How scale-free texture segmentation turns out to be a strongly convex optimization problem ?†.

B. Pascal[1], N. Pustelnik[1], P. Abry[1]

Université Catholique de Louvain
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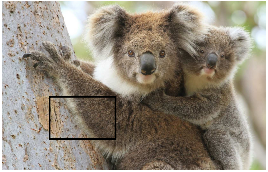
[1] Univ Lyon, ENS de Lyon, Univ Claude Bernard Lyon 1, CNRS,
Laboratoire de Physique, F-69342 Lyon, France, firstname.lastname@ens-lyon.fr

† Supported by Defi Imag'in SIROCCO and ANR-16-CE33-0020 MultiFracs, France.

Describing and interpreting real-world images

Texture as a discriminating feature



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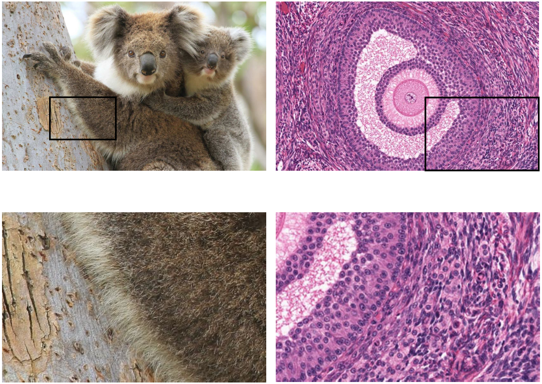

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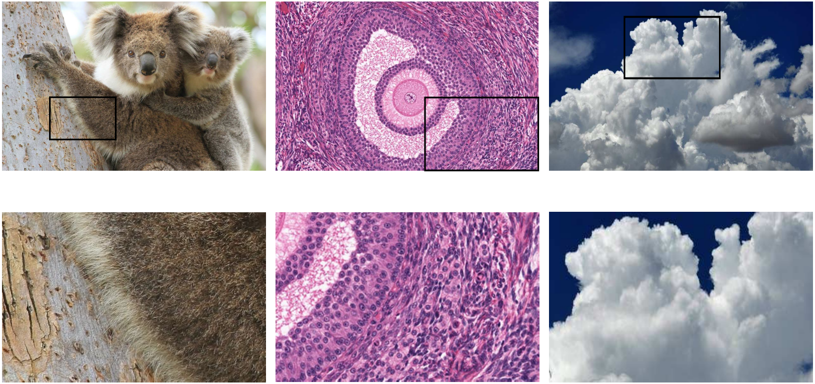

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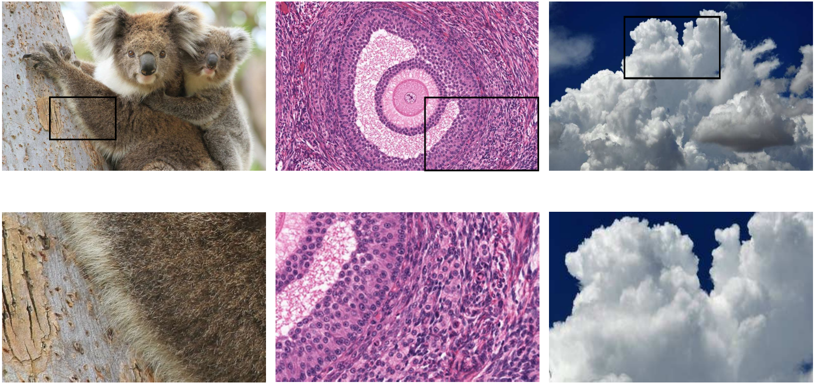

Texture is of utmost importance in complex computer vision tasks.

Describing and interpreting real-world images

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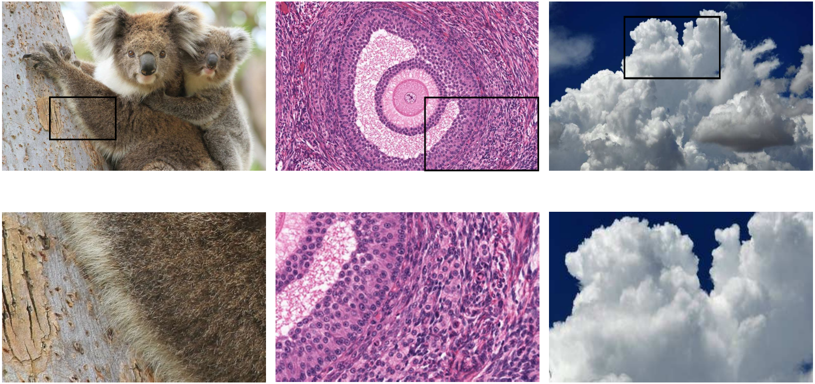

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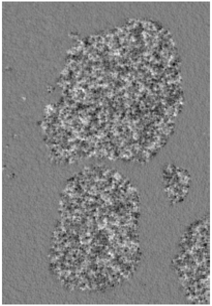
scale-free — segmentation

Formulation of the texture segmentation problem



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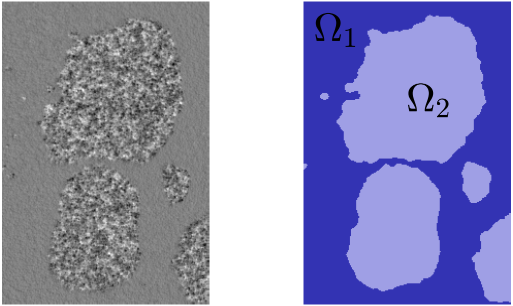


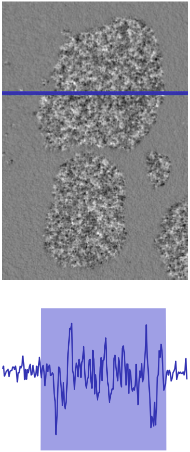
Purpose: obtaining a partition of the image into κ homogeneous regions

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_\kappa$$

Ω_k: pixels corresponding to texture k.

Texture's attributes definition

Piecewise monofractal model



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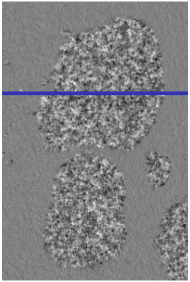
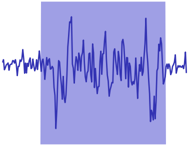
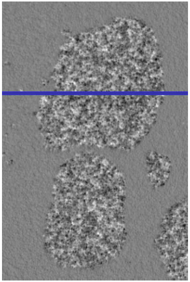


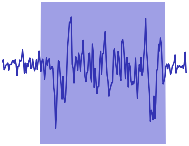


Variance σ^2 *amplitude of variations*

Texture's attributes definition

Piecewise monofractal model





Variance σ^2 — *amplitude of variations*

Local regularity h — *scale-free behavior*

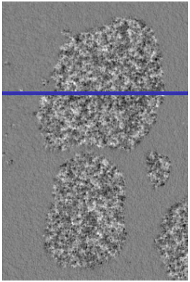
$h(x) \equiv h_1 = 0.9$ $\qquad$ $h(x) \equiv h_2 = 0.3$

Fit local behavior with power law functions

$$|f(x) - f(y)| \leq C|x - y|^{h(x)}$$

Texture's attributes definition

Piecewise monofractal model



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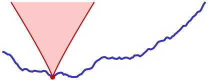

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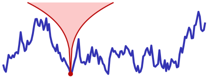
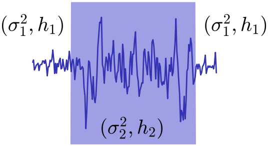


$h(x) \equiv h_2 = 0.3$



Fit local behavior with power law functions

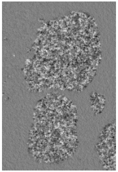
$$|f(x) - f(y)| \leq C|x - y|^{h(x)}$$

Segmentation requires local measurement of σ^2 and h.

Texture's attributes estimation

Multiscale analysis

Textured image



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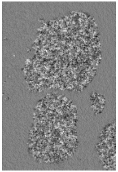
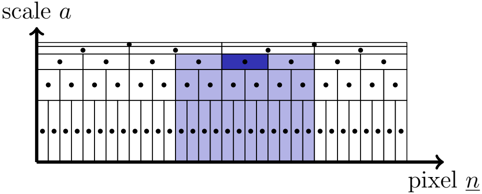
Textured image

Local supremum of wavelet coefficients: *leaders* $\mathcal{L}_{a,\cdot}$





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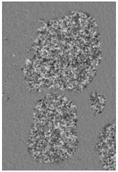
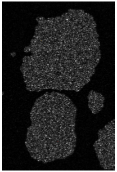
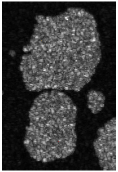


Local supremum of wavelet coefficients: *leaders* $\mathcal{L}_{a,\cdot}$

Scale $a = 2^1$

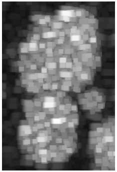


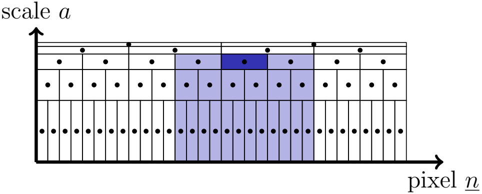
$a = 2^2$



. . .

$a = 2^5$





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Textured image

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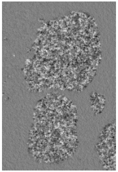

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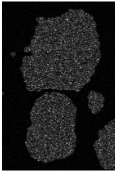

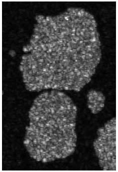

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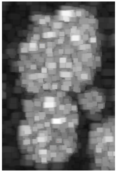

Log-log linear behavior

$$\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \underbrace{\boldsymbol{v}}_{\substack{\sim\log(\boldsymbol{\sigma}^2)\\ \text{(variance)}}} + \log(a) \underbrace{\boldsymbol{h}}_{\textit{regularity}}$$

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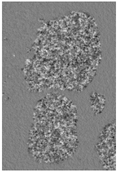

Local supremum of wavelet coefficients: *leaders* $\mathcal{L}_{a,\cdot}$

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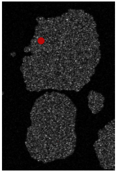

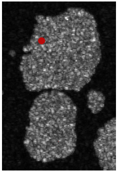

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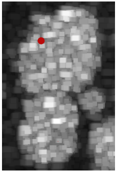

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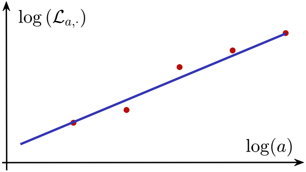

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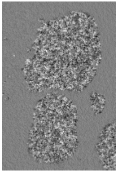

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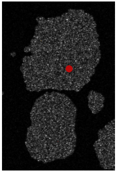

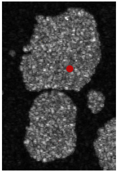

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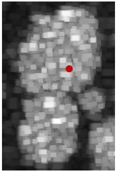

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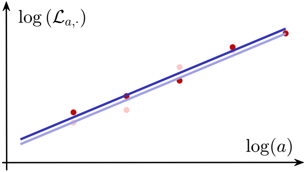

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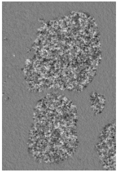
Textured image



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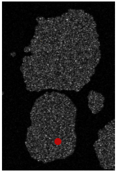

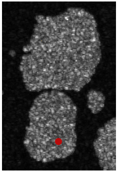

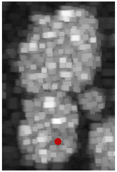

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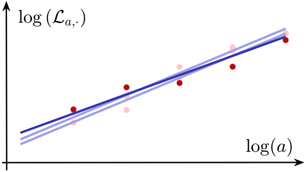

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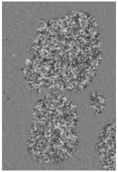

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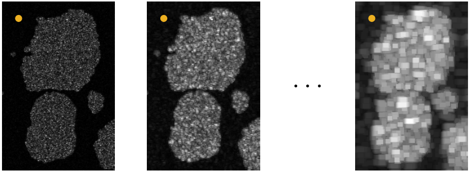

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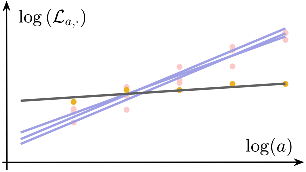

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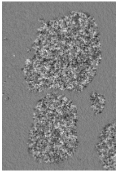

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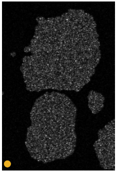

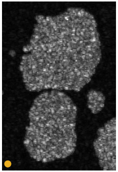

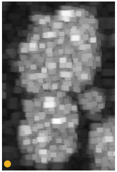

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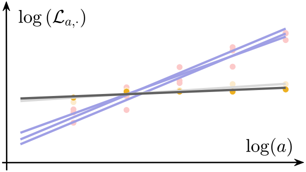

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Multiscale analysis

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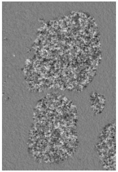

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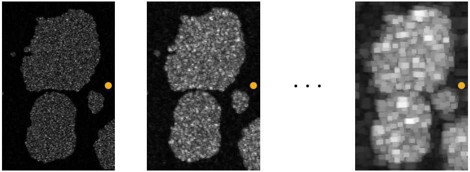

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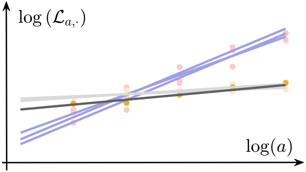


Texture's attributes estimation

Pointwise linear regression

$$\log\left(\mathcal{L}_{a,.}\right) \simeq \underbrace{\boldsymbol{v}}_{\sim\log(\boldsymbol{\sigma}^2)} + \log(a) \underbrace{\boldsymbol{h}}_{\textit{regularity}}$$

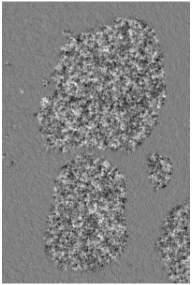
Texture's attributes estimation

Pointwise linear regression

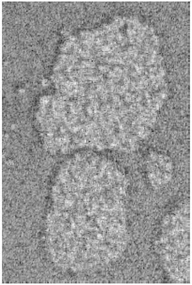
$$\log\left(\boldsymbol{\mathcal{L}}_{a,\cdot}\right) \simeq \underbrace{\boldsymbol{v}}_{\sim \log(\boldsymbol{\sigma}^2)} + \log(a) \underbrace{\boldsymbol{h}}_{\textit{regularity}}$$

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_{a} \left\| \log\left(\boldsymbol{\mathcal{L}}_{a,\cdot}\right) - \boldsymbol{v} - \log(a)\boldsymbol{h} \right\|^2$$

Textured image



Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

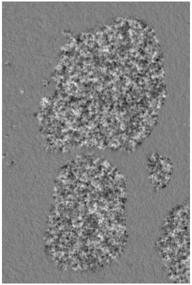
Texture's attributes estimation

Pointwise linear regression

$$\underbrace{\mathbb{E}\log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{expected value}} \simeq \underbrace{\boldsymbol{v}}_{\sim\log(\boldsymbol{\sigma}^2)} + \log(a)\underbrace{\boldsymbol{h}}_{\textit{regularity}}$$

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \boldsymbol{v} - \log(a)\boldsymbol{h}\right\|^2$$

Textured image



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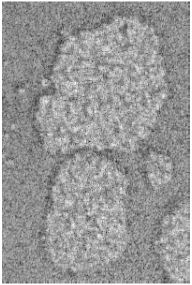


Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

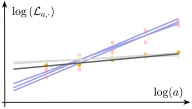
Pointwise linear regression is an estimation from **one** sample!

Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\sum_a \underbrace{\|\log \mathcal{L}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{least-squares}}$$

$\rightarrow$ fidelity to log-linear model



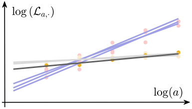
Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{least-squares}} \qquad \lambda \underbrace{\mathcal{R}(\boldsymbol{v}, \boldsymbol{h}; \alpha)}_{\textbf{total variation}}$$

→ fidelity to log-linear model

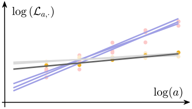
→ enforce piecewise constancy



Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \; \underbrace{\sum_a \left\| \log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h} \right\|^2}_{\substack{\textbf{least-squares} \\ \to \text{ fidelity to log-linear model}}} + \underbrace{\lambda \, \mathcal{R}(\boldsymbol{v}, \boldsymbol{h}; \alpha)}_{\substack{\textbf{total variation} \\ \to \text{ enforce piecewise constancy}}}$$



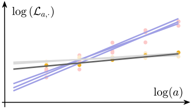
Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\text{least-squares}} + \lambda \underbrace{\mathcal{R}(\boldsymbol{v},\boldsymbol{h};\alpha)}_{\text{total variation}}$$

$\rightarrow$ fidelity to log-linear model

$\rightarrow$ enforce piecewise constancy



joint: $\boldsymbol{v}$, $\boldsymbol{h}$ are **independently** piecewise constant

coupled: $\boldsymbol{v}$, $\boldsymbol{h}$ are **concomitantly** piecewise constant

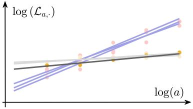
Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\text{least-squares}} + \lambda\, \underbrace{\mathcal{R}(\boldsymbol{v},\boldsymbol{h};\alpha)}_{\text{total variation}}$$

→ fidelity to log-linear model

→ enforce piecewise constancy



Discrete differences $\mathbf{H}\boldsymbol{x}$ (horizontal), $\mathbf{V}\boldsymbol{x}$ (vertical) at each pixel

joint: $\boldsymbol{v}$, $\boldsymbol{h}$ are **independently** piecewise constant

$$\mathcal{R}_{\mathrm{J}}(\boldsymbol{v},\boldsymbol{h};\alpha) = \left(\sum_{\text{pixels}} \sqrt{(\mathbf{H}\boldsymbol{v})^2 + (\mathbf{V}\boldsymbol{v})^2} + \alpha \sum_{\text{pixels}} \sqrt{(\mathbf{H}\boldsymbol{h})^2 + (\mathbf{V}\boldsymbol{h})^2}\right)$$

coupled: $\boldsymbol{v}$, $\boldsymbol{h}$ are **concomitantly** piecewise constant

$$\mathcal{R}_{\mathrm{C}}(\boldsymbol{v},\boldsymbol{h};\alpha) = \sum_{\text{pixels}} \sqrt{(\mathbf{H}\boldsymbol{v})^2 + (\mathbf{V}\boldsymbol{v})^2 + \alpha^2(\mathbf{H}\boldsymbol{h})^2 + \alpha^2(\mathbf{V}\boldsymbol{h})^2}$$

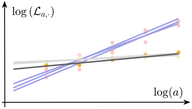
Texture's attributes estimation

One-step joint and coupled segmentation as a convex minimization

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{least-squares}} + \lambda \underbrace{\mathcal{R}(\boldsymbol{v}, \boldsymbol{h}; \alpha)}_{\textbf{total variation}}$$

→ fidelity to log-linear model

→ enforce piecewise constancy



joint: $\boldsymbol{v}$, $\boldsymbol{h}$ are **independently** piecewise constant

$$\mathcal{R}_{\text{J}}(\boldsymbol{v}, \boldsymbol{h}; \alpha) = \mathcal{R}(\boldsymbol{v}) + \alpha\mathcal{R}(\boldsymbol{h})$$

coupled: $\boldsymbol{v}$, $\boldsymbol{h}$ are **concomitantly** piecewise constant

$$\mathcal{R}_{\text{C}}(\boldsymbol{v}, \boldsymbol{h}; \alpha) = \mathcal{R}(\boldsymbol{v}, \alpha\boldsymbol{h})$$

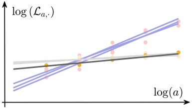
Texture's attributes estimation

Fine tuning of regularization parameters

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{least-squares}} + \lambda \underbrace{\mathcal{R}(\boldsymbol{v},\boldsymbol{h};\alpha)}_{\textbf{total variation}}$$

least-squares → fidelity to log-linear model

total variation → enforce piecewise constancy



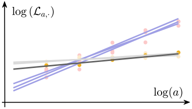
Texture's attributes estimation

Fine tuning of regularization parameters

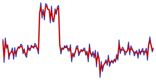
$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{least-squares}} + \lambda \underbrace{\mathcal{R}(\boldsymbol{v}, \boldsymbol{h}; \alpha)}_{\textbf{total variation}}$$

→ fidelity to log-linear model

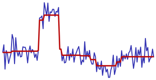
→ enforce piecewise constancy



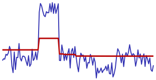
Fine tuning of regularization parameters (λ, α) is necessary . . .



too small



optimal



too large

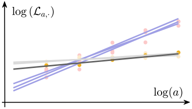
Texture's attributes estimation

Fine tuning of regularization parameters

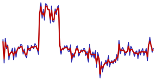
$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{least-squares}} + \lambda\, \underbrace{\mathcal{R}(\boldsymbol{v},\boldsymbol{h};\alpha)}_{\textbf{total variation}}$$

least-squares → fidelity to log-linear model

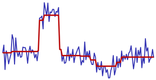
total variation → enforce piecewise constancy



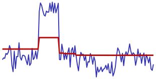
Fine tuning of regularization parameters (λ, α) is necessary ... but **costly**!



too small



optimal



too large

In practice, we explore a log-spaced grid of $15 \times 15 = \mathbf{225}$ hyperparameters (λ, α).

Texture's attributes estimation

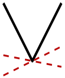
Algorithmic scheme for joint and coupled functionals

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \;\; \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{least-squares}} \;\; + \;\; \lambda \underbrace{\mathcal{R}(\boldsymbol{v}, \boldsymbol{h}; \alpha)}_{\textbf{total variation}}$$

Texture's attributes estimation

Algorithmic scheme for joint and coupled functionals

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\textbf{least-squares}} \; + \; \lambda \underbrace{\mathcal{R}(\boldsymbol{v}, \boldsymbol{h}; \alpha)}_{\substack{\textbf{total variation} \\ \to \text{ non-smooth}}}$$



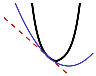
primal-dual algorithm (Chambolle, Pock 11')

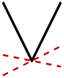
Texture's attributes estimation

Algorithmic scheme for joint and coupled functionals

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\substack{\textbf{least-squares} \\ \to \text{ strongly convex}}} \; + \; \lambda \underbrace{\mathcal{R}(\boldsymbol{v},\boldsymbol{h};\alpha)}_{\substack{\textbf{total variation} \\ \to \text{ non-smooth}}}$$

φ is μ-strongly convex iff $\varphi - \frac{\mu}{2}\|\cdot\|^2$ is convex.





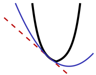
Accelerated primal-dual algorithm (Chambolle, Pock 11')

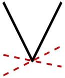
Texture's attributes estimation

Algorithmic scheme for joint and coupled functionals

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2}_{\substack{\textbf{least-squares} \\ \rightarrow \text{ strongly convex}}} + \lambda \underbrace{\mathcal{R}(\boldsymbol{v},\boldsymbol{h};\alpha)}_{\substack{\textbf{total variation} \\ \rightarrow \text{ non-smooth}}}$$

φ is μ-strongly convex iff $\varphi - \frac{\mu}{2}\|\cdot\|^2$ is convex.





Accelerated primal-dual algorithm (Chambolle, Pock 11')

$$\boldsymbol{x}^n = (\boldsymbol{v}^n, \boldsymbol{h}^n), \quad \boldsymbol{y}^n = (\boldsymbol{u}^n, \boldsymbol{\ell}^n)$$

$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma_n \|\cdot\|_{2,1}} \left(\boldsymbol{y}^n + \sigma_n \boldsymbol{\nabla} \bar{\boldsymbol{x}}^n\right)$$

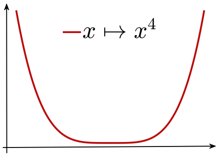
$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau_n \|\boldsymbol{A}\cdot - \boldsymbol{b}\|_2^2} \left(\boldsymbol{x}^n - \tau_n \boldsymbol{\nabla}^* \boldsymbol{y}^{n+1}\right)$$

$$\theta_n = \sqrt{1 + 2\mu\tau_n}, \quad \tau_{n+1} = \tau_n/\theta_n, \quad \sigma_{n+1} = \theta_n \sigma_n$$

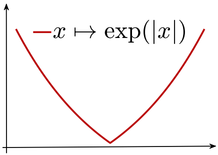
$$\bar{\boldsymbol{x}}^{n+1} = \boldsymbol{x}^{n+1} + \theta_n^{-1}\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

Strong convexity of data fidelity term

φ is μ-strongly convex iff $\varphi - \frac{\mu}{2}\|\cdot\|^2$ is convex.



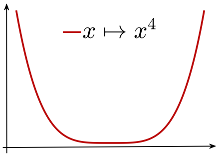
✓ strictly convex
✗ not strongly convex



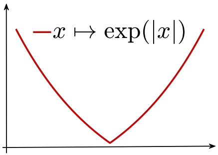
✓ strictly convex
✓ 1-strongly convex

Strong convexity of data fidelity term

φ is μ-strongly convex iff $\varphi - \frac{\mu}{2}\|\cdot\|^2$ is convex.



✓ strictly convex
✗ not strongly convex



✓ strictly convex
✓ 1-strongly convex

If φ is twice-differentiable with Hessian $\boldsymbol{H}\varphi$ and $\mu > 0$,

$$\varphi \text{ is } \mu\text{-strongly convex iif } \forall \eta \in \mathrm{Sp}(\boldsymbol{H}\varphi),\ \eta \geq \mu.$$

In particular φ is $\min \mathrm{Sp}(\boldsymbol{H}\varphi)$-strongly convex.

Strong convexity of data fidelity term

$$\varphi_{\mathbf{A}}(\boldsymbol{v}, \boldsymbol{h}) = \sum_{a=a_{\min}}^{a_{\max}} \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2 = \|\log \mathcal{L} - \mathbf{A}(\boldsymbol{v}, \boldsymbol{h})\|^2$$

where $\mathbf{A} : (\boldsymbol{v}, \boldsymbol{h}) \mapsto \{\boldsymbol{v} + \log(a)\boldsymbol{h}\}_{a=a_{\min}}^{a_{\max}}$ is linear.

Strong convexity of data fidelity term

$$\varphi_{\mathbf{A}}(\boldsymbol{v},\boldsymbol{h}) = \sum_{a=a_{\min}}^{a_{\max}} \|\log \mathcal{L}_{a,.} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2 = \|\log \mathcal{L} - \mathbf{A}(\boldsymbol{v},\boldsymbol{h})\|^2$$

where $\mathbf{A} : (\boldsymbol{v},\boldsymbol{h}) \mapsto \{\boldsymbol{v} + \log(a)\boldsymbol{h}\}_{a=a_{\min}}^{a_{\max}}$ is linear.

$$\boldsymbol{H}\varphi_{\mathbf{A}}(\boldsymbol{v},\boldsymbol{h}) = \mathbf{A}^*\mathbf{A} = \begin{pmatrix} \mathrm{A}_0 \boldsymbol{I} & \mathrm{A}_1 \boldsymbol{I} \\ \mathrm{A}_1 \boldsymbol{I} & \mathrm{A}_2 \boldsymbol{I} \end{pmatrix},\ \mathrm{A}_m = \sum_{a=a_{\min}}^{a_{\max}} (\log a)^m,\ \forall m \in \{0,1,2\}.$$

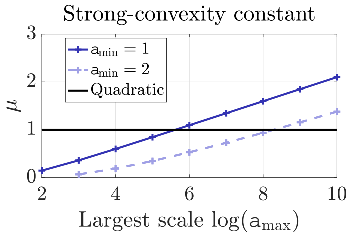
Strong convexity of data fidelity term

$$\varphi_{\mathbf{A}}(\boldsymbol{v}, \boldsymbol{h}) = \sum_{a=a_{\min}}^{a_{\max}} \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2 = \|\log \mathcal{L} - \mathbf{A}(\boldsymbol{v}, \boldsymbol{h})\|^2$$

where $\mathbf{A} : (\boldsymbol{v}, \boldsymbol{h}) \mapsto \{\boldsymbol{v} + \log(a)\boldsymbol{h}\}_{a=a_{\min}}^{a_{\max}}$ is linear.

$$\boldsymbol{H}\varphi_{\mathbf{A}}(\boldsymbol{v}, \boldsymbol{h}) = \mathbf{A}^*\mathbf{A} = \begin{pmatrix} \mathrm{A}_0 \boldsymbol{I} & \mathrm{A}_1 \boldsymbol{I} \\ \mathrm{A}_1 \boldsymbol{I} & \mathrm{A}_2 \boldsymbol{I} \end{pmatrix}, \; \mathrm{A}_m = \sum_{a=a_{\min}}^{a_{\max}} (\log a)^m, \; \forall m \in \{0, 1, 2\}.$$

Prop: $\varphi_{\mathbf{A}}(\boldsymbol{v}, \boldsymbol{h})$ is μ-strongly convex, μ the smallest eigenvalue of $\mathbf{A}^*\mathbf{A}$.

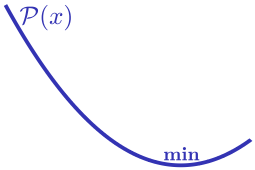


Convergence speed and stopping criterion

Duality gap

Primal problem

$$\widehat{x} = \operatorname*{argmin}_x \varphi_{\mathbf{A}}(x) + \mathcal{G}(\boldsymbol{\nabla} x)$$



Convergence speed and stopping criterion

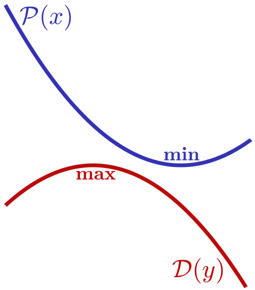
Duality gap

Primal problem

$$\widehat{x} = \underset{x}{\operatorname{argmin}}\, \varphi_{\mathbf{A}}(x) + \mathcal{G}(\nabla x)$$

Dual problem

$$\widehat{y} = \underset{y}{\operatorname{argmax}}\, -\varphi^*_{\mathbf{A}}(-\nabla^* y) - \mathcal{G}^*(y)$$



Convergence speed and stopping criterion

Duality gap

Primal problem

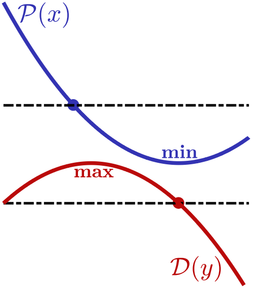
$$\widehat{x} = \underset{x}{\operatorname{argmin}}\, \varphi_{\mathbf{A}}(x) + \mathcal{G}(\nabla x)$$

Dual problem

$$\widehat{y} = \underset{y}{\operatorname{argmax}} - \varphi^*_{\mathbf{A}}(-\nabla^* y) - \mathcal{G}^*(y)$$

Duality gap $\delta(x; y)$

$$\underset{\text{def.}}{=} \varphi_{\mathbf{A}}(x) + \mathcal{G}(\nabla x) + \varphi^*_{\mathbf{A}}(-\nabla^* y) + \mathcal{G}^*(y)$$



Convergence speed and stopping criterion

Duality gap

Primal problem

$$\widehat{x} = \operatorname*{argmin}_x \varphi_{\mathbf{A}}(x) + \mathcal{G}(\boldsymbol{\nabla} x)$$

Dual problem

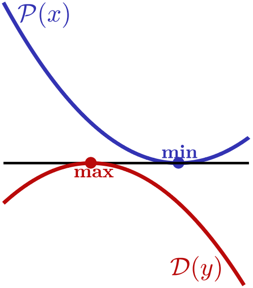
$$\widehat{y} = \operatorname*{argmax}_y \; -\varphi^*_{\mathbf{A}}(-\boldsymbol{\nabla}^* y) - \mathcal{G}^*(y)$$

Duality gap $\delta(x; y)$

$$\underset{\text{def.}}{=} \varphi_{\mathbf{A}}(x) + \mathcal{G}(\boldsymbol{\nabla} x) + \varphi^*_{\mathbf{A}}(-\boldsymbol{\nabla}^* y) + \mathcal{G}^*(y)$$

Characterization of the solution

$$\delta(\widehat{x}; \widehat{y}) \underset{\text{prop.}}{=} 0$$



Computing the duality gap

For **Joint** penalization

$$\delta(\quad ; \quad) = \qquad + $$

Computing the duality gap

For **Joint** penalization

$$\delta(\underbrace{\boldsymbol{v}, \boldsymbol{h}}_{\text{primal}};\quad) = \varphi_{\mathbf{A}}(\boldsymbol{v}, \boldsymbol{h}) + \mathcal{G}(\boldsymbol{\nabla} \boldsymbol{v}, \boldsymbol{\nabla} \boldsymbol{h}) +$$

Data fidelity

$$\varphi_{\mathbf{A}}(\boldsymbol{v}, \boldsymbol{h}) = \sum_{a} \|\boldsymbol{v} + \log(a)\boldsymbol{h} - \boldsymbol{\mathcal{L}}_{a,.}\|_2^2$$

Penalization

$$\mathcal{G}(\boldsymbol{u}, \boldsymbol{\ell}) = \lambda\left(\|\boldsymbol{u}\|_{2,1} + \alpha\|\boldsymbol{\ell}\|_{2,1}\right)$$

Computing the duality gap

For **Joint** penalization

$$\delta(\underbrace{\boldsymbol{v},\boldsymbol{h}}_{\text{primal}};\underbrace{\boldsymbol{u},\boldsymbol{\ell}}_{\text{dual}}) = \varphi_{\mathbf{A}}(\boldsymbol{v},\boldsymbol{h}) + \mathcal{G}(\boldsymbol{\nabla}\boldsymbol{v},\boldsymbol{\nabla}\boldsymbol{h}) + \varphi^*_{\mathbf{A}}(-\boldsymbol{\nabla}^*\boldsymbol{u},-\boldsymbol{\nabla}^*\boldsymbol{\ell}) + \mathcal{G}^*(\boldsymbol{u},\boldsymbol{\ell})$$

Data fidelity

$$\varphi_{\mathbf{A}}(\boldsymbol{v},\boldsymbol{h}) = \sum_a \|\boldsymbol{v} + \log(a)\boldsymbol{h} - \boldsymbol{\mathcal{L}}_{a,.}\|_2^2$$

$$\varphi^*_{\mathbf{A}}(\boldsymbol{v},\boldsymbol{h}) = \frac{1}{4}\langle(\boldsymbol{v},\boldsymbol{h}),(\mathbf{A}^*\mathbf{A})^{-1}(\boldsymbol{v},\boldsymbol{h})\rangle + \langle(\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{T}}),(\mathbf{A}^*\mathbf{A})^{-1}(\boldsymbol{v},\boldsymbol{h})\rangle + \mathcal{C}$$

where $\mathcal{C}$ constant term only depending on $\boldsymbol{\mathcal{L}}_{a,.}$.

Penalization

$$\mathcal{G}(\boldsymbol{u},\boldsymbol{\ell}) = \lambda\left(\|\boldsymbol{u}\|_{2,1} + \alpha\|\boldsymbol{\ell}\|_{2,1}\right)$$

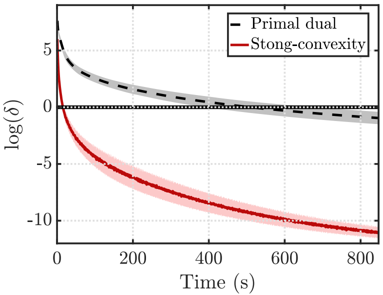
$$\mathcal{G}^*(\boldsymbol{u},\boldsymbol{\ell}) = \iota_{\mathrm{B}_{2,\infty}(\lambda)}(\boldsymbol{u}) + \iota_{\mathrm{B}_{2,\infty}(\lambda\alpha)}(\boldsymbol{\ell})$$

$\mathrm{B}_{2,\infty}(\lambda)$: ball of radius λ w.r.t. $\|.\|_{2,\infty}$.

Computing the duality gap

For **Joint** penalization

$$\underbrace{\delta(\boldsymbol{v},\boldsymbol{h}}_{\text{primal}};\underbrace{\boldsymbol{u},\boldsymbol{\ell}}_{\text{dual}}) = \varphi_{\mathbf{A}}(\boldsymbol{v},\boldsymbol{h}) + \mathcal{G}(\boldsymbol{\nabla}\boldsymbol{v},\boldsymbol{\nabla}\boldsymbol{h}) + \varphi_{\mathbf{A}}^{*}(-\boldsymbol{\nabla}^{*}\boldsymbol{u},-\boldsymbol{\nabla}^{*}\boldsymbol{\ell}) + \mathcal{G}^{*}(\boldsymbol{u},\boldsymbol{\ell})$$

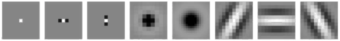


- ✓ Significant convergence acceleration
- ✓ Good stopping criterion: $\underline{\delta(\boldsymbol{v}^n,\boldsymbol{h}^n;\boldsymbol{u}^n,\boldsymbol{\ell}^n) \leq 10^{-3}}$

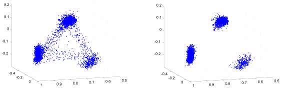
State-of-the-art two-step texture segmentation

Factorization-based segmentation [Yuan *et al.* 15'][†]

(i) local spectral histograms



(ii) matrix factorization



(a) (b)

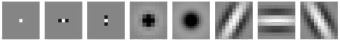
Fig. 2. Scatterplot of features in subspace. (a) Scatterplot of features projected onto the 3-d subspace. (b) Scatterplot after removing features with high edgeness.

[†] `https://sites.google.com/site/factorizationsegmentation/`

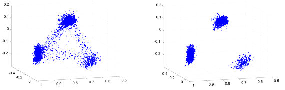
State-of-the-art two-step texture segmentation

Factorization-based segmentation [Yuan *et al.* 15']†

(i) local spectral histograms



(ii) matrix factorization



(a) (b)

Fig. 2. Scatterplot of features in subspace. (a) Scatterplot of features projected onto the 3-d subspace. (b) Scatterplot after removing features with high edgeness.

Threshold-ROF on $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ [Pustelnik 16']

$$\min_{\boldsymbol{h}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|^2 + \lambda \|\boldsymbol{\nabla} \boldsymbol{h}\|_{2,1}$$

Lin. reg.

ROF

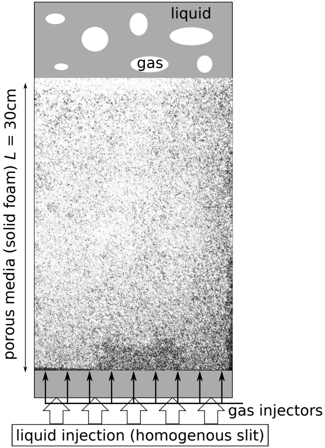
Threshold

Based on regularity $\boldsymbol{h}$ only.

†https://sites.google.com/site/factorizationsegmentation/

Multiphasic (quasi 2D) flow in a porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



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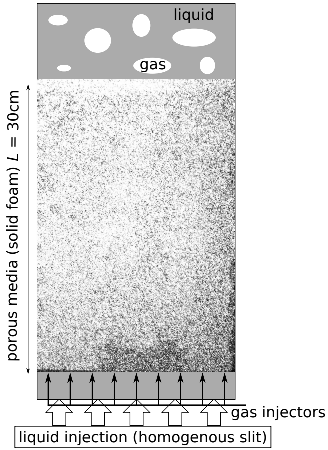

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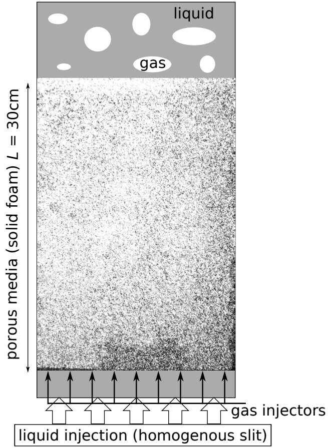
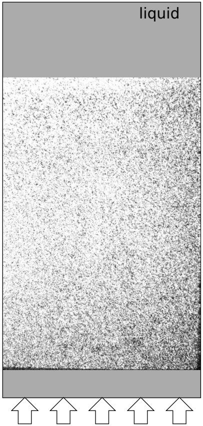
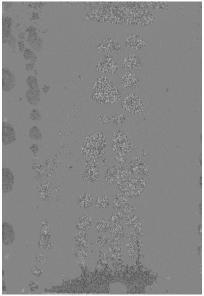




Normalized image



Textured

Multiphasic (quasi 2D) flow in a porous media

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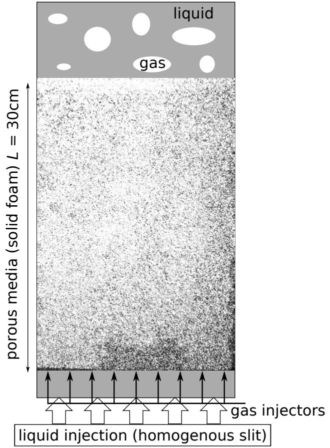
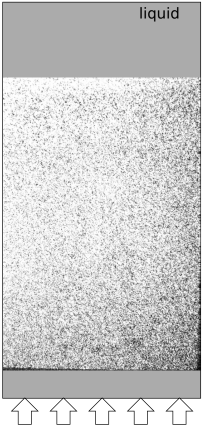
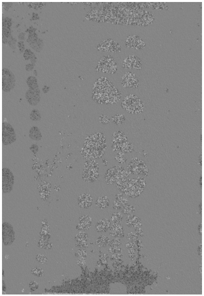




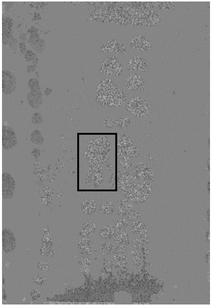
Normalized image



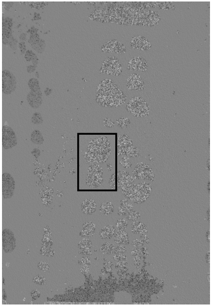
Textured

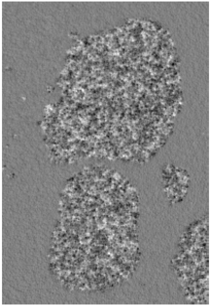
Physical quantities: $\underbrace{\text{gas volume}}_{\text{area}}$ & $\underbrace{\text{contact surface}}_{\text{perimeter}}$.

Texture segmentation

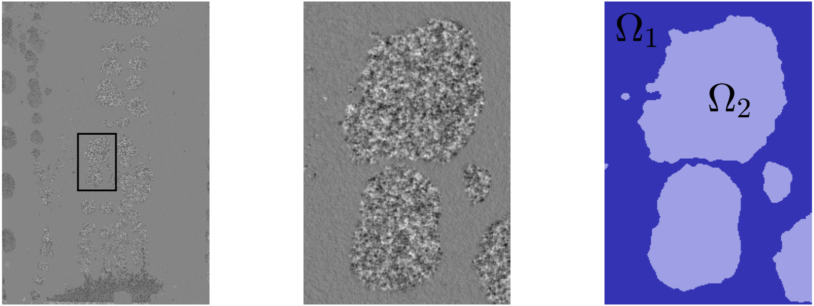


Texture segmentation





Texture segmentation

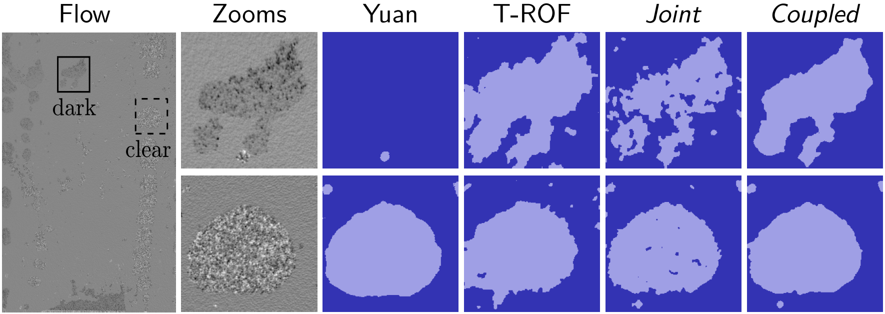


Purpose: obtaining a partition of the image into two regions

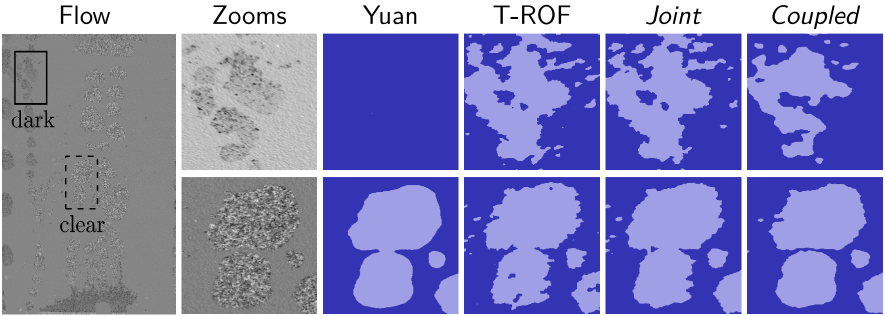
$$\Omega = \Omega_1 \bigsqcup \Omega_2$$

Ω_1: liquid, Ω_2: gas.

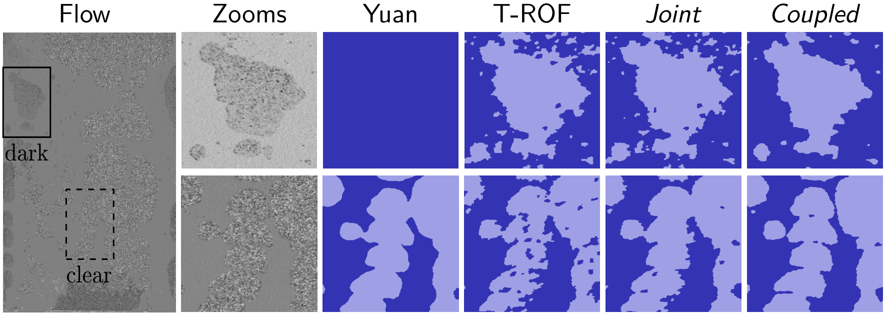
Multiphasic flow. $Q_{\mathrm{G}} = 300$mL/min - $Q_{\mathrm{L}} = 300$mL/min: low activity



Multiphasic flow. $Q_{\mathrm{G}} = 400$mL/min - $Q_{\mathrm{L}} = 700$mL/min: transition



Multiphasic flow. $Q_{\mathrm{G}} = 1200$mL/min - $Q_{\mathrm{L}} = 300$mL/min: high activity



Conclusion

Comparison of the different methods

	Liquid/Gas (regularity change)	Clear/Dark bubbles (variance change)	Smooth contours

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Conclusion

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Yuan	✗	✓	✓
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Coupled is the most satisfactory in term of segmentation quality ...

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Yuan	✗	✓	✓
T-ROF	✓	✓	✗
Joint	✓	✓	∼
Coupled	✓	✓	✓

Coupled is the most satisfactory in term of segmentation quality ...

... but it is the most time consuming (2100s)
Yuan(1s), T-ROF (12s), Joint (700s)

Ongoing work and perspectives

- Video analysis (temporal series of hundreds of images)

Intership of L. Helmlinger

Ongoing work and perspectives

- Video analysis (temporal series of hundreds of images)

  Intership of L. Helmlinger

  ✓ Best (λ, α) tuned on 1^{st} image is sufficiently robust for the entire series.

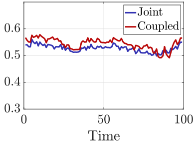
Ongoing work and perspectives

- Video analysis (temporal series of hundreds of images)

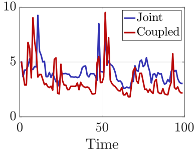
Intership of L. Helmlinger

✓ Best (λ, α) tuned on 1^{st} image is sufficiently robust for the entire series.
✓ Time evolution of physical quantities can be assessed.

Fraction of gas *(area)*



Liquid/gas contact *perimeter*



Ongoing work and perspectives

- Video analysis (temporal series of hundreds of images)

Intership of *L. Helmlinger*

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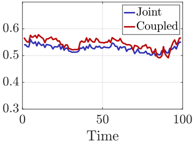

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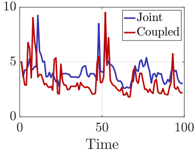
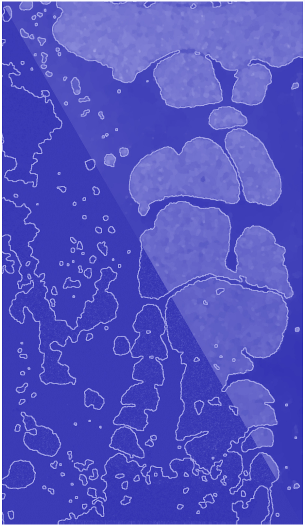


- Automatic tuning of hyperparameters

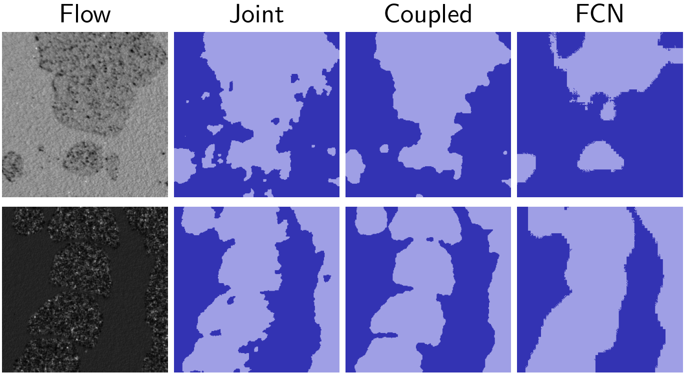
Stein's **U**nbiased **R**isk **E**stimate $\widehat{R}(\lambda, \alpha)$

Stein **U**nbiased **GrA**dient estimator of the **R**isk $\nabla_\lambda \widehat{R}(\lambda, \alpha)$



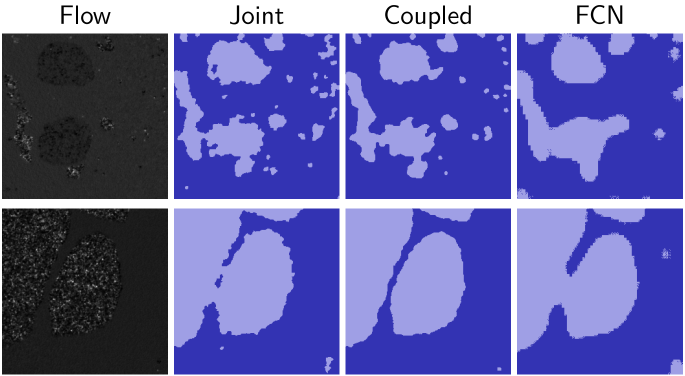
Thank you for your attention!

Fully Convolutional Neural Networks[†]



[†] V. Andrearczyk, `https://arxiv.org/abs/1703.05230`

Fully Convolutional Neural Networks[†]



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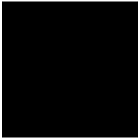
Gas/liquid flow modeled by piecewise monofractal textures

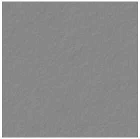
Synthetic textures

Liquid: $h_1 = 0.4$, $\sigma_1^2 = 10^{-2}$

Mask



Texture



Gas/liquid flow modeled by piecewise monofractal textures

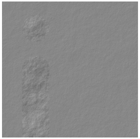
Synthetic textures

Liquid: $h_1 = 0.4$, $\sigma_1^2 = 10^{-2}$

Gas: $h_2 = 0.9$, $\sigma_1^2 = 10^{-2}$ (dark bubbles)

Mask

Texture



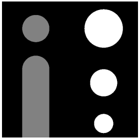
Gas/liquid flow modeled by piecewise monofractal textures

Synthetic textures

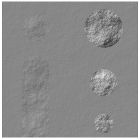
Liquid: $h_1 = 0.4,\ \sigma_1^2 = 10^{-2}$

Gas:
- $h_2 = 0.9,\ \sigma_1^2 = 10^{-2}$ (dark bubbles)
- $h_2 = 0.9,\ \sigma_2^2 = 10^{-1}$ (clear bubbles)

Mask



Texture



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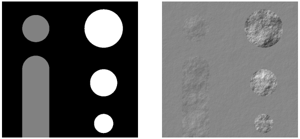
Mask | Texture



Segmentation performance

'Gas/Liquid' | Yuan 88% | T-ROF 88% | Joint 95% | Coupled 95%

Optimization scheme - Monofractal model and piecewise constancy

$$\underset{\boldsymbol{v},\boldsymbol{h}}{\text{minimize}} \sum_a \|\log \mathcal{L}_{a,\cdot} - \boldsymbol{v} - \log(a)\boldsymbol{h}\|^2 + \lambda \mathcal{R}(\boldsymbol{v}, \boldsymbol{h}; \alpha)$$

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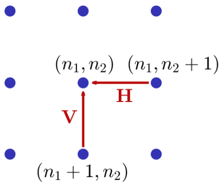
aim: enforce piecewise behavior of estimate

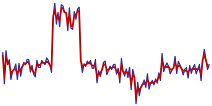


Discrete difference operator

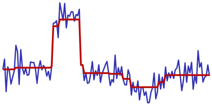
$$(\nabla x)_{n_1,n_2} := \begin{pmatrix} x_{n_1,n_2+1} - x_{n_1,n_2} \\ x_{n_1+1,n_2} - x_{n_1,n_2} \end{pmatrix} := \begin{bmatrix} \mathbf{H}\boldsymbol{x} \\ \mathbf{V}\boldsymbol{x} \end{bmatrix}_{n_1,n_2}$$

Total variation penalization

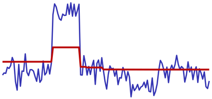
$$\mathcal{R}(\boldsymbol{x}) = \|\nabla \boldsymbol{x}\|_{2,1} = \sum_{n_1=1}^{N-1} \sum_{n_2=1}^{N-1} \sqrt{(\mathbf{H}\boldsymbol{x})^2_{n_1,n_2} + (\mathbf{V}\boldsymbol{x})^2_{n_1,n_2}}$$



Too small



Optimal



Too large

Optimization scheme - Monofractal model and piecewise constancy

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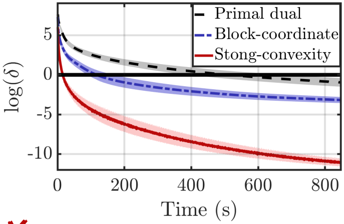
State-of-the-art - Segmentation on h only

$$\underset{\boldsymbol{h}}{\text{minimize}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{LR}}\|_2^2 + \lambda \mathcal{R}(\boldsymbol{h})$$

✓ only one parameter λ

✓ fast algorithms **[Pascal2018]**



✗ poor segmentation performance

$$\underset{\boldsymbol{h},\boldsymbol{\omega}}{\text{minimize}} \|\boldsymbol{h} - \textstyle\sum_a \boldsymbol{\omega}_a \mathcal{L}_{a,\cdot}\|_2^2 + \lambda \mathcal{R}(\boldsymbol{h}, \boldsymbol{\omega}; \alpha_a)$$

✗ additional constraints on $\{\omega\}_a$

✗ time and memory consuming

✓ very good accuracy **[Pustelnik2016]**

Convex conjugate of data fidelity term

$$\varphi_{\mathbf{A}}^*(\boldsymbol{v},\boldsymbol{h}) = \sup_{\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{h}}\in\mathbb{R}^{|\Omega|}} \langle \widetilde{\boldsymbol{v}},\boldsymbol{v}\rangle + \langle \widetilde{\boldsymbol{h}},\boldsymbol{h}\rangle - \varphi_{\mathbf{A}}(\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{h}}) \underset{\text{(if sup is reached)}}{=} \langle \bar{\boldsymbol{v}},\boldsymbol{v}\rangle + \langle \bar{\boldsymbol{h}},\boldsymbol{h}\rangle - \varphi_{\mathbf{A}}(\bar{\boldsymbol{v}},\bar{\boldsymbol{h}}).$$

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Euler condition

$$\begin{cases} \boldsymbol{v} - 2\sum_a \left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log\mathcal{L}_{a,.}\right) = 0 \\ \boldsymbol{h} - 2\sum_a \log(a)\left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log\mathcal{L}_{a,.}\right) = 0 \end{cases}$$

Convex conjugate of data fidelity term

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$$\mathcal{S} = \sum_a \log \mathcal{L}_{a,\cdot} \quad \text{and} \quad \mathcal{T} = \sum_a \log(a) \log \mathcal{L}_{a,\cdot},$$

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$$\mathcal{S} = \sum_a \log \mathcal{L}_{a,.} \quad \text{and} \quad \mathcal{T} = \sum_a \log(a)\log \mathcal{L}_{a,.},$$

$$\forall m = \{0,1,2\},\ \mathrm{A}_m = \sum_a (\log a)^m, \quad \mathbf{A}^*\mathbf{A} = \begin{pmatrix}\mathrm{A}_0\boldsymbol{I} & \mathrm{A}_1\boldsymbol{I}\\ \mathrm{A}_1\boldsymbol{I} & \mathrm{A}_2\boldsymbol{I}\end{pmatrix}$$

Convex conjugate of data fidelity term

$$\varphi^*_{\mathbf{A}}(\boldsymbol{v},\boldsymbol{h}) = \sup_{\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{h}}\in\mathbb{R}^{|\Omega|}} \langle \widetilde{\boldsymbol{v}},\boldsymbol{v}\rangle + \langle \widetilde{\boldsymbol{h}},\boldsymbol{h}\rangle - \varphi_{\mathbf{A}}(\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{h}}) \underset{\text{(if sup is reached)}}{=} \langle \bar{\boldsymbol{v}},\boldsymbol{v}\rangle + \langle \bar{\boldsymbol{h}},\boldsymbol{h}\rangle - \varphi_{\mathbf{A}}(\bar{\boldsymbol{v}},\bar{\boldsymbol{h}}).$$

Euler condition

$$\begin{cases} \boldsymbol{v} - 2\sum_a \left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.}\right) = 0 \\ \boldsymbol{h} - 2\sum_a \log(a)\left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.}\right) = 0 \end{cases} \iff \mathbf{A}^*\mathbf{A}\begin{pmatrix}\bar{\boldsymbol{v}}\\ \bar{\boldsymbol{h}}\end{pmatrix} = \begin{pmatrix}\boldsymbol{v}/2 + \boldsymbol{\mathcal{S}}\\ \boldsymbol{h}/2 + \boldsymbol{\mathcal{T}}\end{pmatrix}$$

$$\boldsymbol{\mathcal{S}} = \sum_a \log \boldsymbol{\mathcal{L}}_{a,.} \quad \text{and} \quad \boldsymbol{\mathcal{T}} = \sum_a \log(a) \log \boldsymbol{\mathcal{L}}_{a,.},$$

$$\forall m = \{0,1,2\},\ \mathrm{A}_m = \sum_a (\log a)^m, \quad \mathbf{A}^*\mathbf{A} = \begin{pmatrix}\mathrm{A}_0\boldsymbol{I} & \mathrm{A}_1\boldsymbol{I}\\ \mathrm{A}_1\boldsymbol{I} & \mathrm{A}_2\boldsymbol{I}\end{pmatrix}$$

$$\varphi^*_{\mathbf{A}}(\boldsymbol{v},\boldsymbol{h}) = \frac{1}{4}\langle(\boldsymbol{v},\boldsymbol{h}),(\mathbf{A}^*\mathbf{A})^{-1}(\boldsymbol{v},\boldsymbol{h})\rangle + \langle(\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{T}}),(\mathbf{A}^*\mathbf{A})^{-1}(\boldsymbol{v},\boldsymbol{h})\rangle + \mathcal{C}$$

where $\mathcal{C}$ constant term only depending on $\boldsymbol{\mathcal{L}}(X)$.