Credibility interval Design for Covid19 Reproduction Number from Nonsmooth Langevin-type Monte Carlo sampling

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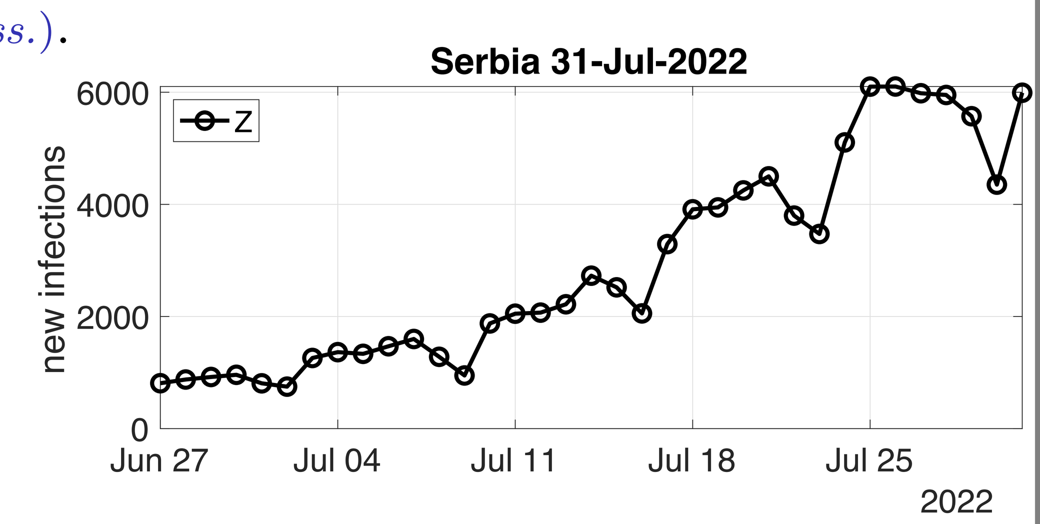
Aims and contributions

Monitoring the Covid19 pandemic is critical to design sanitary policies and to quantify their effectiveness. Reliable estimates of the pandemic *reproduction number* R_t were obtained from a nonsmooth convex optimization procedure designed to fit epidemiology requirements and to be robust to the low quality of the data (outliers, pseudo-seasonalities, ...) (Abry et al., 2020, *PlosOne* ; Pascal et al., 2022, *IEEE Trans. Sig. Process.*).

Purposes:
- grounded monitoring tools, based on sound epidemiological models,
- robust to low quality of the data (outliers data, abnormally small counts on week-ends, ...),
- accompanied by a confidence level, quantifying the uncertainty.

Methods:
- Bayesian paradigm leading to a nonsmooth posterior distribution,
- new Monte Carlo sampler combining Langevin dynamics and proximal operators,
- credibility interval estimate of the reproduction number R_t from Markov chains Monte Carlo samplers.



data from Johns Hopkins University

Bayesian model

(Cori et al., 2013, *Am. Journal of Epidemiology* ; Abry et al., 2020, *PlosOne* ; Pascal et al., 2022, *IEEE Trans. Sig. Process.* ; Fort et al., 2022, *preprint*)

Cori's model with outliers

T observations $\mathbf{Z} = (\mathsf{Z}_1, \ldots, \mathsf{Z}_T)^\top \in \mathbb{N}^T$, Z_t: new infections on day t

$$\mathbb{P}(\mathsf{Z}_t | \mathsf{Z}_1, \ldots, \mathsf{Z}_{t-1}) = \mathrm{Poiss}\left(p_t(\theta)\right) \text{ with } p_t(\theta) = \mathsf{R}_t \left(\Phi \mathsf{Z}\right)_t + \mathsf{O}_t, \quad (\Phi \mathsf{Z})_t = \sum_{u=1}^{\tau_\phi} \Phi_u \mathsf{Z}_{t-u},$$

parameterized by $\theta = (\mathsf{R}_1, \ldots, \mathsf{R}_T, \mathsf{O}_1, \ldots, \mathsf{O}_T) \in (\mathbb{R}_+)^T \times \mathbb{R}^T$

R_t: effective reproduction number ; O_t: models a possible outlier value

Φ: serial interval function $\Rightarrow$ random delay between primary and secondary cases

Negative log-likelihood from Poisson model (by convention $0 \ln 0 = 0$)

$$f(\theta) := \begin{cases} -\sum_{t=1}^T \left(\mathsf{Z}_t \ln p_t(\theta) - p_t(\theta)\right) & \text{if } \theta \in \mathcal{D} = \{\theta \,|\, \forall t,\ p_t(\theta) \geq 0\}, \\ +\infty & \text{otherwise} \end{cases}$$

Negative a priori log-distribution

- reproduction number: $\mathsf{R}_t - 2\mathsf{R}_{t-1} + \mathsf{R}_{t-2} \sim \mathrm{Laplace}(\lambda_{\mathrm{R}})$
- outliers $\mathsf{O}_t \sim \mathrm{Laplace}(\lambda_{\mathrm{O}})$

$$\Longrightarrow \quad g(\theta) = \lambda_{\mathrm{R}} \|\mathsf{D}_2 \mathbf{R}\|_1 + \lambda_{\mathrm{O}} \|\mathbf{O}\|_1,$$

$$\mathsf{D}_2 = \underbrace{\frac{1}{\sqrt{6}} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & 0 & \ldots & 0 \\ \ldots & & & & & & \ldots \\ 0 & \ldots & & & 1 & -2 & 1 \end{bmatrix}}_{\text{Laplacian}}$$

A posteriori distribution $\pi(\theta) \propto \exp\left(-f(\theta) - g(\theta)\right) \mathbb{1}_{\mathcal{D}}(\theta)$

f, g convex, $\quad f$ smooth, g nonsmooth

Monte Carlo sampler

Langevin dynamics

(Kent, 1978, *Adv Appl Probab*)

$\triangleright$ $\theta^{n+\frac{1}{2}} = \mu(\theta^n) + \sqrt{2\gamma} \xi^{n+1}, \quad \xi^{n+1} \sim \mathcal{N}_{2T}(0, \mathsf{I})$

$\mu(\theta)$ adapted to $\pi(\theta) = \exp(-f(\theta) - g(\theta)) \mathbb{1}_{\mathcal{D}}(\theta)$

$\triangleright$ θ^{n+1} via Acceptance-Rejection step

Main challenge:

$$-\ln \pi = f + g = f + h(\mathsf{A} \cdot)$$

closed-form expression of $\mathrm{prox}_{\gamma h}$ but not of $\mathrm{prox}_{\gamma h(\mathsf{A}\cdot)}$

Proximal-gradient dual strategy (Artigas et al., 2022, *EUSIPCO* ; Fort et al., 2022, *preprint*)

1) extend A into **invertible** $\overline{\mathsf{A}}$, and h in $\bar{h}$ such that $\bar{h}(\overline{\mathsf{A}}\theta) = h(\mathsf{A}\theta)$

2) work with the **dual** variable: $\underset{\theta \in \mathbb{R}^{2T}}{\mathrm{argmax}} \ln \pi(\theta) = \underset{\theta \in \mathbb{R}^{2T}}{\mathrm{argmin}}\ f(\theta) + \bar{h}(\overline{\mathsf{A}}\theta) = \mathsf{A}^{-1} \underset{\tilde{\theta} \in \mathbb{R}^{2T}}{\mathrm{argmin}}\ f(\overline{A}^{-1}\tilde{\theta}) + \bar{h}(\tilde{\theta})$

$$\boxed{\Longrightarrow \quad \mu(\theta) = \underbrace{\overline{\mathsf{A}}^{-1}}_{\text{back to } \theta} \underbrace{\mathrm{prox}_{\gamma \bar{h}} \left(\overline{\mathsf{A}}\theta - \gamma \overline{A}^{-\top} \nabla f(\theta)\right)}_{\text{proximal-gradient on } \tilde{\theta} = \overline{A}\theta}}$$

Two strategies to extend $\mathsf{A} = \begin{pmatrix} \mathsf{D}_2 & 0 \\ 0 & \mathsf{I} \end{pmatrix} \in \mathbb{R}^{(2T-1) \times 2T}$ into $\overline{\mathsf{A}} = \begin{pmatrix} \overline{\mathsf{D}} & 0 \\ 0 & \mathsf{I} \end{pmatrix} \in \mathbb{R}^{2T \times 2T}$:

`Invert`

$$\overline{\mathsf{D}}_2 := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 & \cdots & 0 \\ & \mathsf{D}_2 & & & \end{bmatrix}$$

`Ortho`

$$\overline{\mathsf{D}}_o := \begin{bmatrix} v_1^\top \\ v_2^\top \\ \mathsf{D}_2 \end{bmatrix} \qquad \begin{array}{c} v_1, v_2 \in \mathbb{R}^{2T} \\ v_1 \perp v_2,\ v_1, v_2 \in (\mathsf{D}_2^\top)^\perp \end{array}$$

Metropolis-Hastings

Data: $\overline{\mathsf{D}} = \overline{\mathsf{D}}_2$ or $\overline{\mathsf{D}} = \overline{\mathsf{D}}_o$ $\gamma_{\mathrm{R}}, \gamma_{\mathrm{O}} > 0$, $N_{\max} \in \mathbb{N}_\star$, $\theta^0 = (\mathbf{R}^0, \mathbf{O}^0) \in \mathcal{D}$

Result: A $\mathcal{D}$-valued sequence $\{\theta^n = (\mathbf{R}^n, \mathbf{O}^n), n \in 0, \ldots, N_{\max}\}$

for $n = 0, \ldots, N_{\max} - 1$ **do**

Sample indep. $\xi_{\mathrm{R}}^{n+1}, \xi_{\mathrm{O}}^{n+1} \sim \mathcal{N}_T(0, \mathsf{I})$;

$\mathbf{R}^{n+\frac{1}{2}} = \mu_{\mathrm{R}}(\theta^n) + \sqrt{2\gamma_{\mathrm{R}}} \overline{\mathsf{D}}^{-1} \overline{\mathsf{D}}^{-\top} \xi_{\mathrm{R}}^{n+1}$

$\mathbf{O}^{n+\frac{1}{2}} = \mu_{\mathrm{O}}(\theta^n) + \sqrt{2\gamma_{\mathrm{O}}}\, \xi_{\mathrm{O}}^{n+1}$

$\theta^{n+\frac{1}{2}} = (\mathbf{R}^{n+\frac{1}{2}}, \mathbf{O}^{n+\frac{1}{2}})$

$\theta^{n+1} = \theta^{n+\frac{1}{2}}$ with probability

$$1 \wedge \frac{\pi(\theta^{n+\frac{1}{2}})}{\pi(\theta^n)} \frac{q_{\mathrm{R}}(\theta^{n+\frac{1}{2}}, \theta_{\mathrm{R}}^n)}{q_{\mathrm{R}}(\theta^n, \theta_{\mathrm{R}}^{n+\frac{1}{2}})} \frac{q_{\mathrm{O}}(\theta^{n+\frac{1}{2}}, \theta_{\mathrm{O}}^n)}{q_{\mathrm{O}}(\theta^n, \theta_{\mathrm{O}}^{n+\frac{1}{2}})},$$

$q_{\mathrm{R/O}}$: Gaussian kernels

and $\theta^{n+1} = \theta^n$ otherwise.

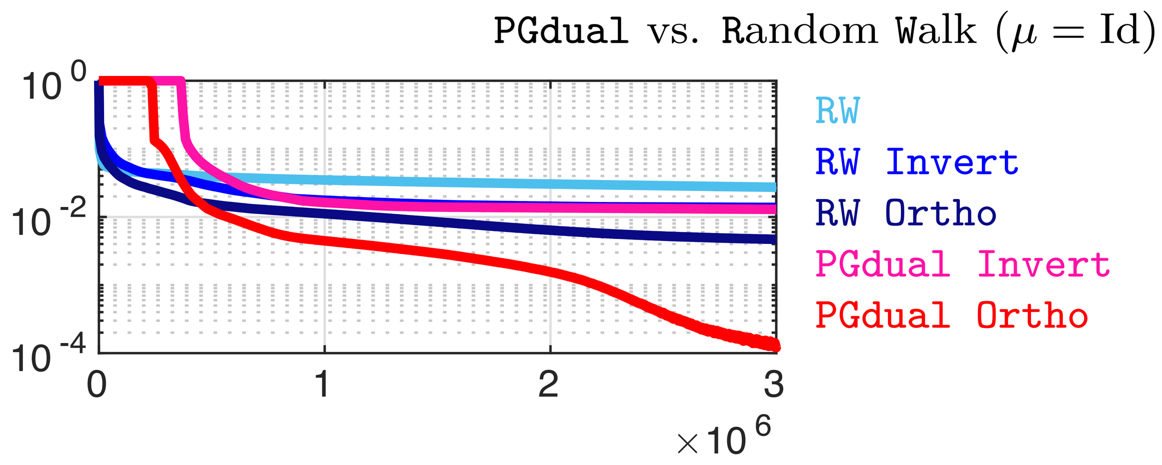
end

Algorithm 1: Proximal-Gradient dual: `PGdual Invert` and `PGdual Ortho`

Numerical experiment

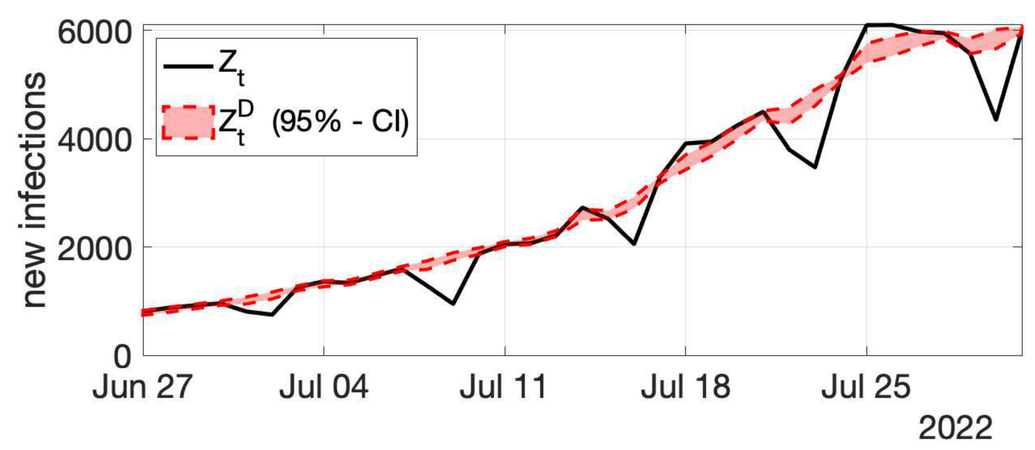
Convergence of the chain $\{\theta^n, n \in \mathbb{N}\}$

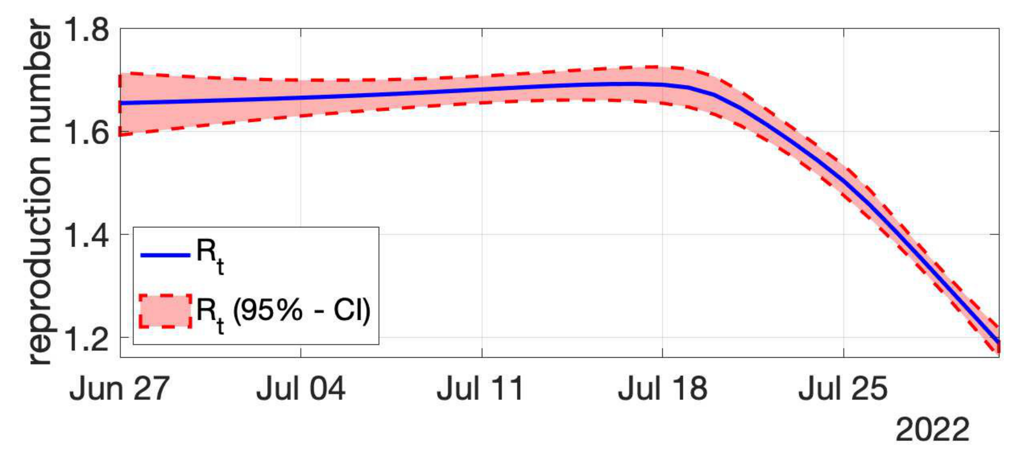
`PGdual` vs. Random Walk ($\mu = \mathrm{Id}$)



$(\ln \pi(\theta^n) - \max \ln \pi) \,/\, \left(\ln \pi(\theta^0) - \max \ln \pi\right)$

Denoised $\mathbf{Z}^{\mathrm{D}} = \mathbf{Z} - \widehat{\mathbf{O}}$ and $\widehat{\mathbf{R}}$





Perspectives

- hyperparameter tuning $(\gamma_{\mathrm{R/O}}, \lambda_{\mathrm{R}}, \lambda_{\mathrm{O}}, \ldots)$;
- handle missing counts: $\mathsf{Z}_{\mathrm{wknd}}$ not reported.