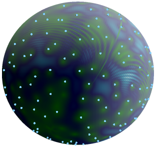
A link between Majorana Stellar representation of pure spin states and Coulomb gas on the sphere

Barbara Pascal

May 28, 2021



from Patrick Bruno, `Physical Review Letters`, 2012

"Quantum Geometric Phase in Majorana's Stellar Representation: Mapping onto a Many-Body Aharonov-Bohm Phase"

Selected for a Viewpoint in *Physics*
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Quantum Geometric Phase in Majorana's Stellar Representation: Mapping onto a Many-Body Aharonov-Bohm Phase

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(Received 10 April 2012; published 11 June 2012)

The (Berry-Aharonov-Anandan) geometric phase acquired during a cyclic quantum evolution of finite-dimensional quantum systems is studied. It is shown that a pure quantum state in a $(2J + 1)$-dimensional Hilbert space (or, equivalently, of a spin-J system) can be mapped onto the partition function of a gas of independent Dirac strings moving on a sphere and subject to the Coulomb repulsion of $2J$ fixed test charges (the Majorana stars) characterizing the quantum state. The geometric phase may be viewed as the Aharonov-Bohm phase acquired by the Majorana stars as they move through the gas of Dirac strings. Expressions for the geometric connection and curvature, for the metric tensor, as well as for the multipole moments (dipole, quadrupole, etc.), are given in terms of the Majorana stars. Finally, the geometric formulation of the quantum dynamics is presented and its application to systems with exotic ordering such as spin nematics is outlined.

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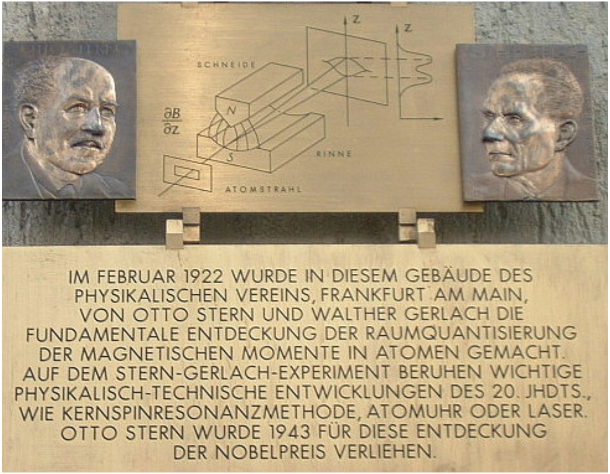
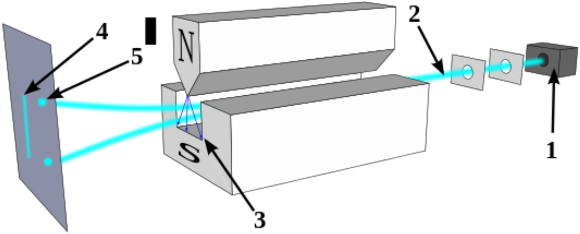
Spin angular momentum: first experimental evidence

Stern (conception 1921)-Gerlach (realization 1922) experiment:



Source: `Wikipedia`, *Peng, CC BY-SA 3.0*

Stern-Gerlach experiment



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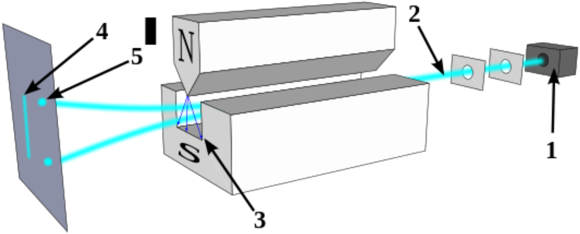
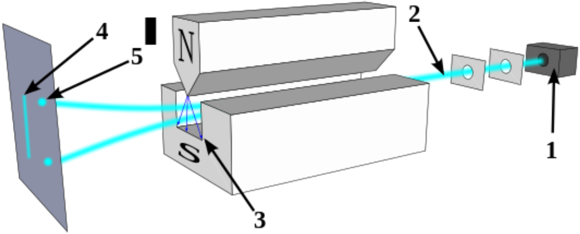


Silver atom beam through a *nonuniform* magnetic field of direction $\widehat{z}$

Stern-Gerlach experiment



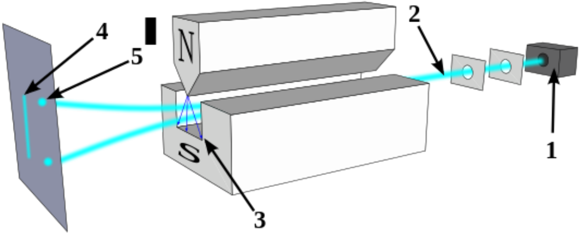
Silver atom beam through a *nonuniform* magnetic field of direction $\widehat{\boldsymbol{z}}$

Force on an object of magnetic moment $\vec{\mu}$ into a magnetic field $\vec{B}$

$$\vec{F} = \left(\vec{\mu} \cdot \vec{\nabla}\right) \vec{B}$$

- silver atoms have no orbital magnetic orbital moment $\vec{L} = \vec{0}$ ...

Stern-Gerlach experiment



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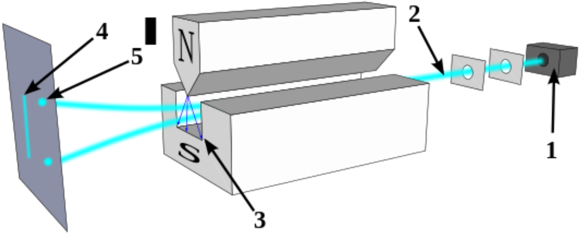

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- ... but the beam is deflected $\vec{\boldsymbol{\mu}} \neq \vec{\mathbf{0}}$**!**
- resulting in **two** accumulation points on the screen.

Total angular moment $\quad \vec{\mu} = \vec{L} + \vec{J}, \quad \vec{J}$: intrinsic quantum *spin* momentum

The *spin* from a physicist point of view

Elementary and composite particle *ex: electron, proton, neutron*

- mass
- electric charge
- ***spin***: *intrinsic angular momentum which is quantized* ***(new)***

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Silver atoms in Stern-Gerlach experiment

- two possible measurements: “spin *up*” $\uparrow$ or “spin *down*” $\downarrow$
- equal probability $\mathbb{P}_{1\ \text{atom}}(\uparrow) = \mathbb{P}_{1\ \text{atom}}(\downarrow) = 1/2$
- same deflection amplitude $|\vec{J}|_{\uparrow} = |\vec{J}|_{\downarrow}$.

Elements of quantum mechanics

State of the system Quantum Hilbert space

$$|\Psi\rangle \in \mathcal{H}$$

Superpositions: $|\Psi\rangle = \gamma_1|\Psi_1\rangle + \gamma_2|\Psi_2\rangle \in \mathcal{H}$, $\gamma_1, \gamma_2 \in \mathbb{C}$, $|\Psi_1\rangle, |\Psi_2\rangle \in \mathcal{H}$

Equivalence: If $|\Psi_2\rangle = \mathrm{e}^{\mathrm{i}\varphi}|\Psi_1\rangle$, for some $\varphi \in \mathbb{R}$ then

$$|\Psi_2\rangle \sim |\Psi_1\rangle \text{ describe the } \textit{same} \text{ physical state.}$$

Physical quantum state: $\mathcal{P} := \mathcal{H}/\sim$ *projective* space

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Hamiltonian dynamics Schrödinger equation

$$\frac{\mathrm{d}|\Psi(t)\rangle}{\mathrm{d}t} = \mathbf{H}(t)|\Psi\rangle$$

Time evolution: $|\Psi(t)\rangle = \exp\left(\mathrm{i}\int_0^t \mathbf{H}(t')\,\mathrm{d}t'\right)|\Psi(0)\rangle$

Elementary intrinsic angular momentum: spin-$1/2$

General spin-$1/2$ state superposition of "spin up" and "spin down"

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, \quad \alpha, \beta \in \mathbb{C}$$

Quantum Hilbert space: $|\psi\rangle \in \mathcal{H}^{(1/2)} := \mathbb{C}^2$

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Measurement probabilities

$$\mathbb{P}(\uparrow) = |\langle\uparrow|\,\psi\rangle|^2 = |\alpha|^2 \quad \text{and} \quad \mathbb{P}(\downarrow) = |\langle\downarrow|\,\psi\rangle|^2 = |\beta|^2$$

Normalization: $\mathbb{P}(\uparrow) + \mathbb{P}(\downarrow) = |\alpha|^2 + |\beta|^2 = 1$

Bloch's sphere

Spin-$1/2$ $\quad |\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$

$$|\psi\rangle = \cos\frac{\vartheta}{2}|\uparrow\rangle + \sin\frac{\vartheta}{2}\mathrm{e}^{\mathrm{i}\varphi}|\downarrow\rangle$$

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Appropriate variables to describe spins are *angles*.

REVIEWS OF MODERN PHYSICS VOLUME 17, NUMBERS 2 AND 3 APRIL–JULY, 1945

Atoms in Variable Magnetic Fields*

F. Bloch
Stanford University, Stanford University, California

AND

I. I. Rabi
Columbia University, New York, New York

Majorana star: $\widehat{\boldsymbol{u}} := -(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ on Bloch's sphere

Schwinger bosons: second quantization

One particle of spin-$1/2$ $\mathcal{H}^{(J)} = \mathbb{C}^2$

Let $\widehat{\boldsymbol{u}} := -(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, the associated spin-$1/2$ state is

$$|\widehat{\boldsymbol{u}}\rangle := \left(\cos\frac{\vartheta}{2}\mathbf{c}_\uparrow^\dagger + \sin\frac{\vartheta}{2}\mathrm{e}^{\mathrm{i}\varphi}\mathbf{c}_\downarrow^\dagger\right)|\emptyset\rangle := \mathbf{c}_{\widehat{\boldsymbol{u}}}^\dagger|\emptyset\rangle$$

$\mathbf{c}_\uparrow^\dagger$ (resp. $\mathbf{c}_\downarrow^\dagger$) *operator* creating a state $|\uparrow\rangle$ (resp. $|\downarrow\rangle$)

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Spin-J, $J \in \mathbb{N}^*/2$ as a $2J$ spin-$1/2$ particle state $\mathcal{H}^{(J)} = \mathbb{C}^{2J+1}$

Let $\boldsymbol{U} := \{\widehat{\boldsymbol{u}}_1, \dots, \widehat{\boldsymbol{u}}_{2J}\}$ $2J$ points on Bloch's sphere and define

$$|\Psi_{\boldsymbol{U}}^{(J)}\rangle := \frac{1}{\sqrt{(2J)!}}\left(\prod_{i=1}^{2J}\mathbf{c}_{\widehat{\boldsymbol{u}}_i}^\dagger\right)|\emptyset\rangle$$

Spin-J coherent states

Particular case *maximally degenerate constellations*

$$|\widehat{\boldsymbol{n}}^{(J)}\rangle := \frac{1}{\sqrt{(2J)!}}\left(\boldsymbol{c}_{-\widehat{\boldsymbol{n}}}\right)^{2J}|\emptyset\rangle, \qquad \boldsymbol{U} := \{-\underbrace{\widehat{\boldsymbol{n}},\ldots,-\widehat{\boldsymbol{n}}}_{2J \text{ copies}}\}$$

Not a basis of the quantum Hilbert space $\mathcal{H}^{(J)} = \mathbb{C}^{2J+1}$ ...

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Not a basis of the quantum Hilbert space $\mathcal{H}^{(J)} = \mathbb{C}^{2J+1}$ ...
... but a family indexed by S^2 which is well-suited for computations!

Scalar product of two coherent states

$$\left\langle \widehat{\boldsymbol{n}}_1^{(J)} \mid \widehat{\boldsymbol{n}}_2^{(J)} \right\rangle = \left(\frac{1 + \widehat{\boldsymbol{n}}_1 \cdot \widehat{\boldsymbol{n}}_2}{2}\right)^J \exp\left\{\mathrm{i}J\Sigma(\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{n}}_1, \widehat{\boldsymbol{n}}_2)\right\}$$

$\Sigma(\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{n}}_1, \widehat{\boldsymbol{n}}_2)$: oriented spherical area of the triangle $(\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{n}}_1, \widehat{\boldsymbol{n}}_2)$

Coherent state representation

Generative family of the quantum Hilbert space

$$\mathbf{1}_J = \frac{2J+1}{4\pi} \int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}} \; |\widehat{\boldsymbol{n}}^{(J)}\rangle\langle\widehat{\boldsymbol{n}}^{(J)}| \qquad \textit{Resolution of identity}$$

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Wavefunction on the sphere S^2

$$\forall \widehat{\boldsymbol{n}}, \quad \Psi_{\boldsymbol{U}}^{(J)}(\widehat{\boldsymbol{n}}) := \left\langle \widehat{\boldsymbol{n}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \right\rangle = \prod_{i=1}^{2J} \sqrt{\frac{1-\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{u}}_i}{2}} \exp\left\{\mathrm{i}J\Sigma(\widehat{\boldsymbol{z}},\widehat{\boldsymbol{n}},-\widehat{\boldsymbol{u}}_i)\right\}$$

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$$\mathbf{1}_J = \frac{2J+1}{4\pi} \int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}} \, |\widehat{\boldsymbol{n}}^{(J)}\rangle\langle\widehat{\boldsymbol{n}}^{(J)}| \qquad \textit{Resolution of identity}$$

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Probability distribution

$$Q_{\boldsymbol{U}}^{(J)}(\widehat{\boldsymbol{n}}) : = \left| \Psi_{\boldsymbol{U}}^{(J)}(\widehat{\boldsymbol{n}}) \right|^2 \qquad \textit{Husimi function}$$

$$\textit{(algebraic manipulations)} \quad = \prod_{i=1}^{2J} \frac{1 - \widehat{\boldsymbol{n}} \cdot \widehat{\boldsymbol{u}}_i}{2}$$

Majorana stars $\boldsymbol{U} = \{\widehat{\boldsymbol{u}}_1, \ldots, \widehat{\boldsymbol{u}}_{2J}\}$ are the zeros of the Husimi function.

$\mathcal{P}^{(J)}$ unambiguously parametrized by Majorana's stars "constellation":

$$\boldsymbol{U} = \{\widehat{\boldsymbol{u}}_i,\ i = 1, \ldots, 2J\}$$

Reminder: $\mathbf{c}^\dagger_{\widehat{\boldsymbol{u}}} := \cos\frac{\vartheta}{2}\mathbf{c}^\dagger_\uparrow + \sin\frac{\vartheta}{2}\mathrm{e}^{\mathrm{i}\varphi}\mathbf{c}^\dagger_\downarrow$

By construction

$$|\Psi^{(J)}_{\boldsymbol{U}}\rangle := \frac{1}{\sqrt{(2J)!}}\left(\prod_{i=1}^{2J}\mathbf{c}^\dagger_{\widehat{\boldsymbol{u}}_i}\right)|\emptyset\rangle = \mathtt{Polynomial}_\Psi(\mathbf{c}^\dagger_\uparrow, \mathbf{c}^\dagger_\downarrow)|\emptyset\rangle$$

$\mathtt{Polynomial}_\Psi$: homogeneous polynomial.

Reciprocally

Factorization of the Husimi function provides the Majorana stars

$$\boldsymbol{U} = \{\widehat{\boldsymbol{u}}_1, \ldots, \widehat{\boldsymbol{u}}_{2J}\}$$

System of interacting particles on the sphere

2D Coulomb potential on the sphere:

$$V(\widehat{\boldsymbol{u}}_1, \widehat{\boldsymbol{u}}_2) = -\ln d_{12}$$

$d_{12} = \sin^2(\vartheta_{12}/2) = \dfrac{1 - \widehat{\boldsymbol{u}}_1 \cdot \widehat{\boldsymbol{u}}_2}{2}$: chordal distance on S^2

Internal energy: for a configuration of Majorana stars $\boldsymbol{U} = \{\widehat{\boldsymbol{u}}_1, \ldots, \widehat{\boldsymbol{u}}_{2J}\}$

$$\forall \widehat{\boldsymbol{n}}, \quad E_{\boldsymbol{U}}(\widehat{\boldsymbol{n}}) := \sum_{i=1}^{2J} V(\widehat{\boldsymbol{n}}, \widehat{\boldsymbol{u}}_i)$$

Partition function: $Z(\boldsymbol{U}) := \dfrac{1}{4\pi} \displaystyle\int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}} \exp\{-E_{\boldsymbol{U}}(\widehat{\boldsymbol{n}})/T\}$

Free energy: $F(\boldsymbol{U}) := T \ln Z(\boldsymbol{U})$

Mapping between spin-J system and gas on the sphere

2D Coulomb potential on the sphere: $V(\widehat{\boldsymbol{u}}_1, \widehat{\boldsymbol{u}}_2) = -\ln \frac{1 - \widehat{\boldsymbol{u}}_1 \cdot \widehat{\boldsymbol{u}}_2}{2}$

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Boltzmann density for fixed "temperature" $T = 1$

$$\exp\{-E_{\boldsymbol{U}}(\widehat{\boldsymbol{n}})/T\} = \exp\left\{-\sum_{i=1}^{2J} -\ln\left(\frac{1 - \widehat{\boldsymbol{n}} \cdot \widehat{\boldsymbol{u}}_i}{2}\right)\right\} = \prod_{i=1}^{2J}\left(\frac{1 - \widehat{\boldsymbol{n}} \cdot \widehat{\boldsymbol{u}}_i}{2}\right)$$

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Partition function

$$Z(\boldsymbol{U}) := \frac{1}{4\pi}\int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}}\, \exp\{-E_{\boldsymbol{U}}(\widehat{\boldsymbol{n}})/T\} = \frac{1}{4\pi}\int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}}\, Q_{\boldsymbol{U}}^{(J)}(\widehat{\boldsymbol{n}})$$

$Q_{\boldsymbol{U}}^{(J)}(\widehat{\boldsymbol{n}})$: Husimi function associated to $|\Psi_{\boldsymbol{U}}^{(J)}\rangle$

Resolution of identity: $\mathbf{1}_J = \frac{2J+1}{4\pi} \int_{S^2} \mathrm{d}^2\hat{\boldsymbol{n}} \, |\hat{\boldsymbol{n}}^{(J)}\rangle\langle\hat{\boldsymbol{n}}^{(J)}|$

From partition function to vector norm

$$\begin{aligned}
Z(\boldsymbol{U}) &= \frac{1}{4\pi} \int_{S^2} \mathrm{d}^2\hat{\boldsymbol{n}} \, Q_{\boldsymbol{U}}^{(J)}(\hat{\boldsymbol{n}}) \\
&= \frac{1}{4\pi} \int_{S^2} \mathrm{d}^2\hat{\boldsymbol{n}} \left| \langle \hat{\boldsymbol{n}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle \right|^2 \\
&= \frac{1}{4\pi} \int_{S^2} \mathrm{d}^2\hat{\boldsymbol{n}} \, \langle \Psi_{\boldsymbol{U}}^{(J)} \mid \hat{\boldsymbol{n}}^{(J)} \rangle \langle \hat{\boldsymbol{n}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle \\
&= \langle \Psi_{\boldsymbol{U}}^{(J)} \mid \left(\frac{1}{4\pi} \int_{S^2} \mathrm{d}^2\hat{\boldsymbol{n}} \, |\hat{\boldsymbol{n}}^{(J)}\rangle\langle\hat{\boldsymbol{n}}^{(J)}| \right) \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle = \frac{\langle \Psi_{\boldsymbol{U}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle}{2J+1}
\end{aligned}$$

Mapping between spin-J system and gas on the sphere

$$Z(\boldsymbol{U}) = \frac{1}{4\pi} \int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}}\, Q_{\boldsymbol{U}}^{(J)}(\widehat{\boldsymbol{n}}) = \frac{\langle \Psi_{\boldsymbol{U}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle}{2J+1}$$

- Fictitious *classical* gas of independent particles living on the *sphere*
- With density $Q_{\boldsymbol{U}}^{(J)}$
- Interacting via Coulomb repulsion with $2J$ charges located at the $\widehat{\boldsymbol{u}}_i$'s

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Fictitious *indirect* interaction between the Majorana stars $\{\widehat{\boldsymbol{u}}_i\}_{i=1}^{2J}$

indirect: mediated by gas particles at "thermal equilibrium" at $T = 1$

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indirect: mediated by gas particles at “thermal equilibrium” at $T = 1$

Mapping of a spin-J *quantum* state onto a $2J$-body *classical* system.

Mapping between spin-J system and gas on the sphere

Observables in quantum mechanics $\langle \mathbf{O} \rangle_{\boldsymbol{U}} := \langle \Psi_{\boldsymbol{U}}^{(J)} \mid \mathbf{O} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle / \| \Psi_{\boldsymbol{U}}^{(J)} \|^2$

$$\langle \mathbf{O} \rangle_{\boldsymbol{U}} = \left(\frac{2J+1}{4\pi} \right)^2 \int_{S^2} \int_{S^2} \mathrm{d}^2 \widehat{\boldsymbol{n}} \, \mathrm{d}^2 \widehat{\boldsymbol{n}'} \, \frac{\langle \Psi_{\boldsymbol{U}}^{(J)} | \widehat{\boldsymbol{n}}^{(J)} \rangle \langle \widehat{\boldsymbol{n}}^{(J)} \mid \mathbf{O} | \widehat{\boldsymbol{n}'}^{(J)} \rangle \langle \widehat{\boldsymbol{n}'}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle}{\langle \Psi_{\boldsymbol{U}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle}$$

<u>If $\mathbf{O}$ acts diagonally on</u> $\{ | \widehat{\boldsymbol{n}}^{(J)} \rangle \}_{\widehat{\boldsymbol{n}} \in S^2}$: $\langle \widehat{\boldsymbol{n}}^{(J)} \mid \mathbf{O} | \widehat{\boldsymbol{n}}^{(J)} \rangle := \mathrm{O}(\widehat{\boldsymbol{n}})$

Averaged quantity

$$\begin{aligned} \langle \mathbf{O} \rangle_{\boldsymbol{U}} &= \frac{2J+1}{4\pi} \frac{1}{\langle \Psi_{\boldsymbol{U}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle} \int_{S^2} \mathrm{d}^2 \widehat{\boldsymbol{n}} \, \langle \Psi_{\boldsymbol{U}}^{(J)} | \widehat{\boldsymbol{n}}^{(J)} \rangle \mathrm{O}(\widehat{\boldsymbol{n}}) \langle \widehat{\boldsymbol{n}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle \\ &= \frac{2J+1}{4\pi} \frac{1}{\langle \Psi_{\boldsymbol{U}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle} \int_{S^2} \mathrm{d}^2 \widehat{\boldsymbol{n}} \, \mathrm{O}(\widehat{\boldsymbol{n}}) \left| \langle \widehat{\boldsymbol{n}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle \right|^2 \\ &= \frac{2J+1}{4\pi} \frac{1}{\langle \Psi_{\boldsymbol{U}}^{(J)} \mid \Psi_{\boldsymbol{U}}^{(J)} \rangle} \int_{S^2} \mathrm{d}^2 \widehat{\boldsymbol{n}} \, \mathrm{O}(\widehat{\boldsymbol{n}}) Q_{\boldsymbol{U}}^{(J)}(\widehat{\boldsymbol{n}}) \end{aligned}$$

Mapping between spin-J system and gas on the sphere

Reminder

$$Z(\boldsymbol{U}) = \int_{S^2} \frac{\mathrm{d}^2\widehat{\boldsymbol{n}}}{4\pi} \exp\left\{-\frac{E_U(\widehat{\boldsymbol{n}})}{T}\right\} = \int_{S^2} \frac{\mathrm{d}^2\widehat{\boldsymbol{n}}}{4\pi} Q^{(J)}_{\boldsymbol{U}}(\widehat{\boldsymbol{n}}) = \frac{\langle \Psi^{(J)}_{\boldsymbol{U}} \mid \Psi^{(J)}_{\boldsymbol{U}} \rangle}{2J+1}$$

Averaged quantum observable

$$\langle \mathbf{O} \rangle_{\boldsymbol{U}} = \frac{2J+1}{4\pi} \frac{1}{\langle \Psi^{(J)}_{\boldsymbol{U}} \mid \Psi^{(J)}_{\boldsymbol{U}} \rangle} \int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}}\, \mathrm{O}(\widehat{\boldsymbol{n}}) Q^{(J)}_{\boldsymbol{U}}(\widehat{\boldsymbol{n}}) = \frac{\int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}}\, \mathrm{O}(\widehat{\boldsymbol{n}}) Q^{(J)}_{\boldsymbol{U}}(\widehat{\boldsymbol{n}})}{\int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}}\, Q^{(J)}_{\boldsymbol{U}}(\widehat{\boldsymbol{n}})}$$

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Expectation value in statistical physics:

$$\langle f(\widehat{\boldsymbol{n}}) \rangle := \frac{\int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}}\, f(\widehat{\boldsymbol{n}}) \exp\{-E_{\boldsymbol{U}}(\widehat{\boldsymbol{n}})/T\}}{\int_{S^2} \mathrm{d}^2\widehat{\boldsymbol{n}}\, \exp\{-E_{\boldsymbol{U}}(\widehat{\boldsymbol{n}})/T\}}$$

Diagrammatic expressions

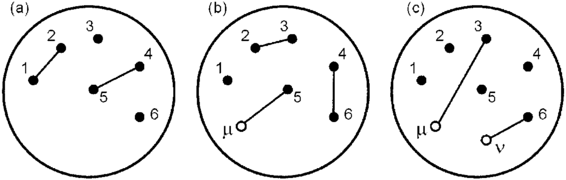
Statistical physics tools applied to quantum averaging

- $f(\widehat{\boldsymbol{n}}) = 1, \qquad Z(\boldsymbol{U}) = \dfrac{1}{2J+1} \displaystyle\sum_{n=0}^{[J]} (-1)^n \dfrac{(2J-n)!}{(2J)!} D_{\boldsymbol{U}}^{(J,n)}$
- $f(\widehat{\boldsymbol{n}}) = \widehat{n}_\nu, \qquad \langle \widehat{n}_\nu \rangle = \dfrac{1}{2(J+1)} \dfrac{\sum_{n=0}^{[J]} (-1)^n \frac{(2J-n)!}{(2J)!} D_{\boldsymbol{U}\nu}^{(J,n)}}{\sum_{n=0}^{[J]} (-1)^n \frac{(2J-n)!}{(2J)!} D_{\boldsymbol{U}}^{(J,n)}}$
- $f(\widehat{\boldsymbol{n}}) = \widehat{n}_\nu \widehat{n}_\rho, \qquad \langle \widehat{n}_\nu \widehat{n}_\rho \rangle = \dfrac{1}{2(J+1)(J+3)} \dfrac{\sum_{n=0}^{[J]} (-1)^n \frac{(2J-n)!}{(2J)!} D_{\boldsymbol{U}\nu\rho}^{(J,n)}}{\sum_{n=0}^{[J]} (-1)^n \frac{(2J-n)!}{(2J)!} D_{\boldsymbol{U}}^{(J,n)}}$

$$[J] := \begin{cases} J & \text{for } 2J \text{ even} \\ J - 1/2 & \text{for } 2J \text{ odd} \end{cases}$$

$D_{\boldsymbol{U}}^{(J,n)}$, $D_{\boldsymbol{U}\nu}^{(J,n)}$ and $D_{\boldsymbol{U}\nu\rho}^{(J,n)}$ computed from *diagrams*.

Diagrammatic rules to compute $D_{\boldsymbol{U}}^{(J,n)}$, $D_{\boldsymbol{U}\nu}^{(J,n)}$, $D_{\boldsymbol{U}\nu\rho}^{(J,n)}$



$\bullet$: Majorana stars, $\circ$: auxiliary stars

(i) draw all possible distinct diagrams with n pairing links

(ii) calculate the contribution of each possible diagram

- unlinked $\bullet$: factor 1
- unlinked $\circ$: factor 0
- link between $\bullet_i$ and $\bullet_j$: factor d_{ij}
- link between $\bullet_i$ and $\circ_\nu$: factor $(\widehat{u}_i)_\nu$
- link between $\circ_\mu$ and $\circ_\nu$: factor $-2\delta_{\mu\nu}$

E.g.: (a) $d_{12}d_{45}$, (b) $(\widehat{u}_5)_\mu\, d_{23}d_{46}$, (c) $(\widehat{u}_3)_\mu\, (\widehat{u}_6)_\nu$

(iii) sum all the contributions

Computation of physical quantities of interest

Magnetic energy $E_{\text{magnet.}} \propto -\vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{B}}$

$\vec{\boldsymbol{J}}$: angular momentum, $\vec{\boldsymbol{B}}$: magnetic field

Dipole moment J:

J_μ: μ-th component of spin operator $\vec{\mathbf{J}}$

$$\langle J_\mu \rangle = (J+1)\langle \widehat{n}_\mu \rangle$$

E.g.: for spin-1, $2J = 2$ Majorana stars $\widehat{\boldsymbol{u}}_1$ and $\widehat{\boldsymbol{u}}_2$, $\langle J_\nu \rangle = -\dfrac{(\widehat{u}_1)_\nu + (\widehat{u}_2)_\nu}{2 - d_{12}}$

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Quadrupole moment M

$$\boldsymbol{M}_{\nu,\rho} := \frac{\boldsymbol{J}_\nu \boldsymbol{J}_\rho + \boldsymbol{J}_\rho \boldsymbol{J}_\nu}{2} - \frac{J(J+1)}{3}\delta_{\nu\rho}$$

$$\langle \boldsymbol{M}_{\nu\rho} \rangle = (J+1)\left(J + \frac{3}{2}\right)\left(\langle \widehat{n}_\nu \widehat{n}_\rho \rangle - \frac{\delta_{\nu\rho}}{3}\right)$$

E.g.: $\langle \boldsymbol{M}_{\nu\rho} \rangle = \dfrac{1}{2 - d_{12}}\left(\dfrac{(\widehat{u}_1)_\nu (\widehat{u}_2)_\rho + (\widehat{u}_2)_\nu (\widehat{u}_1)_\rho}{2} - \widehat{\boldsymbol{u}}_1 \cdot \widehat{\boldsymbol{u}}_2 \dfrac{\delta_{\nu\rho}}{3}\right)$