \mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)$

- **Monte Carlo strategy** (MC) — **Large size matrix $\mathbf{M} \in \mathbb{R}^{P\times P}$**

$$\mathrm{tr}(\mathbf{M}) = \mathbb{E}_{\boldsymbol{\varepsilon}}\left\langle \mathbf{M}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}\right\rangle, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_P)$$

- **Finite Differences** (FD) — **Inaccessible Jacobian matrix**

$$\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}\left[\boldsymbol{\varepsilon}\right] \underset{\nu\to 0}{\simeq} \frac{1}{\nu}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)$$

Proposition *(Pascal, 2020)*

Let $(\boldsymbol{y};\boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

- uniformly Lipschitz continuous w.r.t. $\boldsymbol{y}$,
- such that $\forall \boldsymbol{\Lambda} \in \mathbb{R}^L$, $\widehat{\boldsymbol{x}}(\mathbf{0}_P;\boldsymbol{\Lambda}) = \mathbf{0}_N$. Then

$$\mathbb{E}_{\boldsymbol{\zeta}}\left[\mathrm{dof}\right] = \lim_{\nu\to 0}\mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}}\left[\frac{1}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle\right]$$

Stein Unbiased Risk Estimate (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

Projected estimation error $R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Generalized Finite Difference Monte Carlo SURE

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) \triangleq \left\|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\right\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

Stein Unbiased Risk Estimate (Computation)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

Projected estimation error $R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

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$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) \triangleq \|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

Theorem *(Pascal, 2020)*

Let $(\boldsymbol{y};\boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

- uniformly Lipschitz continuous w.r.t. $\boldsymbol{y}$,
- such that $\forall \boldsymbol{\Lambda} \in \mathbb{R}^L$, $\widehat{\boldsymbol{x}}(\mathbf{0}_P;\boldsymbol{\Lambda}) = \mathbf{0}_N$. Then

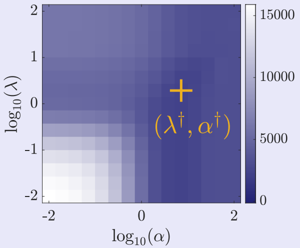
$$R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \lim_{\nu\to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}}\left[\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}})\right]$$

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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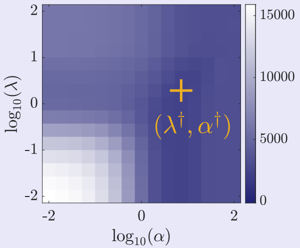
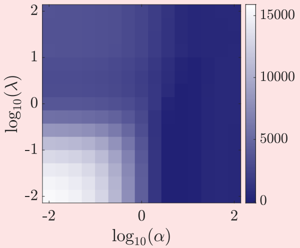
$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

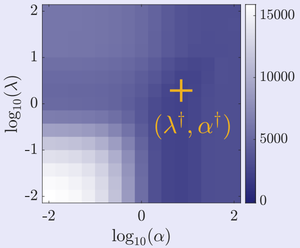


Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

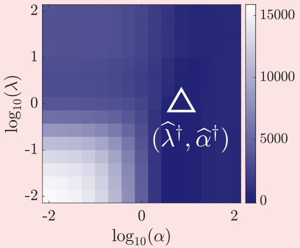
$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

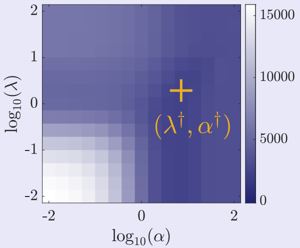


Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

$\bar{\boldsymbol{h}}$: *true* regularity

$$\mathcal{R}(\lambda, \alpha) = \left\| \widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha) - \bar{\boldsymbol{h}} \right\|^2$$



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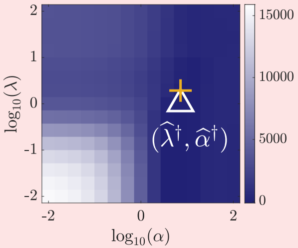
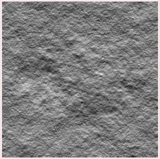
$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$

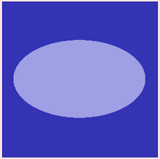


Grid search selection of regularization parameters

$$\left(\widehat{\boldsymbol{h}}^{\mathsf{F}}, \widehat{\boldsymbol{v}}^{\mathsf{F}}\right)(\boldsymbol{\mathcal{L}}; \boldsymbol{\Lambda}) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathsf{F}}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

Example





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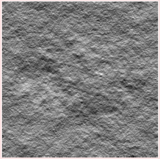
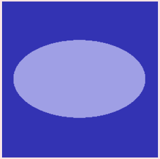
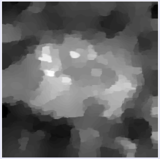




$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}}; \lambda^\dagger, \alpha^\dagger)$
(grid)



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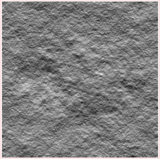
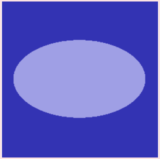
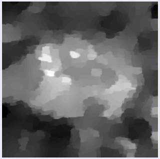
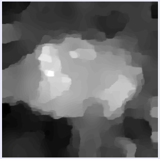




$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}}; \lambda^\dagger, \alpha^\dagger)$ (grid)



$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^\dagger, \widehat{\alpha}^\dagger)$ (grid)



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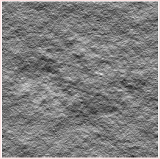

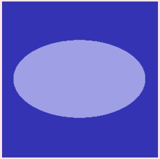

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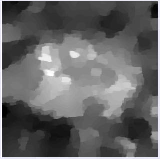
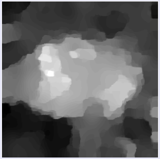


$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\dagger}, \widehat{\alpha}^{\dagger})$ (grid)



$15 \times 15 =$ **225** parameters $\longrightarrow$ grid search is very costly!

Automatic minimization of Generalized SURE

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

Generalized FDMC SURE $\quad \lim_{\nu \to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) = R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda})$

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But : $\underset{\boldsymbol{\Lambda}}{\text{minimize}}\ \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}})$ for given $\boldsymbol{y}$, $\boldsymbol{\mathcal{S}}$

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Broyden-Fletcher-Goldfarb-Shanno Quasi-Newton *(Nocedal, 2006)*

for $t = 0, 1, \ldots$

$\boldsymbol{d}^{[t]} = -\boldsymbol{H}^{[t]} \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *descent direction*

$\alpha^{[t]} \in \underset{\alpha \in \mathbb{R}}{\text{Argmin}} \ \widehat{R}(\boldsymbol{\Lambda}^{[t]} + \alpha \boldsymbol{d}^{[t]})$ *line search*

$\boldsymbol{\Lambda}^{[t+1]} = \boldsymbol{\Lambda}^{[t]} + \alpha^{[t]} \boldsymbol{d}^{[t]}$

$\boldsymbol{u}^{[t]} = \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t+1]}) - \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *gradient increment*

$\boldsymbol{H}^{[t+1]} = \text{BFGS}(\boldsymbol{H}^{[t]}, \boldsymbol{d}^{[t]}, \boldsymbol{u}^{[t]})$ *"inverse Hessian" update*

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Stein Unbiased GrAdient Risk estimate

Generalized FDMC SURE

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = \left\|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\right\|^2 +$$
$$\frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

Stein Unbiased GrAdient Risk estimate

Generalized FDMC SURE

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Generalized Finite Difference Monte Carlo SUGAR

$$\begin{aligned}\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\varepsilon}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = {} & 2\left(\mathbf{A}\boldsymbol{\Phi}\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)^\top\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right) \\ & + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle\end{aligned}$$

Stein Unbiased GrAdient Risk estimate

Generalized FDMC SURE

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = \|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

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Theorem *(Pascal, 2020)*

Let $(\boldsymbol{y};\boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})$ an estimator of $\bar{\boldsymbol{x}}$

- uniformly Lipschitz continuous w.r.t. $\boldsymbol{y}$
- such that $\forall \boldsymbol{\Lambda} \in \mathbb{R}^L$, $\widehat{\boldsymbol{x}}(\mathbf{0}_P;\boldsymbol{\Lambda}) = \mathbf{0}_N$,
- uniformly L-Lipschitz continuous w.r.t. $\boldsymbol{\Lambda}$, L ind. of $\boldsymbol{y}$. Then

$$\partial_{\boldsymbol{\Lambda}} R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \lim_{\nu\to 0}\mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}}\left[\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}})\right]$$

Parameter tuning (Automatic selection)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)$$

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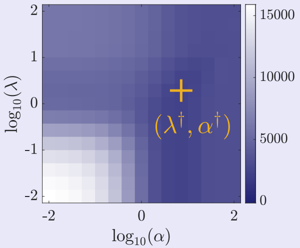
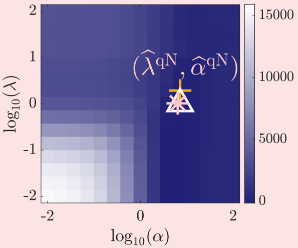


$\bar{\boldsymbol{h}}$: unknown!

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha | \boldsymbol{\mathcal{S}})$$



Automated selection of regularization parameters

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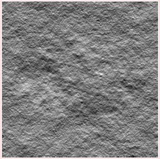
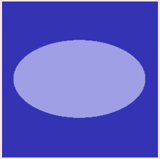
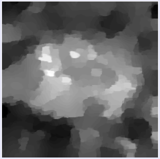
Example





$\widehat{\boldsymbol{h}}^{\mathrm{F}}(\boldsymbol{\mathcal{L}}; \lambda^{\dagger}, \alpha^{\dagger})$
(grid)



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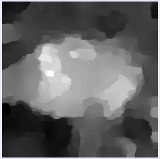

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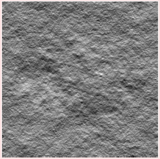
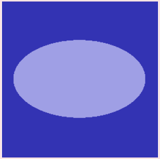
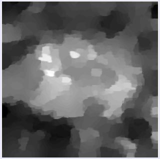
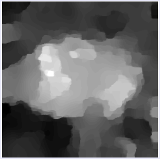




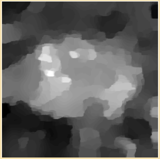
$\widehat{\boldsymbol{h}}^{\mathrm{F}}(\boldsymbol{\mathcal{L}}; \lambda^\dagger, \alpha^\dagger)$ (grid)



$\widehat{\boldsymbol{h}}^{\mathrm{F}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^\dagger, \widehat{\alpha}^\dagger)$ (grid)



$\widehat{\boldsymbol{h}}^{\mathrm{F}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\mathrm{qN}}, \widehat{\alpha}^{\mathrm{qN}})$ (quasi-Newton)



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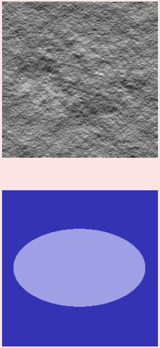
Example



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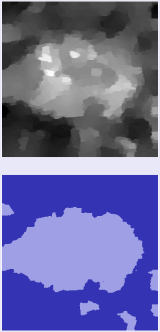
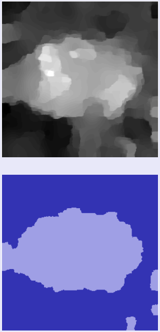
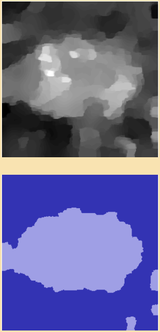


$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\dagger}, \widehat{\alpha}^{\dagger})$ (grid)



$\widehat{\boldsymbol{h}}^{\mathsf{F}}(\boldsymbol{\mathcal{L}}; \widehat{\lambda}^{\mathsf{qN}}, \widehat{\alpha}^{\mathsf{qN}})$ (quasi-Newton)



40 calls of the estimator v.s. **225** over a grid

Take home messages

- Fractal texture model based on local *regularity* and *variance*
  - ∗ appropriate for real-world texture characterization
  - ∗ complementary attributes → able to finely discriminate

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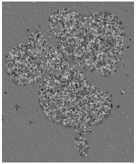
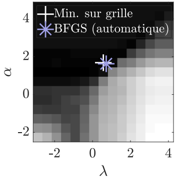
  - $*$ appropriate for real-world texture characterization
  - $*$ complementary attributes $\rightarrow$ able to finely discriminate
- Simultaneous estimation and regularization
  - $*$ significant decrease of the estimation error
  - $*$ accurate and regular contours thanks to co-localized penalization

Take home messages

- Fractal texture model based on local *regularity* and *variance*
  - appropriate for real-world texture characterization
  - complementary attributes → able to finely discriminate
- Simultaneous estimation and regularization
  - significant decrease of the estimation error
  - accurate and regular contours thanks to co-localized penalization
- Fast algorithms for automated tuning of hyperparameters
  - possibility to manage huge amount of data
  - amenable to process data corrupted by *correlated* Gaussian noise
  - ensured objectivity and reproducibility

Thank you for your attention.





Automated analysis of images from multiphase flow experiments