Estimation régularisée d'attributs fractals par minimisation convexe pour la segmentation de textures.

Barbara Pascal

30 septembre 2020

Laboratoire de Physique à l'École Normale Supérieure de Lyon

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M. Bruno Torrésani	Rapporteur
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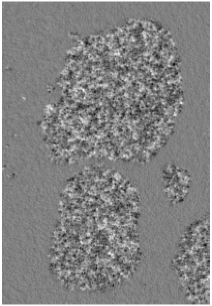
Segmentation d'image



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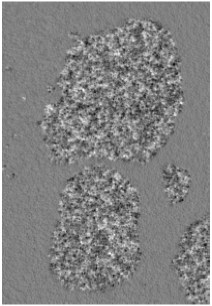

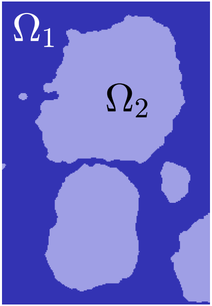

Objectif : obtenir une partition de l'image en K régions homogènes

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_K$$

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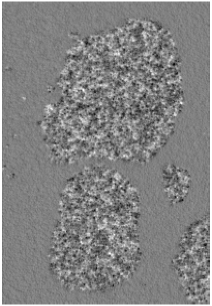

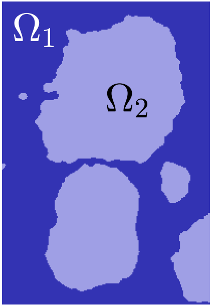

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Textures

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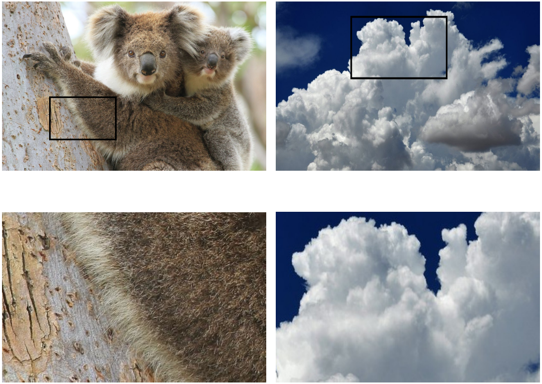

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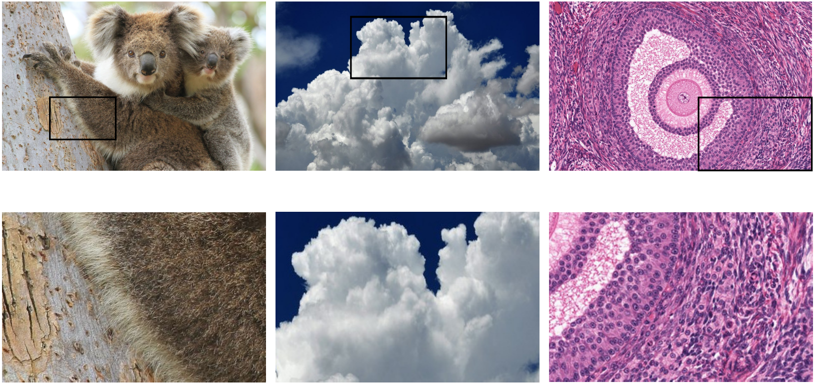

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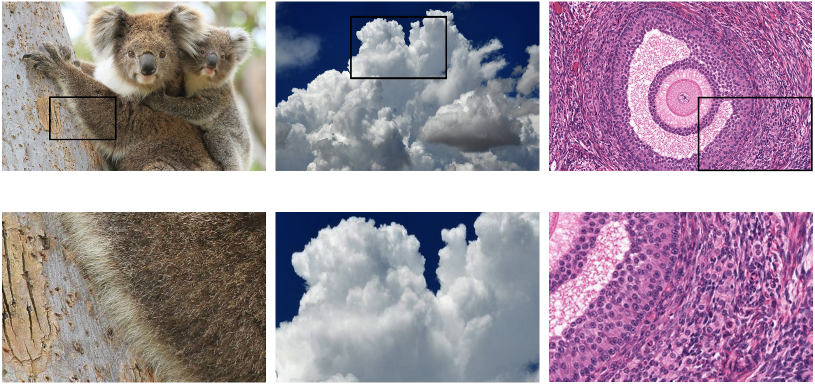
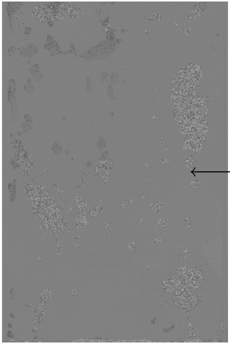


Crucial pour décrire les images réelles

Écoulement multiphasiques en milieu poreux

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



Mousse solide

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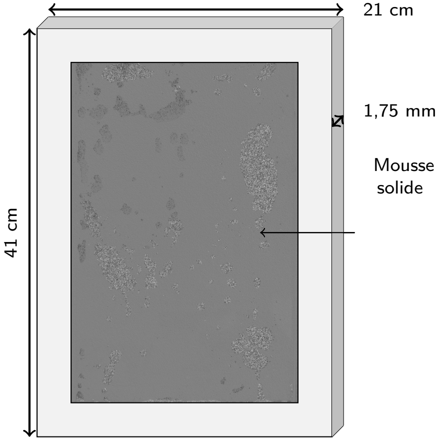

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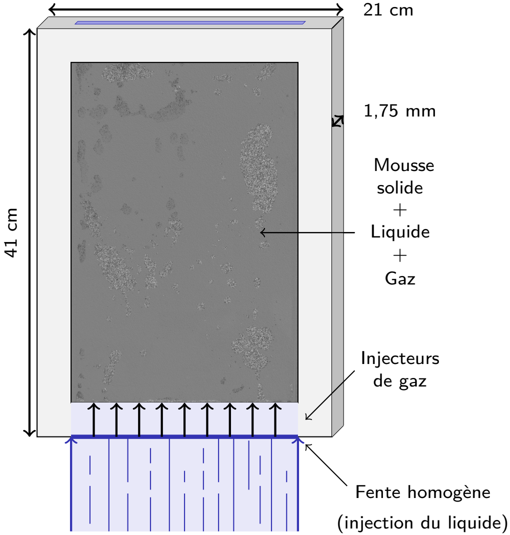

- → 1600 × 1100 pixels
- → vidéo : ∼ 1000 images
- → plan de phases : ∼ 10 débits

Plan de l'exposé

1. Caractérisation de textures

[Filtres de Gabor (Dunn, 1995)]
[Amplitude et fréquence locales (Havlicek, 1996)]
[Histogrammes spectraux (Yuan, 2015)]

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1. Caractérisation de textures

$\rightarrow$ attributs fractals

- variance locale σ^2
- régularité locale h

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[SURE DFMC (Ramani, 2008)]
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4. Réglage des hyperparamètres

→ SURE avec bruit gaussien corrélé
- erreur d'estimation projetée
- minimisation par quasi-Newton
  ↪ SUGAR généralisé

[SURE (Stein, 1981)]
[SURE DFMC (Ramani, 2008)]
[GSURE (Eldar, 2008)]
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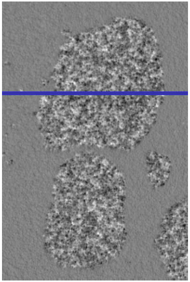
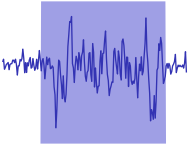
Modèle monofractal par morceaux

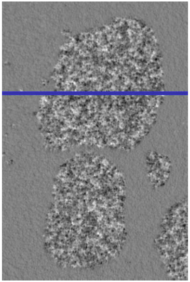


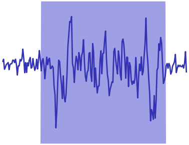


Modèle monofractal par morceaux

Attributs fractals

- variance σ^2 *amplitude des variations*

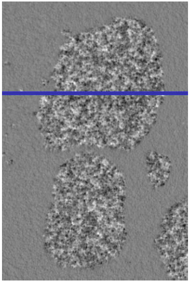


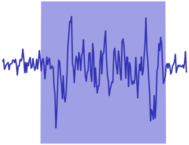


Modèle monofractal par morceaux

Attributs fractals

- variance σ^2 — *amplitude des variations*
- régularité locale h — *invariance d'échelle*



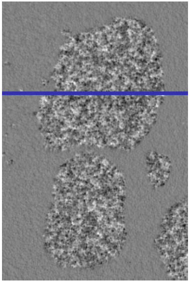


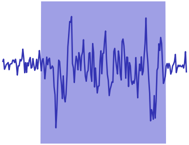
Modèle monofractal par morceaux

Attributs fractals

- variance σ^2 *amplitude des variations*
- régularité locale h *invariance d'échelle*

$$|f(x) - f(y)| \leq \sigma(x)|x - y|^{h(x)}$$





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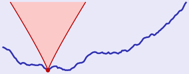
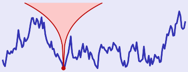
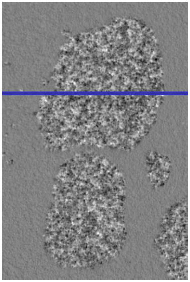
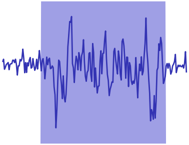


$h(x) \equiv h_1 = 0.9$



$h(x) \equiv h_2 = 0.3$





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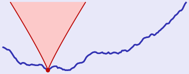

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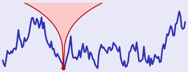

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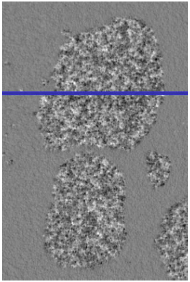

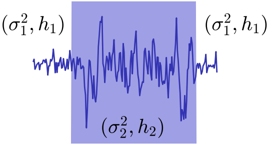

Segmentation

- h et σ^2 constants par morceaux

Modèle monofractal par morceaux

Attributs fractals

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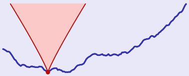

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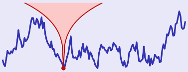

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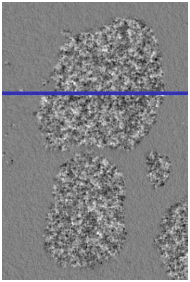
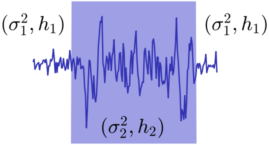


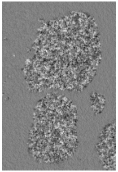


Segmentation

- h et σ^2 constants par morceaux
- région Ω_k caractérisée par (h_k, σ_k^2)

Analyse multi-échelle

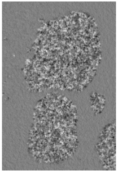
Image texturée

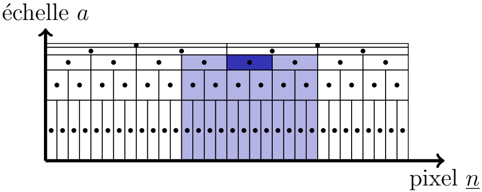


Analyse multi-échelle

Image texturée

Maximum local des coefficients d'ondelettes : $\mathcal{L}_{a,\cdot}$





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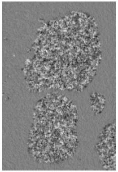
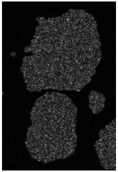
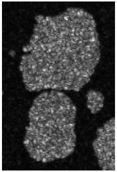
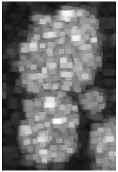


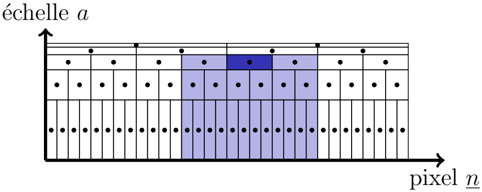
Maximum local des coefficients d'ondelettes : $\mathcal{L}_{a,\cdot}$

Échelle $a = 2^1$ $a = 2^2$ … $a = 2^5$









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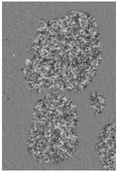

Maximum local des coefficients d'ondelettes : $\mathcal{L}_{a,\cdot}$

Échelle

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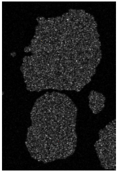
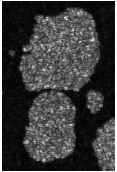
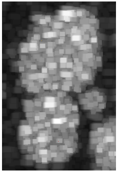


$a = 2^2$



$\ldots$

$a = 2^5$



Proposition *(Jaffard, 2004), (Wendt, 2008)*

$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underbrace{\boldsymbol{h}}_{\text{régularité}} + \underbrace{\boldsymbol{v}}_{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}$$

Analyse multi-échelle

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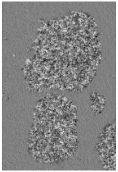

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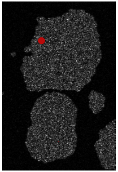

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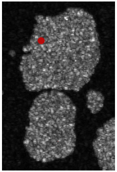

$\ldots$

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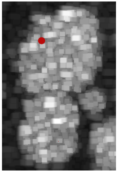

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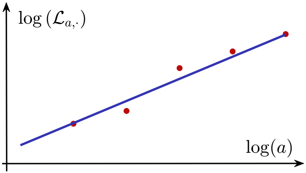
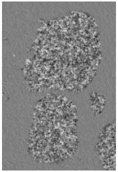
$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underbrace{\boldsymbol{h}}_{\text{régularité}} + \underbrace{\boldsymbol{v}}_{\substack{\propto \log(\boldsymbol{\sigma}^2)\\ \text{(variance)}}}$$



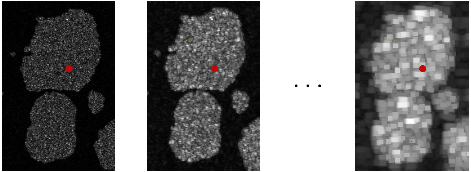
Analyse multi-échelle

Image texturée



Maximum local des coefficients d'ondelettes : $\mathcal{L}_{a,\cdot}$

Échelle $a = 2^1$ $a = 2^2$ … $a = 2^5$



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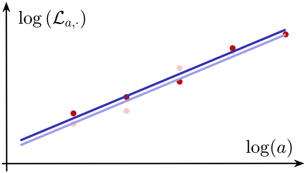

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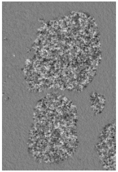

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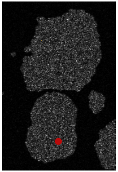

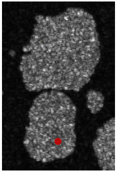

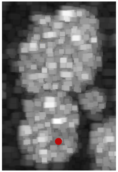

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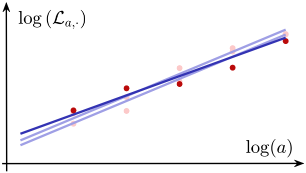

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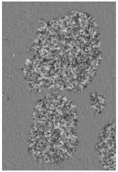
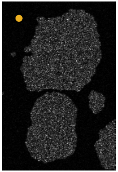


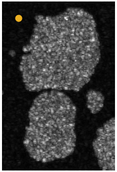
Maximum local des coefficients d'ondelettes : $\mathcal{L}_{a,\cdot}$

Échelle

$a = 2^1$

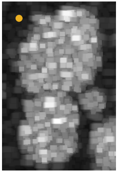


$a = 2^2$



$\cdots$

$a = 2^5$



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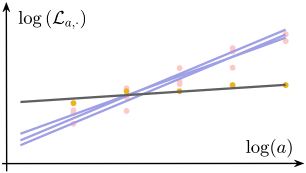

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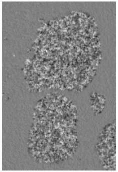

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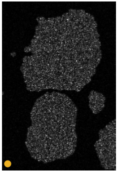

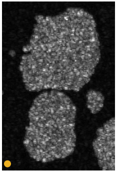

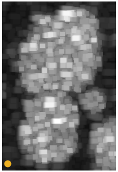

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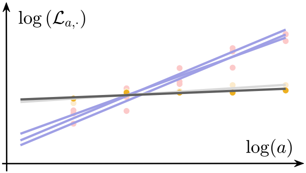

Analyse multi-échelle

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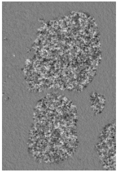

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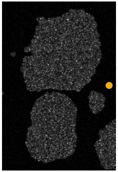

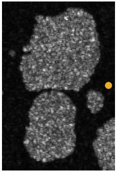

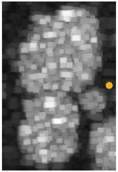

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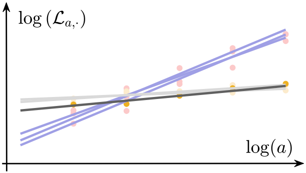
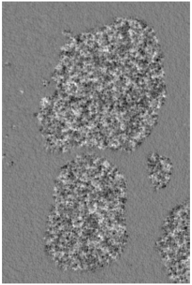


Estimation directe ponctuelle

Régression linéaire $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{régularité}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

Image texturée



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$$\left(\widehat{\boldsymbol{h}}^{\mathrm{RL}}, \widehat{\boldsymbol{v}}^{\mathrm{RL}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Image texturée

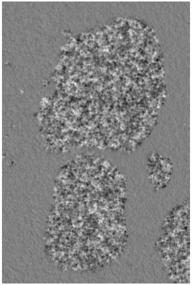

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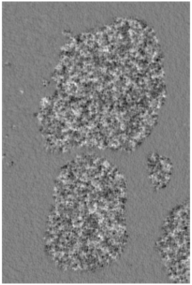
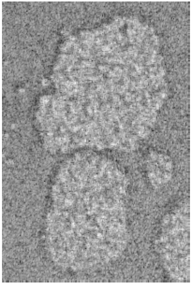
Image texturée



Régularité locale $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Puissance locale $\widehat{\boldsymbol{v}}^{\mathrm{RL}}$

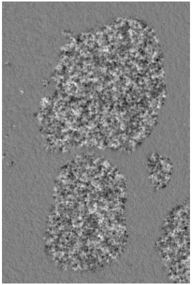


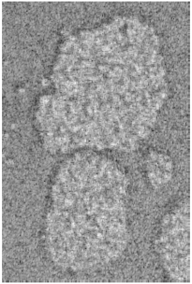
Estimation directe ponctuelle

Régression linéaire $\underbrace{\mathbb{E}\log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{espérance}} = \log(a)\underbrace{\bar{\boldsymbol{h}}}_{\text{régularité}} + \underbrace{\bar{\boldsymbol{v}}}_{\propto\log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{RL}},\widehat{\boldsymbol{v}}^{\mathrm{RL}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Image texturée | Régularité locale $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$ | Puissance locale $\widehat{\boldsymbol{v}}^{\mathrm{RL}}$





$\longrightarrow$ variance d'estimation élevée

Régularisation *a posteriori*

Régression linéaire $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

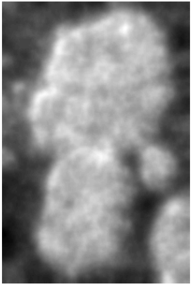
Régularisation *a posteriori*

Lissage par filtrage (linéaire)

$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{RL}}$$

Régression linéaire $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Lissage



Régularisation *a posteriori*

Lissage par filtrage (linéaire)

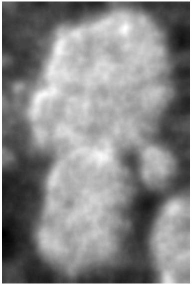
$$\left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \widehat{\boldsymbol{h}}^{\mathrm{RL}}$$

Débruitage ROF (non linéaire)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \; \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{RL}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Régression linéaire $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Lissage



ROF

Régularisation *a posteriori*

Lissage par filtrage (linéaire)

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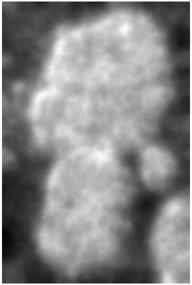


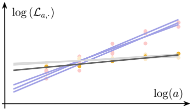
ROF

$\longrightarrow$ cumul de la variance d'estimation et du biais de régularisation

Fonctionnelles à contours libres et à contours co-localisés

$$\sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}}$$

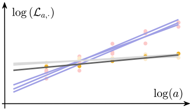
$\rightarrow$ fidélité au modèle log-linéaire



Fonctionnelles à contours libres et à contours co-localisés

$$\sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

$\rightarrow$ fidélité au modèle log-linéaire



$\rightarrow$ favorise la constance par morceaux

Fonctionnelles à contours libres et à contours co-localisés

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

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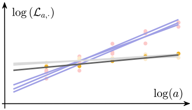

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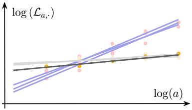

Différences finies $\mathbf{D}_1\boldsymbol{x}$ (horizontales), $\mathbf{D}_2\boldsymbol{x}$ (verticales) en chaque pixel

Fonctionnelles à contours libres et à contours co-localisés

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

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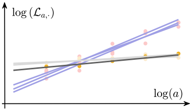


$\rightarrow$ favorise la constance par morceaux

Différences finies $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x},\mathbf{D}_2\boldsymbol{x}]$

libres : $\boldsymbol{h}$, $\boldsymbol{v}$ sont **indépendamment** constantes par morceaux

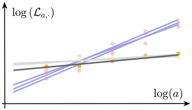
$$\mathcal{Q}_{\mathrm{L}}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

Fonctionnelles à contours libres et à contours co-localisés

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

→ fidélité au modèle log-linéaire

→ favorise la constance par morceaux



Différences finies $\mathbf{D}\boldsymbol{x} = [\mathbf{D}_1\boldsymbol{x},\mathbf{D}_2\boldsymbol{x}]$

<u>libres :</u> $\boldsymbol{h}$, $\boldsymbol{v}$ sont **indépendamment** constantes par morceaux

$$\mathcal{Q}_{\mathrm{L}}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha) = \alpha\|\mathbf{D}\boldsymbol{h}\|_{2,1} + \|\mathbf{D}\boldsymbol{v}\|_{2,1}$$

<u>co-localisés :</u> $\boldsymbol{h}$, $\boldsymbol{v}$ sont **concomitamment** constantes par morceaux

$$\mathcal{Q}_{\mathrm{C}}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha) = \|[\alpha\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v}]\|_{2,1}$$

Fonctionnelles à contours libres et à contours co-localisés

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

→ fidélité au modèle log-linéaire

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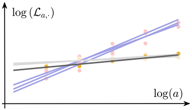
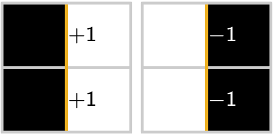


Contours disjoints

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

Contours communs



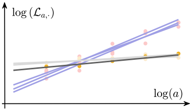
$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

Fonctionnelles à contours libres et à contours co-localisés

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

$\rightarrow$ fidélité au modèle log-linéaire

$\rightarrow$ favorise la constance par morceaux



Contours disjoints

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};1) = 4$

Contours communs

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

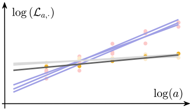
$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};1) = 4$

Fonctionnelles à contours libres et à contours co-localisés

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

→ fidélité au modèle log-linéaire

→ favorise la constance par morceaux



Contours disjoints

$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

$\mathcal{Q}_C(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 2 + \sqrt{2} \simeq 3{,}4$

Contours communs

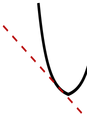
$\boldsymbol{h} \in \mathbb{R}^{2\times 2}$ $\quad$ $\boldsymbol{v} \in \mathbb{R}^{2\times 2}$

$\mathcal{Q}_L(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 4$

$\mathcal{Q}_C(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; 1) = 2\sqrt{2} \simeq 2{,}8$

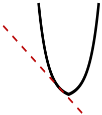
Minimisation de fonctionnelle

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$



Minimisation de fonctionnelle

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \ + \ \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

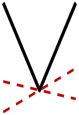


- descente de gradient $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau\nabla\varphi(\boldsymbol{x}^n)$

Minimisation de fonctionnelle

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

non lisse



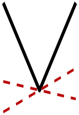
- descente de gradient $\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \tau \nabla \varphi(\boldsymbol{x}^n)$
- descente de sous-gradient implicite : algorithme du point proximal

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n - \boldsymbol{u}^n,\; \boldsymbol{u}^n \in \partial\varphi(\boldsymbol{x}^{n+1}) \;\Leftrightarrow\; \boldsymbol{x}^{n+1} = \mathrm{prox}_{\tau\varphi}(\boldsymbol{x}^n)$$

Minimisation de fonctionnelle

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

non lisse



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- algorithme proximal scindé

$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

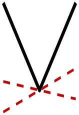
$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau\|\boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau \mathbf{D}^\top \boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

Minimisation de fonctionnelle

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

non lisse



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- algorithme proximal scindé $\quad \operatorname{prox}_{\tau\varphi}(\boldsymbol{x}) = \underset{\boldsymbol{u}}{\operatorname{argmin}} \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{u}\|^2 + \tau\varphi(\boldsymbol{u})$

$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma\mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

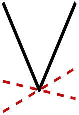
$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau\|\mathcal{L}-\boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau\mathbf{D}^\top\boldsymbol{y}^{n+1}\right), \qquad \boldsymbol{\Phi} : (\boldsymbol{h},\boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$$

$$\bar{\boldsymbol{x}}^{n+1} = 2\boldsymbol{x}^{n+1} - \boldsymbol{x}^n$$

Calcul des opérateurs proximaux

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underset{\textbf{Moindres Carrés}}{\underline{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}} \quad + \quad \lambda \underset{\textbf{Variation Totale}}{\underline{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}}$$

non lisse



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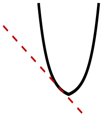
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \; + \; \lambda \; \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$



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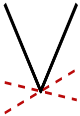


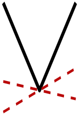
Ex. Norme mixte : pour $\boldsymbol{z} = [\boldsymbol{z}_1; \ldots,; \boldsymbol{z}_I]$

$$\mathcal{Q}(\boldsymbol{z}) = \|\boldsymbol{z}\|_{2,1} = \sum_{\underline{n}\in\Omega} \sqrt{\sum_{i=1}^{I} z_i^2(\underline{n})} = \sum_{\underline{n}\in\Omega} \|\boldsymbol{z}(\underline{n})\|_2$$

Calcul des opérateurs proximaux

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

non lisse



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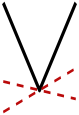
$$\boldsymbol{p} = \mathrm{prox}_{\lambda\|\cdot\|_{2,1}}(\boldsymbol{z}) \quad \Leftrightarrow \quad p_i(\underline{n}) = \max\left(0,1 - \frac{\lambda}{\|\boldsymbol{z}(\underline{n})\|_2}\right) z_i(\underline{n})$$

Calcul des opérateurs proximaux

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

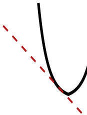
non lisse



Moindres carrés : $\|\log \boldsymbol{\mathcal{L}} - \Phi(\boldsymbol{h},\boldsymbol{v})\|^2, \quad \Phi : (\boldsymbol{h},\boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Calcul des opérateurs proximaux

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



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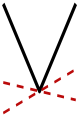


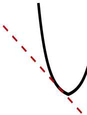
Moindres carrés : $\|\log \boldsymbol{\mathcal{L}} - \boldsymbol{\Phi}(\boldsymbol{h},\boldsymbol{v})\|^2, \quad \boldsymbol{\Phi} : (\boldsymbol{h},\boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Proposition *(Pascal, 2019)*

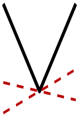
$$(\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) = \mathrm{prox}_{\tau\|\boldsymbol{\mathcal{L}}-\boldsymbol{\Phi}\cdot\|^2}(\boldsymbol{h},\boldsymbol{v}) \Longleftrightarrow (\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) = \left(\mathbf{I} + \tau\boldsymbol{\Phi}^\top\boldsymbol{\Phi}\right)^{-1}\left((\boldsymbol{h},\boldsymbol{v}) + \tau\boldsymbol{\Phi}^\top \log \boldsymbol{\mathcal{L}}\right)$$

Calcul des opérateurs proximaux

$$\operatorname*{minimiser}_{\boldsymbol{h},\boldsymbol{v}} \sum_a \underbrace{\|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$



non lisse



Moindres carrés : $\|\log \mathcal{L} - \Phi(\boldsymbol{h},\boldsymbol{v})\|^2, \quad \Phi : (\boldsymbol{h},\boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

Proposition *(Pascal, 2019)*

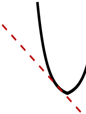
Soit $S_m = \sum_a \log^m(a), \;\; \mathcal{D} = (1+\tau S_2)(1+\tau S_0) - \tau^2 S_1^2,$
$\boldsymbol{\mathcal{T}} = \sum_a \log \mathcal{L}_a$ et $\boldsymbol{\mathcal{G}} = \sum_a \log(a) \log \mathcal{L}_a$, alors

$$(\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) = \mathrm{prox}_{\tau\|\mathcal{L}-\Phi\cdot\|^2}(\boldsymbol{h},\boldsymbol{v}) \Longleftrightarrow (\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) = \left(\mathbf{I} + \tau \Phi^\top \Phi\right)^{-1} \left((\boldsymbol{h},\boldsymbol{v}) + \tau \Phi^\top \log \mathcal{L}\right)$$

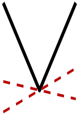
$$\Longleftrightarrow \begin{cases} \widetilde{\boldsymbol{h}} = \mathcal{D}^{-1}\left((1+\tau S_0)(\tau \boldsymbol{\mathcal{G}} + \boldsymbol{h}) - \tau S_1(\tau \boldsymbol{\mathcal{T}} + \boldsymbol{v})\right) \\ \widetilde{\boldsymbol{v}} = \mathcal{D}^{-1}\left((1+\tau S_2)(\tau \boldsymbol{\mathcal{T}} + \boldsymbol{v}) - \tau S_1(\tau \boldsymbol{\mathcal{G}} + \boldsymbol{h})\right) \end{cases}$$

Algorithme accéléré par forte-convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$



non lisse

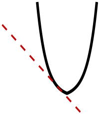


Algorithme primal-dual *(Chambolle, 2011)*

$$\delta \text{ : gap de dualité, } \delta(\boldsymbol{x}^n,\boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$$

Algorithme accéléré par forte-convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \ \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$



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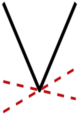
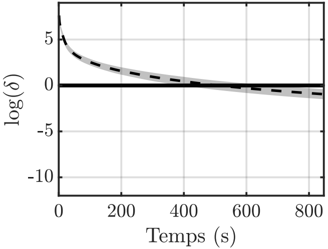


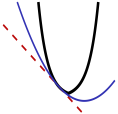
Algorithme primal-dual *(Chambolle, 2011)*

δ : gap de dualité, $\delta(\boldsymbol{x}^n,\boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$



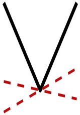
Propriétés de convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underbrace{\left\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2}_{\textbf{Moindres Carrés}} + \lambda\, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$



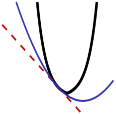
$\boldsymbol{\mu}$-fortement convexe

non lisse



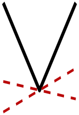
Propriétés de convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



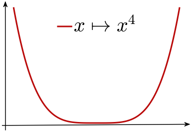
$\boldsymbol{\mu}$-fortement convexe

non lisse

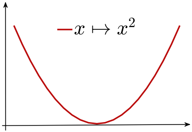


Forte convexité

- φ $\boldsymbol{\mu}$-fortement convexe ssi $\varphi - \dfrac{\boldsymbol{\mu}}{2}\|\cdot\|^2$ convexe



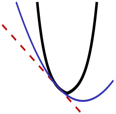
✓ strictement convexe
✗ non fortement convexe



✓ strictement convexe
✓ 1-fortement convexe

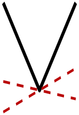
Propriétés de convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \; + \; \underbrace{\lambda\, \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$



μ-fortement convexe

non lisse

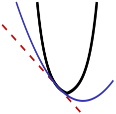


Forte convexité

- φ μ-fortement convexe ssi $\varphi - \frac{\mu}{2}\|\cdot\|^2$ convexe
- φ $\mathcal{C}^2$ de hessienne $\boldsymbol{H}\varphi \succeq 0 \implies \mu = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

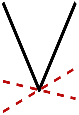
Propriétés de convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \ \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \ + \ \lambda \ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$



μ-fortement convexe

non lisse



Forte convexité

- φ μ-fortement convexe ssi $\varphi - \dfrac{\mu}{2}\|\cdot\|^2$ convexe
- φ $\mathcal{C}^2$ de hessienne $\boldsymbol{H}\varphi \succeq 0 \implies \mu = \min \mathrm{Sp}(\boldsymbol{H}\varphi)$

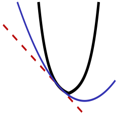
Proposition *(Pascal, 2019)*

$\sum_a \|\log \boldsymbol{\mathcal{L}} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2$ est μ-fortement convexe.

$a_{\min} = 2^1$, $a_{\max}$	2^2	2^3	2^4	2^5	2^6
$\mu = \min \mathrm{Sp}\left(2\boldsymbol{\Phi}^\top\boldsymbol{\Phi}\right)$	0.29	**0.72**	1.20	1.69	2.20

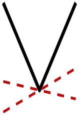
Algorithme accéléré par forte-convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \ \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$



μ-fortement convexe

non lisse



Algorithme primal-dual **accéléré** *(Chambolle, 2011)*

for $n = 0,1,\ldots$ $\qquad \boldsymbol{x} = (\boldsymbol{h},\boldsymbol{v})$

$$\boldsymbol{y}^{n+1} = \operatorname{prox}_{\sigma_n(\lambda\mathcal{Q})^*}\left(\boldsymbol{y}^n + \sigma_n \mathbf{D}\bar{\boldsymbol{x}}^n\right)$$

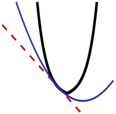
$$\boldsymbol{x}^{n+1} = \operatorname{prox}_{\tau_n\|\boldsymbol{\mathcal{L}}-\boldsymbol{\Phi}\cdot\|_2^2}\left(\boldsymbol{x}^n - \tau_n \mathbf{D}^\top \boldsymbol{y}^{n+1}\right)$$

$$\theta_n = \sqrt{1+2\mu\tau_n}, \quad \tau_{n+1} = \tau_n/\theta_n, \quad \sigma_{n+1} = \theta_n\sigma_n$$

$$\bar{\boldsymbol{x}}^{n+1} = \boldsymbol{x}^{n+1} + \theta_n^{-1}\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

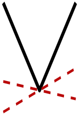
Algorithme accéléré par forte-convexité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$



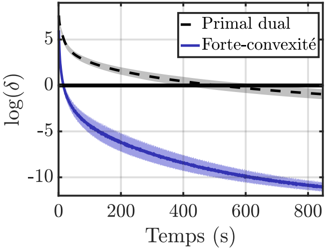
$\boldsymbol{\mu}$-fortement convexe

non lisse



Algorithme primal-dual **accéléré** *(Chambolle, 2011)*

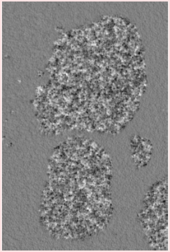
$$\delta \text{ : gap de dualité, } \delta(\boldsymbol{x}^n,\boldsymbol{y}^n) \underset{n\to\infty}{\longrightarrow} 0$$



Segmentation par seuillage itéré

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \sum_{a} \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

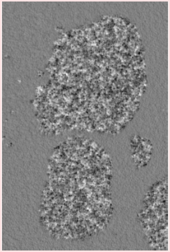
Image texturée



Segmentation par seuillage itéré

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \; + \; \underbrace{\lambda \, \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

Image texturée

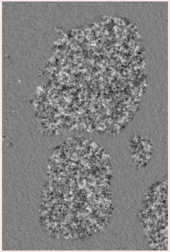


Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Segmentation par seuillage itéré

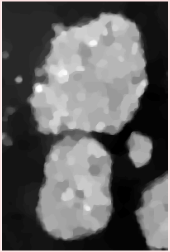
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \underbrace{\lambda\, \mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

Image texturée



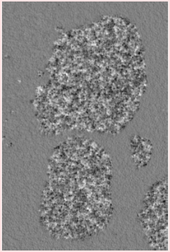
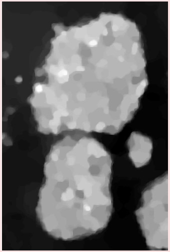
Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Contours co-localisés $\widehat{\boldsymbol{h}}^{\mathrm{C}}$



Segmentation par seuillage itéré

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \quad \underbrace{\sum_a \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\text{Moindres Carrés}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\text{Variation Totale}}$$

Image texturée	Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$	Contours co-localisés $\widehat{\boldsymbol{h}}^{\mathrm{C}}$	Estimée seuillée[†] $S\widehat{\boldsymbol{h}}^{\mathrm{C}}$
			

[†] *(Cai, 2013)*

Méthodes de l'état-de-l'art en segmentation de texture

ROF-Seuillé sur $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

(Nafornita, 2014), (Pustelnik, 2016)

$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{RL}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

S'appuie uniquement sur la régularité $\boldsymbol{h}$.

† https://sites.google.com/site/factorizationsegmentation/

Méthodes de l'état-de-l'art en segmentation de texture

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(Nafornita, 2014), (Pustelnik, 2016)

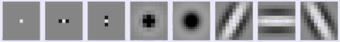
$$\underset{\boldsymbol{h}}{\operatorname{argmin}} \|\boldsymbol{h} - \widehat{\boldsymbol{h}}^{\mathrm{RL}}\|^2 + \lambda \|\mathbf{D}\boldsymbol{h}\|_{2,1}$$

Rég. lin. | ROF | Seuillage

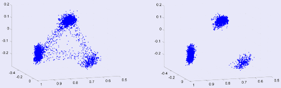
S'appuie uniquement sur la régularité $\boldsymbol{h}$.

Segmentation par factorisation matricielle[†] *(Yuan, 2015)*

(i) histogrammes locaux



(ii) factorisation matricielle



(a) (b)

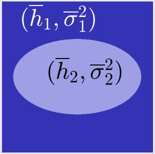
Fig. 2. Scatterplot of features in subspace. (a) Scatterplot of features projected onto the 3-d subspace. (b) Scatterplot after removing features with high edgeness.

[†] `https://sites.google.com/site/factorizationsegmentation/`

Performances comparées sur des textures synthétiques

Synthèse de texture monofractale par morceaux *(Pascal, 2019)*

- masque : $\Omega = \Omega_1 \sqcup \Omega_2$,
- attributs : $(\overline{h}_k, \overline{\sigma}_k^2)_{k=1,2}$



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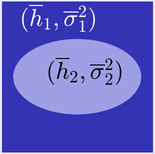
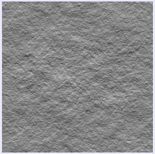
Ex. $\bar{h}_1 = 0{,}5$, $\bar{\sigma}_1^2 = 0{,}6$
$\bar{h}_2 = 0{,}6$, $\bar{\sigma}_2^2 = 0{,}7$





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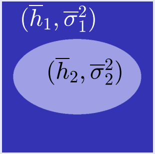
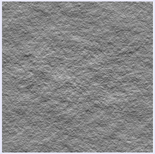
Ex. $\overline{h}_1 = 0{,}5,\ \overline{\sigma}_1^2 = 0{,}6$
$\overline{h}_2 = 0{,}6,\ \overline{\sigma}_2^2 = 0{,}7$

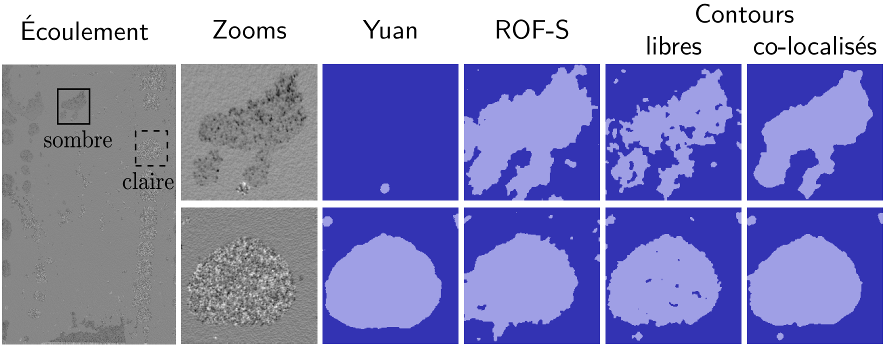




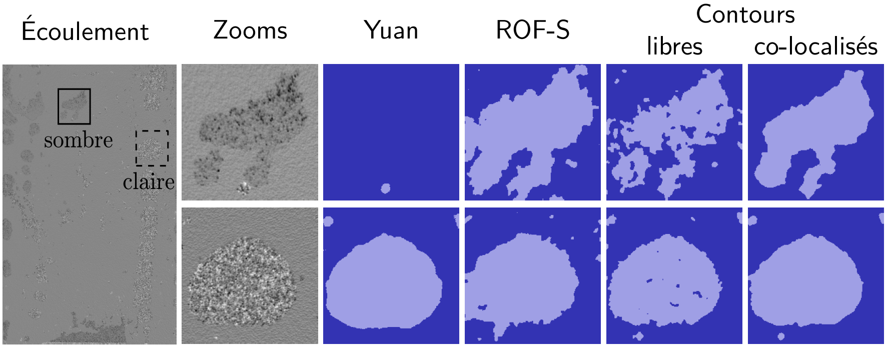
Performances de segmentation moyennées sur 5 réalisations

Yuan	ROF-S	Contours libres	Contours co-localisés
			
$71{,}1 \pm 1{,}3\%$	$78{,}5 \pm 1{,}1\%$	$90{,}2 \pm 1{,}9\%$	$91{,}1 \pm 1{,}5\%$

Faible activité : $Q_{\mathrm{G}} = 300\text{mL}/\text{min}$ - $Q_{\mathrm{L}} = 300\text{mL}/\text{min}$



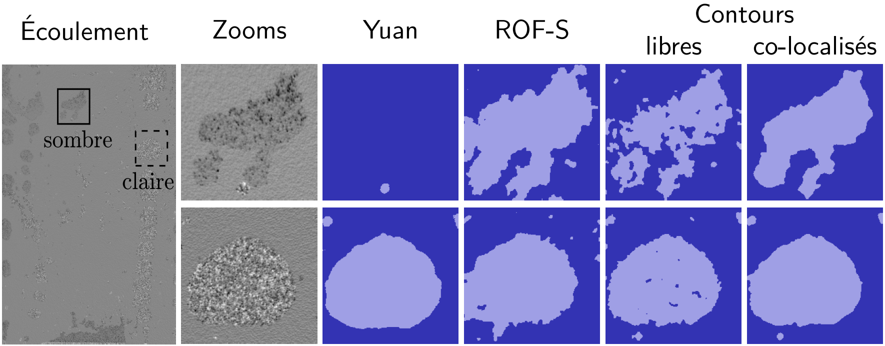
Faible activité : $Q_\mathrm{G} = 300\mathrm{mL/min}$ - $Q_\mathrm{L} = 300\mathrm{mL/min}$



Liquide : $h_\mathrm{L} = 0{,}4$

Gaz : $h_\mathrm{G} = 0{,}9$

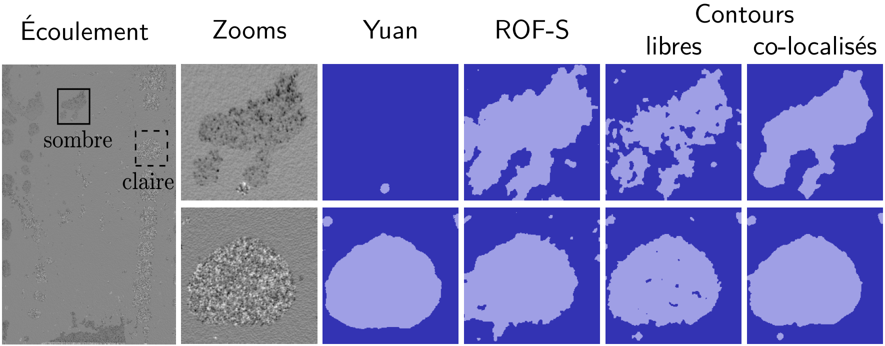
Faible activité : $Q_{\mathrm{G}} = 300\text{mL}/\text{min}$ - $Q_{\mathrm{L}} = 300\text{mL}/\text{min}$



Liquide : $h_{\mathrm{L}} = 0{,}4$ $\sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$

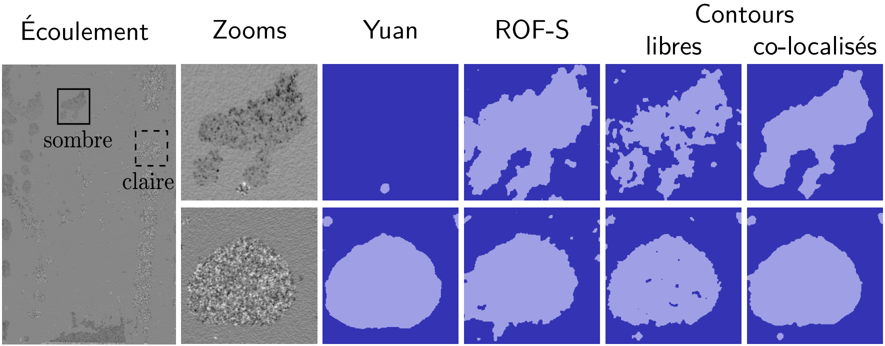
Faible activité : $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$



Liquide : $h_{\mathrm{L}} = 0{,}4$ $\sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$

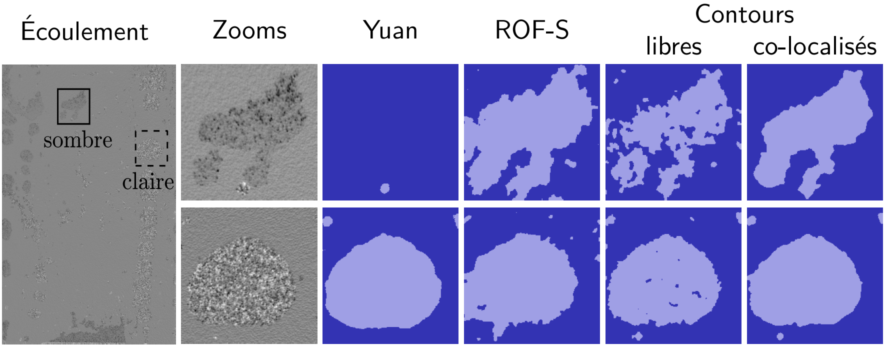
Faible activité : $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$



Liquide : $h_{\mathrm{L}} = 0{,}4$ $\quad \sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$ $\quad \sigma^2_{\mathrm{sombre}} = 10^{-2}$ (bulles sombres)

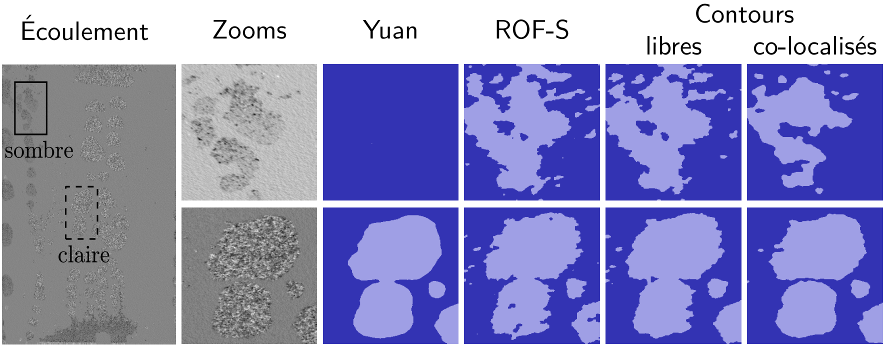
Faible activité : $Q_{\mathrm{G}} = 300\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 300\mathrm{mL}/\mathrm{min}$



Liquide : $h_{\mathrm{L}} = 0{,}4 \qquad \sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$ $\left| \begin{array}{ll} \sigma^2_{\mathrm{sombre}} = 10^{-2} & \text{(bulles sombres)} \\ \sigma^2_{\mathrm{claire}} = 10^{-1} & \text{(bulles claires)} \end{array} \right.$

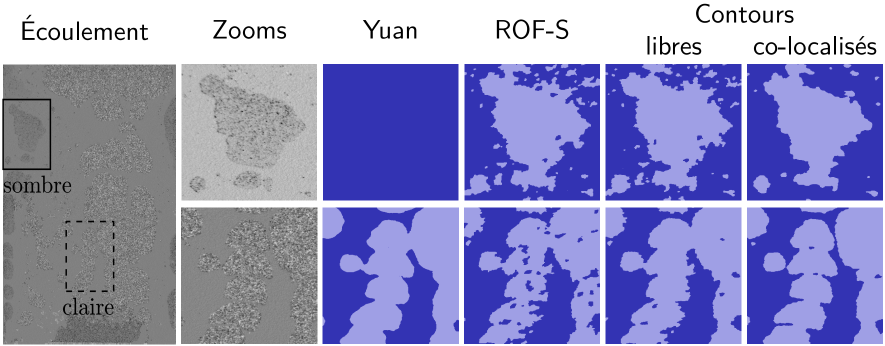
Transition : $Q_{\mathrm{G}} = 400\mathrm{mL}/\mathrm{min}$ - $Q_{\mathrm{L}} = 700\mathrm{mL}/\mathrm{min}$



Liquide : $h_{\mathrm{L}} = 0{,}4 \qquad \sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$ $\left| \begin{array}{ll} \sigma^2_{\mathrm{sombre}} = 10^{-2} & \text{(bulles sombres)} \\ \sigma^2_{\mathrm{claire}} = 10^{-1} & \text{(bulles claires).} \end{array} \right.$

Forte activité : $Q_{\mathrm{G}} = 1200$mL/min - $Q_{\mathrm{L}} = 300$mL/min

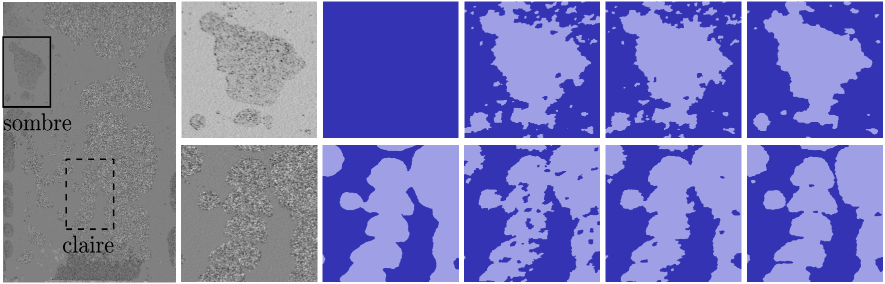


Liquide : $h_{\mathrm{L}} = 0{,}4 \qquad \sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9 \quad \left| \begin{array}{ll} \sigma^2_{\mathrm{sombre}} = 10^{-2} & \text{(bulles sombres)} \\ \sigma^2_{\mathrm{claire}} = 10^{-1} & \text{(bulles claires).} \end{array} \right.$

Forte activité : $Q_{\mathrm{G}} = 1200\text{mL}/\text{min}$ - $Q_{\mathrm{L}} = 300\text{mL}/\text{min}$

Écoulement	Zooms	Yuan	ROF-S	Contours libres	Contours co-localisés



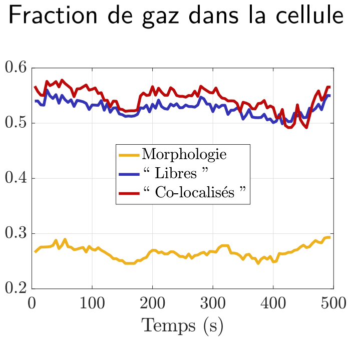
Temps de calcul		1s	12s	700s	**2100s**

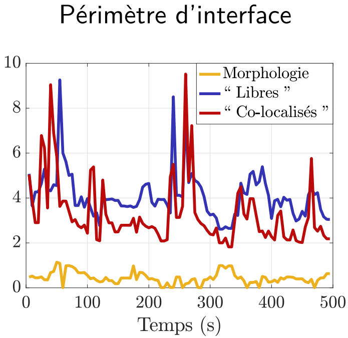
Liquide : $h_{\mathrm{L}} = 0{,}4 \qquad \sigma^2_{\mathrm{sombre}} = 10^{-2}$

Gaz : $h_{\mathrm{G}} = 0{,}9$ $\Big|$ $\sigma^2_{\mathrm{sombre}} = 10^{-2}$ (bulles sombres)
$\sigma^2_{\mathrm{claire}} = 10^{-1}$ (bulles claires).

Écoulement multiphasiques en milieu poreux

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)





Réglage des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Réglage des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

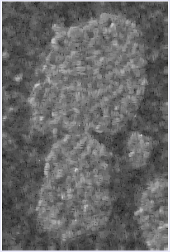
$(\lambda;\alpha) = (0;0)$

Réglage des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$ Estimée $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ à contours co-localisés

$(\lambda;\alpha) = (0;0)$ $(\lambda,\alpha) = (0{,}5;0{,}5)$



trop faible

Réglage des paramètres de régularisation

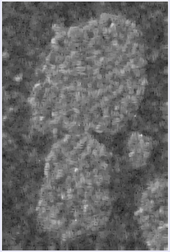
$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$

Estimée $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ à contours co-localisés

$(\lambda;\alpha) = (0;0)$

$(\lambda,\alpha) = (0{,}5;0{,}5)$



trop faible

$(\lambda;\alpha) = (500;500)$

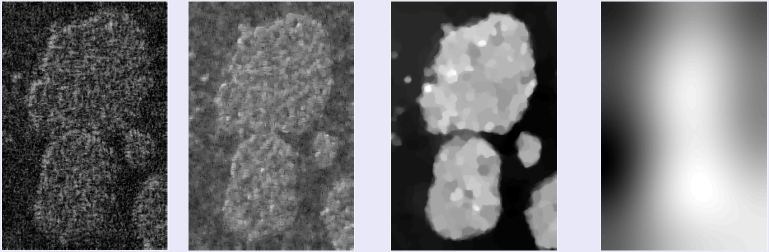
trop grand

Réglage des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\mathrm{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$ — Estimée $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ à contours co-localisés

$(\lambda;\alpha) = (0;0)$ | $(\lambda,\alpha) = (0{,}5;0{,}5)$ | $(\lambda^\dagger,\alpha^\dagger) = (11{,}5;0{,}8)$ | $(\lambda;\alpha) = (500;500)$



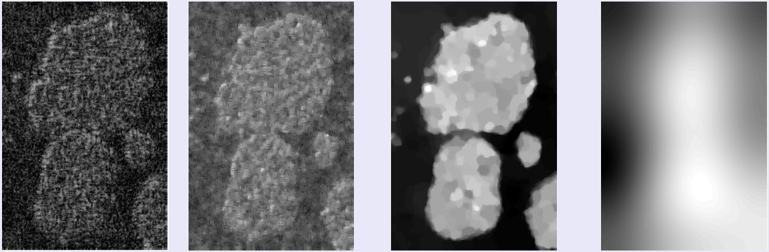
trop faible | optimal | trop grand

Réglage des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$ | Estimée $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ à contours co-localisés

$(\lambda;\alpha) = (0;0)$ | $(\lambda,\alpha) = (0{,}5;0{,}5)$ | $(\lambda^\dagger,\alpha^\dagger) = (11{,}5;0{,}8)$ | $(\lambda;\alpha) = (500;500)$



trop faible | optimal | trop grand

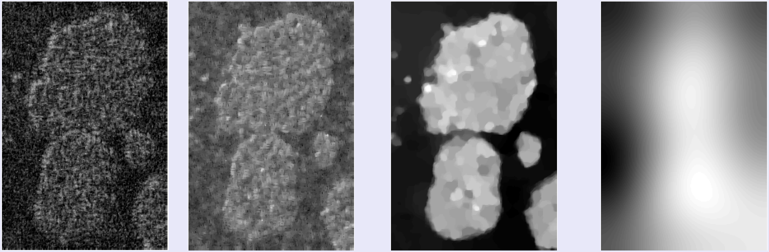
Que signifie *optimal* ?

Réglage des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Rég. lin. $\widehat{\boldsymbol{h}}^{\mathrm{RL}}$ — Estimée $\widehat{\boldsymbol{h}}^{\mathrm{C}}$ à contours co-localisés

$(\lambda;\alpha) = (0;0)$ — $(\lambda,\alpha) = (0{,}5;0{,}5)$ — $(\lambda^\dagger,\alpha^\dagger) = (11{,}5;0{,}8)$ — $(\lambda;\alpha) = (500;500)$



trop faible — optimal — trop grand

Que signifie *optimal* ? Comment déterminer $\lambda^\dagger$ et $\alpha^\dagger$?

Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

$\boldsymbol{h}$: *discriminant*, $\boldsymbol{v}$: *auxiliaire*

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$\bar{\boldsymbol{h}}$: *vraie* régularité

$$\mathcal{R}(\lambda,\alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}};\lambda,\alpha) - \bar{\boldsymbol{h}}\right\|^2$$

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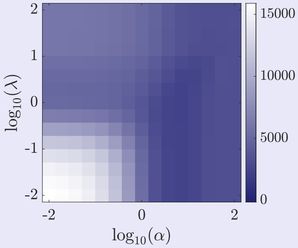

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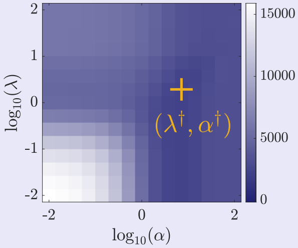

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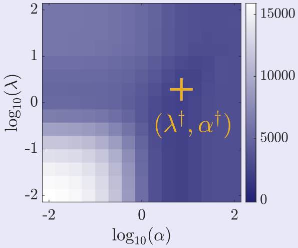

$\bar{\boldsymbol{h}}$: inconnue !

?

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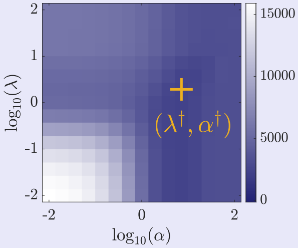

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?

Stein Unbiased Risk Estimate
(SURE)

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: vérité et $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

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Estimateur paramétrique $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \, \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$$

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Erreur quadratique $R(\lambda) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \overset{?}{=} \mathbb{E}_{\boldsymbol{\zeta}} \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ inconnue

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Théorème *(Stein, 1981)*

Soit $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$ un estimateur de $\bar{\boldsymbol{x}}$

- différentiable au sens faible par rapport à $\boldsymbol{y}$,
- tel que $\zeta \mapsto \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \zeta; \lambda), \zeta \rangle$ est intégrable par rapport à $\mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$.

$$\widehat{R}(\boldsymbol{y}; \lambda) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\rho^2 \operatorname{tr}\left(\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)\right) - \rho^2 P$$
$$\Longrightarrow R(\lambda) = \mathbb{E}_\zeta[\widehat{R}(\boldsymbol{y}; \lambda)].$$

Stein Unbiased Risk Estimate généralisé

Observations $\boldsymbol{y} = \Phi \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \Phi : \mathbb{R}^{P \times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

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Observations $\boldsymbol{y} = \Phi\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\Phi : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0},\mathcal{S})$

Ex. des estimateurs $\widehat{h}(\mathcal{L}; \lambda,\alpha)$ à contours libres ou co-localisés

$\log \boldsymbol{\mathcal{L}} = \Phi(\bar{\boldsymbol{h}},\bar{\boldsymbol{v}}) + \zeta$

$\Phi : (\boldsymbol{h},\boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$

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Observations $\boldsymbol{y} = \Phi\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N,\ \Phi : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0},\mathcal{S})$

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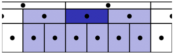

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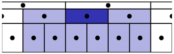

$\Pi : (\boldsymbol{h},\boldsymbol{v}) \mapsto (\boldsymbol{h},\boldsymbol{0})$

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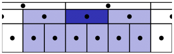
$\boldsymbol{\Phi} : (\boldsymbol{h},\boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h},\boldsymbol{v}) \mapsto (\boldsymbol{h},\mathbf{0})$

Erreur d'estimation projetée $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N,\ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

$$R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \| \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}} \|^2$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0},\mathcal{S})$

$$R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$$

$$= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\| \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}}) \right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2
\end{aligned}
$$

Stein Unbiased Risk Estimate généralisé (Calcul)

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$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\zeta} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
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&= \mathbb{E}_{\zeta} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
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\end{aligned}$$

Stein Unbiased Risk Estimate généralisé (Calcul)

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&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \zeta)\|^2 \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\zeta\right\rangle + \|\mathbf{A}\zeta\|^2\right]
\end{aligned}
$$

Stein Unbiased Risk Estimate généralisé (Calcul)

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&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right]
\end{aligned}
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Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\quad \boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ et $\zeta \sim \mathcal{N}(\boldsymbol{0}, \mathcal{S})$

$$
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&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
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&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
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\end{aligned}
$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0},\mathcal{S})$

$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \zeta)\|^2 \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
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\end{aligned}$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \zeta)\|^2 \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A} \qquad \zeta, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathbb{E}_\zeta \langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - \mathbb{E}_\zeta \|\mathbf{A}\zeta\|^2
\end{aligned}
$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A} \qquad \boldsymbol{\zeta}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathbb{E}_{\boldsymbol{\zeta}} \left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)
\end{aligned}$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\quad \boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A} \qquad \boldsymbol{\zeta}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathbb{E}_{\boldsymbol{\zeta}} \left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle \underbrace{- \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \zeta)\|^2 \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A} \qquad \zeta, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \underbrace{\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2}_{\text{accessible}} + 2\mathbb{E}_\zeta \langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - \underbrace{\operatorname{tr}\left(\mathbf{A}\mathcal{S}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}
$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

$$\mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top$$

$$\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \zeta)\|^2 \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A} \qquad \zeta, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \underbrace{\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2}_{\text{accessible}} + 2\mathbb{E}_\zeta \langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - \underbrace{\mathrm{tr}\left(\mathbf{A}\mathcal{S}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{\zeta})\|^2 \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - 2\left\langle \mathbf{A} \quad \boldsymbol{\zeta}, \mathbf{A}\boldsymbol{\zeta}\right\rangle + \|\mathbf{A}\boldsymbol{\zeta}\|^2\right] \\
&= \mathbb{E}_{\boldsymbol{\zeta}} \underbrace{\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2}_{\text{accessible}} + 2\mathbb{E}_{\boldsymbol{\zeta}} \left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle - \underbrace{\mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}
$$

$$
\mathbb{E}_{\boldsymbol{\zeta}} \left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle = \mathbb{E}_{\boldsymbol{\zeta}} \left\langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\boldsymbol{\zeta}\right\rangle
$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \zeta)\|^2 \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}), \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A}\boldsymbol{y}, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - 2\langle \mathbf{A} \qquad \zeta, \mathbf{A}\zeta\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \underbrace{\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2}_{\text{accessible}} + 2\mathbb{E}_\zeta \langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle - \underbrace{\operatorname{tr}\left(\mathbf{A}\mathcal{S}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_\zeta \langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle &= \mathbb{E}_\zeta \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}), \mathbf{A}\zeta\rangle \\
&= \int \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta;\lambda), \mathbf{A}\zeta\rangle \exp\left(-\frac{\zeta^\top\mathcal{S}^{-1}\zeta}{2}\right) \mathrm{d}\zeta
\end{aligned}
$$

Stein Unbiased Risk Estimate généralisé (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\boldsymbol{0},\mathcal{S})$

$$
\begin{aligned}
R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) &\triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \left\|\boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top\boldsymbol{\Phi}(\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \bar{\boldsymbol{x}})\right\|^2 \qquad \mathbf{A} \triangleq \boldsymbol{\Pi}(\boldsymbol{\Phi}^\top\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\top \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \boldsymbol{y} - \boldsymbol{\Phi}\bar{\boldsymbol{x}})\|^2 \\
&= \mathbb{E}_\zeta \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y} + \zeta)\|^2 \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}),\mathbf{A}\zeta\right\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}),\mathbf{A}\zeta\right\rangle - 2\left\langle \mathbf{A}\boldsymbol{y},\mathbf{A}\zeta\right\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \left[\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}),\mathbf{A}\zeta\right\rangle - 2\left\langle \mathbf{A} \qquad \zeta\ ,\mathbf{A}\zeta\right\rangle + \|\mathbf{A}\zeta\|^2\right] \\
&= \mathbb{E}_\zeta \underbrace{\|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y})\|^2}_{\text{accessible}} + 2\mathbb{E}_\zeta \left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}),\mathbf{A}\zeta\right\rangle - \underbrace{\operatorname{tr}\left(\mathbf{A}\mathcal{S}\mathbf{A}^\top\right)}_{\text{accessible}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_\zeta \left\langle \mathbf{A}\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}),\mathbf{A}\zeta\right\rangle &= \mathbb{E}_\zeta \left\langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}),\mathbf{A}\zeta\right\rangle \\
&= \int \left\langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta;\lambda),\mathbf{A}\zeta\right\rangle \exp\left(-\frac{\zeta^\top\mathcal{S}^{-1}\zeta}{2}\right) \mathrm{d}\zeta \\
\text{(I.P.P. gén.)} &= \mathbb{E}_\zeta \operatorname{tr}\left(\mathcal{S}\mathbf{A}^\top\boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)
\end{aligned}
$$

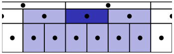
Stein Unbiased Risk Estimate généralisé

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

Ex. des estimateurs $\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha)$ à contours libres ou co-localisés

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\quad \boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$ $\quad \mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Erreur d'estimation projetée $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Théorème *(Pascal, 2020)*

Soit $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ un estimateur de $\bar{\boldsymbol{x}}$

- différentiable au sens faible par rapport à $\boldsymbol{y}$,
- tel que $\boldsymbol{\zeta} \mapsto \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \mathbf{A}\boldsymbol{\zeta} \rangle$ est intégrable par rapport à $\mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$.

$$\widehat{R}(\boldsymbol{\Lambda}) \triangleq \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi} \partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right) - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

$$\Longrightarrow R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{\Lambda})].$$

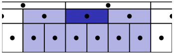
Stein Unbiased Risk Estimate généralisé

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

Ex. des estimateurs $\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha)$ à contours libres ou co-localisés

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\qquad$ $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$ $\qquad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \boldsymbol{0})$

Erreur d'estimation projetée $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

Théorème *(Pascal, 2020)*

Soit $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ un estimateur de $\bar{\boldsymbol{x}}$

- différentiable au sens faible par rapport à $\boldsymbol{y}$,
- tel que $\boldsymbol{\zeta} \mapsto \langle \boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \mathbf{A}\boldsymbol{\zeta}\rangle$ est intégrable par rapport à $\mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$.

$$\widehat{R}(\boldsymbol{\Lambda}) \triangleq \|\mathbf{A}(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{y})\|^2 + 2\mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\partial_{\boldsymbol{y}}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right) - \mathrm{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

$$\Longrightarrow R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{\Lambda})].$$

Calcul des degrés de liberté

Degrés de liberté $\quad \mathrm{dof} \triangleq \mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi} \partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right)$

Calcul des degrés de liberté

Degrés de liberté $\quad \mathrm{dof} \triangleq \mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi} \partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right)$

- **Stratégie de Monte Carlo** (MC) $\quad$ **$\mathbf{M} \in \mathbb{R}^{P\times P}$ de grande taille**

$$\mathrm{tr}(\mathbf{M}) = \mathbb{E}_{\boldsymbol{\varepsilon}} \langle \mathbf{M}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} \rangle, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_P)$$

Calcul des degrés de liberté

Degrés de liberté $\quad \mathrm{dof} \triangleq \mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi} \partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)$

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- **Différences Finies** (DF) $\quad$ **Jacobienne inaccessible**

$$\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}[\boldsymbol{\varepsilon}] \underset{\nu\to 0}{\simeq} \frac{1}{\nu}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)$$

Calcul des degrés de liberté

Degrés de liberté $\quad \mathrm{dof} \triangleq \mathrm{tr}\left(\boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi} \partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right)$

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$$\mathrm{tr}(\mathbf{M}) = \mathbb{E}_{\boldsymbol{\varepsilon}} \left\langle \mathbf{M}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} \right\rangle, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_P)$$

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$$\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}\left[\boldsymbol{\varepsilon}\right] \underset{\nu \to 0}{\simeq} \frac{1}{\nu}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y} + \nu\boldsymbol{\varepsilon}; \boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right)$$

Proposition *(Pascal, 2020)*

Soit $(\boldsymbol{y}; \boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})$ un estimateur de $\bar{\boldsymbol{x}}$

- uniformément lipschitzien par rapport à $\boldsymbol{y}$,
- tel que $\forall \boldsymbol{\Lambda} \in \mathbb{R}^L$, $\widehat{\boldsymbol{x}}(\mathbf{0}_P; \boldsymbol{\Lambda}) = \mathbf{0}_N$. Alors

$$\mathbb{E}_{\boldsymbol{\zeta}}\left[\mathrm{dof}\right] = \lim_{\nu \to 0} \mathbb{E}_{\boldsymbol{\zeta}, \boldsymbol{\varepsilon}}\left[\frac{1}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top \boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y} + \nu\boldsymbol{\varepsilon}; \boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle\right]$$

Stein Unbiased Risk Estimate (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

Erreur d'estimation projetée $R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

SURE généralisé Différences Finies Monte Carlo

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) \triangleq \left\|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\right\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

Stein Unbiased Risk Estimate (Calcul)

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\zeta \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

Erreur d'estimation projetée $R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_\zeta \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

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$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) \triangleq \|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

Théorème *(Pascal, 2020)*

Soit $(\boldsymbol{y};\boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})$ un estimateur de $\bar{\boldsymbol{x}}$

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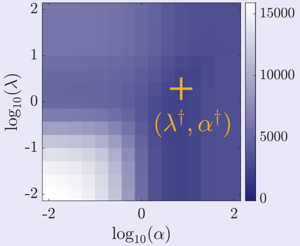
$$R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \lim_{\nu\to 0} \mathbb{E}_{\zeta,\boldsymbol{\varepsilon}}\left[\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}})\right]$$

Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

$\bar{\boldsymbol{h}}$: *vraie* régularité

$$\mathcal{R}(\lambda,\alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}};\lambda,\alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: inconnue !

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}};\lambda,\alpha|\,\boldsymbol{\mathcal{S}})$$

Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

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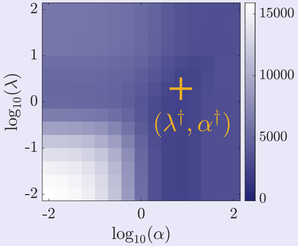
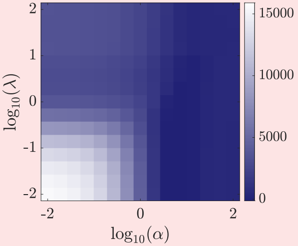
$$\mathcal{R}(\lambda,\alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}};\lambda,\alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: inconnue !

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}};\lambda,\alpha|\,\boldsymbol{\mathcal{S}})$$



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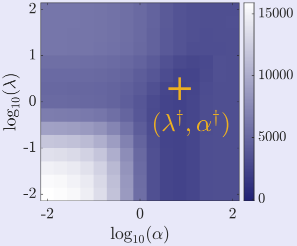
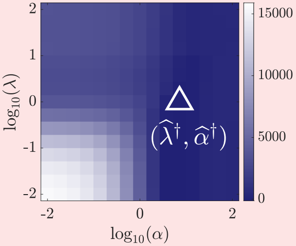


$\bar{\boldsymbol{h}}$: inconnue !

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}};\lambda,\alpha|\,\boldsymbol{\mathcal{S}})$$

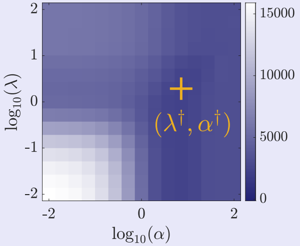


Réglage des paramètres (Recherche systématique)

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha)=\underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}}\sum_{a}\left\|\log\boldsymbol{\mathcal{L}}_{a,\cdot}-\log(a)\boldsymbol{h}-\boldsymbol{v}\right\|^2+\lambda\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

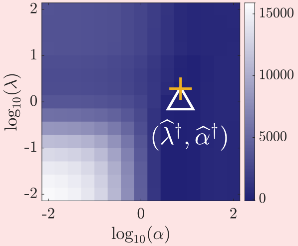
$\bar{\boldsymbol{h}}$: *vraie* régularité

$$\mathcal{R}(\lambda,\alpha)=\left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}};\lambda,\alpha)-\bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: inconnue !

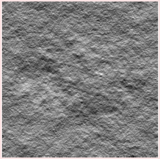
$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}};\lambda,\alpha|\,\boldsymbol{\mathcal{S}})$$

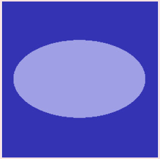


Recherche systématique des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}}^{\mathsf{L}}, \widehat{\boldsymbol{v}}^{\mathsf{L}}\right)(\mathcal{L};\boldsymbol{\Lambda}) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \mathcal{L}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathsf{L}}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Exemple





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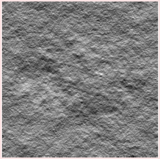
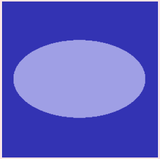
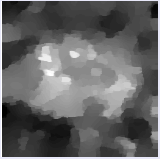
Exemple





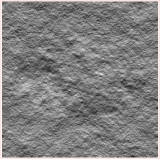
$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}};\lambda^\dagger,\alpha^\dagger)$ (grille)

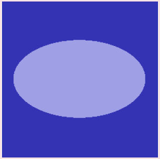


Recherche systématique des paramètres de régularisation

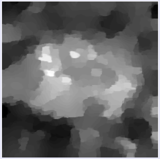
$$\left(\widehat{\boldsymbol{h}}^{\mathsf{L}},\widehat{\boldsymbol{v}}^{\mathsf{L}}\right)(\boldsymbol{\mathcal{L}};\boldsymbol{\Lambda}) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathsf{L}}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Exemple

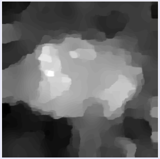




$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}};\lambda^\dagger,\alpha^\dagger)$ (grille)



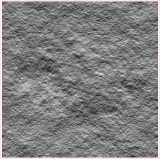
$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}};\widehat{\lambda}^\dagger,\widehat{\alpha}^\dagger)$ (grille)

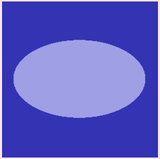


Recherche systématique des paramètres de régularisation

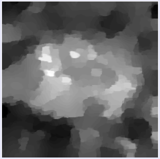
$$\left(\widehat{\boldsymbol{h}}^{\mathrm{L}},\widehat{\boldsymbol{v}}^{\mathrm{L}}\right)(\boldsymbol{\mathcal{L}};\boldsymbol{\Lambda}) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathrm{L}}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Exemple

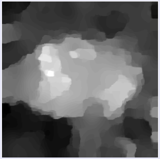




$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}};\lambda^{\dagger},\alpha^{\dagger})$ (grille)



$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}};\widehat{\lambda}^{\dagger},\widehat{\alpha}^{\dagger})$ (grille)



$15 \times 15 =$ **225** paramètres $\longrightarrow$ recherche sur grille très coûteuse

Minimisation automatique de l'estimateur SURE

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

SURE généralisé DFMC $\lim_{\nu \to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) = R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda})$

Minimisation automatique de l'estimateur SURE

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N, \ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\mathcal{S}})$

SURE généralisé DFMC $\lim_{\nu\to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) = R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda})$

But : $\underset{\boldsymbol{\Lambda}}{\operatorname{minimiser}} \ \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}})$ pour $\boldsymbol{y}$, $\boldsymbol{\mathcal{S}}$ donnés

Minimisation automatique de l'estimateur SURE

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N,\ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

SURE généralisé DFMC $\lim\limits_{\nu\to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) = R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda})$

But : $\underset{\boldsymbol{\Lambda}}{\mathrm{minimiser}}\ \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) \equiv \widehat{R}(\boldsymbol{\Lambda})$ pour $\boldsymbol{y}$, $\boldsymbol{\mathcal{S}}$ donnés

Minimisation automatique de l'estimateur SURE

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{S}})$

SURE généralisé DFMC $\lim_{\nu\to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) = R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda})$

But : $\underset{\boldsymbol{\Lambda}}{\text{minimiser}}\ \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) \equiv \widehat{R}(\boldsymbol{\Lambda})$ pour $\boldsymbol{y}$, $\boldsymbol{\mathcal{S}}$ donnés

Quasi-Newton de Broyden-Fletcher-Goldfarb-Shanno *(Nocedal, 2006)*

for $t = 0, 1, \ldots$

$\boldsymbol{d}^{[t]} = -\boldsymbol{H}^{[t]} \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *direction de descente*

$\alpha^{[t]} \in \underset{\alpha\in\mathbb{R}}{\text{Argmin}}\ \widehat{R}(\boldsymbol{\Lambda}^{[t]} + \alpha \boldsymbol{d}^{[t]})$ *recherche sur une ligne*

$\boldsymbol{\Lambda}^{[t+1]} = \boldsymbol{\Lambda}^{[t]} + \alpha^{[t]} \boldsymbol{d}^{[t]}$

$\boldsymbol{u}^{[t]} = \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t+1]}) - \partial_{\boldsymbol{\Lambda}} \widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *variation du gradient*

$\boldsymbol{H}^{[t+1]} = \text{BFGS}(\boldsymbol{H}^{[t]}, \boldsymbol{d}^{[t]}, \boldsymbol{u}^{[t]})$ *mise à jour "hessienne inverse"*

Minimisation automatique de l'estimateur SURE

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^N,\ \boldsymbol{\Phi} : \mathbb{R}^{P\times N}$ et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0},\boldsymbol{\mathcal{S}})$

SURE généralisé DFMC $\lim_{\nu\to 0} \mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) = R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda})$

But : $\operatorname*{minimiser}_{\boldsymbol{\Lambda}} \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y}; \boldsymbol{\Lambda} \,|\, \boldsymbol{\mathcal{S}}) \equiv \widehat{R}(\boldsymbol{\Lambda})$ pour $\boldsymbol{y}$, $\boldsymbol{\mathcal{S}}$ donnés

Quasi-Newton de Broyden-Fletcher-Goldfarb-Shanno *(Nocedal, 2006)*

for $t = 0,1,\dots$

$\boldsymbol{d}^{[t]} = -\boldsymbol{H}^{[t]}\partial_{\boldsymbol{\Lambda}}\widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *direction de descente*

$\alpha^{[t]} \in \operatorname*{Argmin}_{\alpha\in\mathbb{R}} \widehat{R}(\boldsymbol{\Lambda}^{[t]} + \alpha \boldsymbol{d}^{[t]})$ *recherche sur une ligne*

$\boldsymbol{\Lambda}^{[t+1]} = \boldsymbol{\Lambda}^{[t]} + \alpha^{[t]}\boldsymbol{d}^{[t]}$

$\boldsymbol{u}^{[t]} = \partial_{\boldsymbol{\Lambda}}\widehat{R}(\boldsymbol{\Lambda}^{[t+1]}) - \partial_{\boldsymbol{\Lambda}}\widehat{R}(\boldsymbol{\Lambda}^{[t]})$ *variation du gradient*

$\boldsymbol{H}^{[t+1]} = \mathrm{BFGS}(\boldsymbol{H}^{[t]},\boldsymbol{d}^{[t]},\boldsymbol{u}^{[t]})$ *mise à jour "hessienne inverse"*

Stein Unbiased GrAdient Risk estimate

SURE généralisé DFMC

$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) = \left\|\mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\right\|^2 + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle - \operatorname{tr}\left(\mathbf{A}\boldsymbol{\mathcal{S}}\mathbf{A}^\top\right)$$

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SUGAR généralisé Différences Finies Monte Carlo

$$\begin{aligned}\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}}) &= 2\left(\mathbf{A}\boldsymbol{\Phi}\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right)^\top \mathbf{A}\left(\boldsymbol{\Phi}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda}) - \boldsymbol{y}\right)\\ &\quad + \frac{2}{\nu}\left\langle \boldsymbol{\mathcal{S}}\mathbf{A}^\top\boldsymbol{\Pi}\left(\partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y}+\nu\boldsymbol{\varepsilon};\boldsymbol{\Lambda}) - \partial_{\boldsymbol{\Lambda}}\widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})\right), \boldsymbol{\varepsilon}\right\rangle\end{aligned}$$

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Théorème *(Pascal, 2020)*

Soit $(\boldsymbol{y};\boldsymbol{\Lambda}) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y};\boldsymbol{\Lambda})$ un estimateur de $\bar{\boldsymbol{x}}$

- uniformément lipschitzien par rapport à $\boldsymbol{y}$
- tel que $\forall\boldsymbol{\Lambda}\in\mathbb{R}^L$, $\widehat{\boldsymbol{x}}(\mathbf{0}_P;\boldsymbol{\Lambda}) = \mathbf{0}_N$,
- uniformément L-lipschitzien par rapport à $\boldsymbol{\Lambda}$, L indép. de $\boldsymbol{y}$. Alors

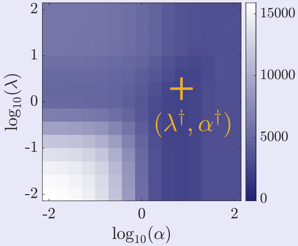
$$\partial_{\boldsymbol{\Lambda}}R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) = \lim_{\nu\to 0}\mathbb{E}_{\boldsymbol{\zeta},\boldsymbol{\varepsilon}}\left[\partial_{\boldsymbol{\Lambda}}\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{y};\boldsymbol{\Lambda}\,|\,\boldsymbol{\mathcal{S}})\right]$$

Réglage des paramètres (Recherche automatique)

$$\left(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}};\lambda,\alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

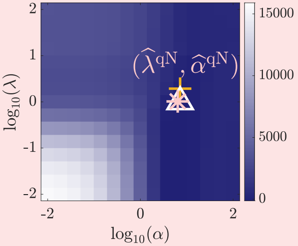
$\bar{\boldsymbol{h}}$: *vraie* régularité

$$\mathcal{R}(\lambda,\alpha) = \left\|\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}};\lambda,\alpha) - \bar{\boldsymbol{h}}\right\|^2$$



$\bar{\boldsymbol{h}}$: inconnue !

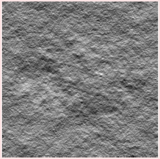
$$\widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}};\lambda,\alpha|\,\boldsymbol{\mathcal{S}})$$

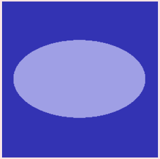


Recherche automatique des paramètres de régularisation

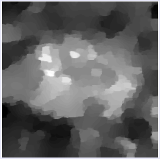
$$\left(\widehat{\boldsymbol{h}}^{\mathsf{L}},\widehat{\boldsymbol{v}}^{\mathsf{L}}\right)(\boldsymbol{\mathcal{L}};\boldsymbol{\Lambda}) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathsf{L}}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Exemple

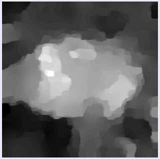




$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}};\lambda^\dagger,\alpha^\dagger)$ (grille)



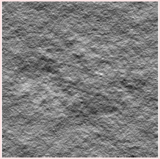
$\widehat{\boldsymbol{h}}^{\mathsf{L}}(\boldsymbol{\mathcal{L}};\widehat{\lambda}^\dagger,\widehat{\alpha}^\dagger)$ (grille)

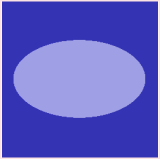


Recherche automatique des paramètres de régularisation

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{L}},\widehat{\boldsymbol{v}}^{\mathrm{L}}\right)(\boldsymbol{\mathcal{L}};\boldsymbol{\Lambda}) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathrm{L}}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Exemple





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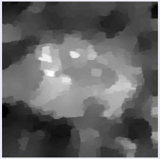
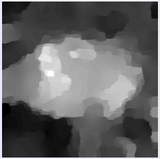
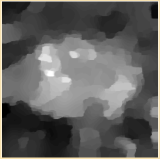


$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}};\widehat{\lambda}^{\dagger},\widehat{\alpha}^{\dagger})$ (grille)



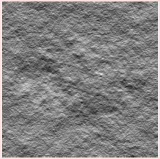
$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}};\widehat{\lambda}^{\mathrm{qN}},\widehat{\alpha}^{\mathrm{qN}})$ (quasi-Newton)

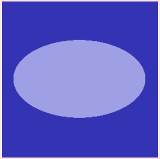


Recherche automatique des paramètres de régularisation

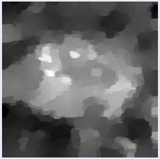
$$\left(\widehat{\boldsymbol{h}}^{\mathrm{L}},\widehat{\boldsymbol{v}}^{\mathrm{L}}\right)(\boldsymbol{\mathcal{L}};\boldsymbol{\Lambda}) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}_{\mathrm{L}}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)$$

Exemple





$\widehat{\boldsymbol{h}}^{\mathrm{L}}(\boldsymbol{\mathcal{L}};\lambda^{\dagger},\alpha^{\dagger})$ (grille)



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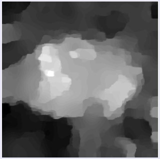

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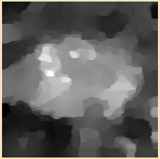

40 appels de l'estimateurs v.s. **225** sur une grille

Bilan de la présentation

- Régularité et variance locale [ICIP, 2018]

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  - aptes à caractériser des textures réelles

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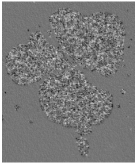
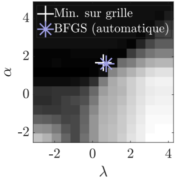
$\longrightarrow$ **En cours** : traitement automatisé de séries temporelles issues de l'étude des écoulements multiphasiques [Ann. Telecom, 2020]





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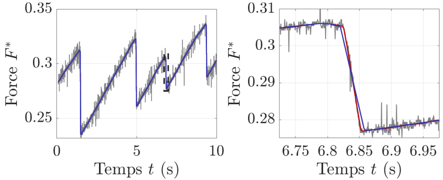
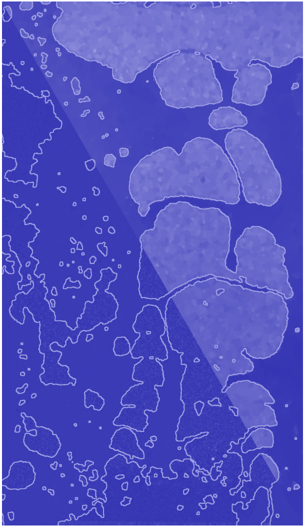


Merci



Grazie

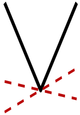
Gracias

Thank you

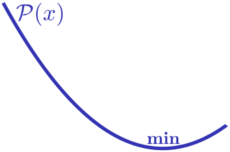
Définition du gap de dualité

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \sum_{a} \underbrace{\left\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2}_{\textbf{Moindres Carrés}} \quad + \quad \lambda\, \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h},\mathbf{D}\boldsymbol{v};\alpha)}_{\textbf{Variation Totale}}$$

non lisse



Primal $\quad \min_{\boldsymbol{x}} \; \mathrm{MC}(\boldsymbol{x}) + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x})$



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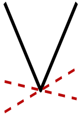
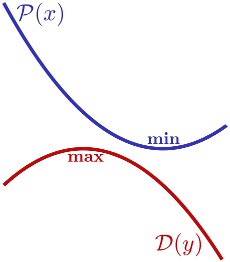


Primal $\quad \min_{\boldsymbol{x}} \; \mathrm{MC}(\boldsymbol{x}) + \lambda\mathcal{Q}(\mathbf{D}\boldsymbol{x})$

Dual $\quad \max_{\boldsymbol{y}} \; -\mathrm{MC}^*(-\mathbf{D}^\top \boldsymbol{y}) - (\lambda\mathcal{Q})^*(\boldsymbol{y})$



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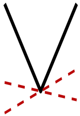

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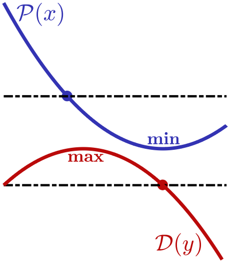
Proposition *(Bauschke, 2011)*

Soit $\delta(\boldsymbol{x};\boldsymbol{y}) \triangleq \mathcal{P}(\boldsymbol{x}) - \mathcal{D}(\boldsymbol{y})$ le gap de dualité,



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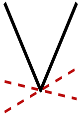
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimiser}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Moindres Carrés}} \ + \ \lambda \ \underbrace{\mathcal{Q}(\mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}; \alpha)}_{\textbf{Variation Totale}}$$

non lisse



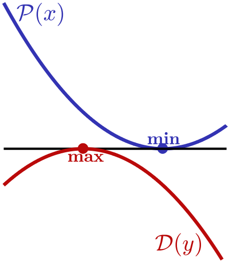
Primal $\min_{\boldsymbol{x}} \ \mathrm{MC}(\boldsymbol{x}) + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x})$

Dual $\max_{\boldsymbol{y}} \ -\mathrm{MC}^*(-\mathbf{D}^\top \boldsymbol{y}) - (\lambda\mathcal{Q})^*(\boldsymbol{y})$

Proposition *(Bauschke, 2011)*

Soit $\delta(\boldsymbol{x}; \boldsymbol{y}) \triangleq \mathcal{P}(\boldsymbol{x}) - \mathcal{D}(\boldsymbol{y})$ le gap de dualité,

$$\delta(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{y}}) = \mathcal{P}(\widehat{\boldsymbol{x}}) - \mathcal{D}(\widehat{\boldsymbol{y}}) = 0$$



Conjuguée convexe des moindres carrés

$$\mathrm{MC}^*(\boldsymbol{h},\boldsymbol{v}) = \sup_{\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}} \langle\widetilde{\boldsymbol{h}},\boldsymbol{h}\rangle + \langle\widetilde{\boldsymbol{v}},\boldsymbol{v}\rangle - \mathrm{MC}(\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) \underset{\text{(si le sup est atteint)}}{=} \langle\bar{\boldsymbol{h}},\boldsymbol{h}\rangle + \langle\bar{\boldsymbol{v}},\boldsymbol{v}\rangle - \mathrm{MC}(\bar{\boldsymbol{h}},\bar{\boldsymbol{v}}).$$

Conjuguée convexe des moindres carrés

$$\mathrm{MC}^*(\boldsymbol{h},\boldsymbol{v}) = \sup_{\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}} \langle \widetilde{\boldsymbol{h}},\boldsymbol{h}\rangle + \langle \widetilde{\boldsymbol{v}},\boldsymbol{v}\rangle - \mathrm{MC}(\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) \underset{\text{(si le sup est atteint)}}{=} \langle \bar{\boldsymbol{h}},\boldsymbol{h}\rangle + \langle \bar{\boldsymbol{v}},\boldsymbol{v}\rangle - \mathrm{MC}(\bar{\boldsymbol{h}},\bar{\boldsymbol{v}}).$$

Condition d'optimalité

$$\begin{cases} \boldsymbol{h} - 2\sum_a \log(a)\left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.}\right) = 0 \\ \boldsymbol{v} - 2\sum_a \left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.}\right) = 0 \end{cases}$$

Conjuguée convexe des moindres carrés

$$\mathrm{MC}^*(\boldsymbol{h},\boldsymbol{v}) = \sup_{\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}} \langle \widetilde{\boldsymbol{h}},\boldsymbol{h}\rangle + \langle \widetilde{\boldsymbol{v}},\boldsymbol{v}\rangle - \mathrm{MC}(\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) \underset{\text{(si le sup est atteint)}}{=} \langle \bar{\boldsymbol{h}},\boldsymbol{h}\rangle + \langle \bar{\boldsymbol{v}},\boldsymbol{v}\rangle - \mathrm{MC}(\bar{\boldsymbol{h}},\bar{\boldsymbol{v}}).$$

Condition d'optimalité

$$\begin{cases} \boldsymbol{h} - 2\sum_a \log(a)\left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \mathcal{L}_{a,.}\right) = 0 \\ \boldsymbol{v} - 2\sum_a \left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \mathcal{L}_{a,.}\right) = 0 \end{cases} \iff \boldsymbol{\Phi}^*\boldsymbol{\Phi}\begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{v}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{h}/2 + \mathcal{G} \\ \boldsymbol{v}/2 + \mathcal{T} \end{pmatrix}$$

$$\mathcal{T} = \sum_a \log \mathcal{L}_{a,.} \quad \text{and} \quad \mathcal{G} = \sum_a \log(a) \log \mathcal{L}_{a,.},$$

Conjuguée convexe des moindres carrés

$$\mathrm{MC}^*(\boldsymbol{h},\boldsymbol{v}) = \sup_{\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}} \langle \widetilde{\boldsymbol{h}},\boldsymbol{h}\rangle + \langle \widetilde{\boldsymbol{v}},\boldsymbol{v}\rangle - \mathrm{MC}(\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) \underset{\text{(si le sup est atteint)}}{=} \langle \bar{\boldsymbol{h}},\boldsymbol{h}\rangle + \langle \bar{\boldsymbol{v}},\boldsymbol{v}\rangle - \mathrm{MC}(\bar{\boldsymbol{h}},\bar{\boldsymbol{v}}).$$

Condition d'optimalité

$$\begin{cases} \boldsymbol{h} - 2\sum_a \log(a)\left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.}\right) = 0 \\ \boldsymbol{v} - 2\sum_a \left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.}\right) = 0 \end{cases} \iff \boldsymbol{\Phi}^*\boldsymbol{\Phi}\begin{pmatrix}\bar{\boldsymbol{h}} \\ \bar{\boldsymbol{v}}\end{pmatrix} = \begin{pmatrix}\boldsymbol{h}/2 + \boldsymbol{\mathcal{G}} \\ \boldsymbol{v}/2 + \boldsymbol{\mathcal{T}}\end{pmatrix}$$

$$\boldsymbol{\mathcal{T}} = \sum_a \log \boldsymbol{\mathcal{L}}_{a,.} \quad \text{and} \quad \boldsymbol{\mathcal{G}} = \sum_a \log(a) \log \boldsymbol{\mathcal{L}}_{a,.},$$

$$\forall m = \{0,1,2\}, S_m = \sum_a (\log a)^m, \quad \boldsymbol{\Phi}^*\boldsymbol{\Phi} = \begin{pmatrix} S_2\boldsymbol{I} & S_1\boldsymbol{I} \\ S_1\boldsymbol{I} & S_0\boldsymbol{I}\end{pmatrix}$$

Conjuguée convexe des moindres carrés

$$\mathrm{MC}^*(\boldsymbol{h},\boldsymbol{v}) = \sup_{\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}} \langle \widetilde{\boldsymbol{h}},\boldsymbol{h}\rangle + \langle \widetilde{\boldsymbol{v}},\boldsymbol{v}\rangle - \mathrm{MC}(\widetilde{\boldsymbol{h}},\widetilde{\boldsymbol{v}}) \underset{\text{(si le sup est atteint)}}{=} \langle \bar{\boldsymbol{h}},\boldsymbol{h}\rangle + \langle \bar{\boldsymbol{v}},\boldsymbol{v}\rangle - \mathrm{MC}(\bar{\boldsymbol{h}},\bar{\boldsymbol{v}}).$$

Condition d'optimalité

$$\left\{ \begin{array}{l} \boldsymbol{h} - 2\sum_a \log(a)\left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.}\right) = 0 \\ \boldsymbol{v} - 2\sum_a \left(\bar{\boldsymbol{v}} + \log(a)\bar{\boldsymbol{h}} - \log \boldsymbol{\mathcal{L}}_{a,.}\right) = 0 \end{array} \right. \iff \boldsymbol{\Phi}^*\boldsymbol{\Phi} \begin{pmatrix} \bar{\boldsymbol{h}} \\ \bar{\boldsymbol{v}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{h}/2 + \boldsymbol{\mathcal{G}} \\ \boldsymbol{v}/2 + \boldsymbol{\mathcal{T}} \end{pmatrix}$$

$$\boldsymbol{\mathcal{T}} = \sum_a \log \boldsymbol{\mathcal{L}}_{a,.} \quad \text{and} \quad \boldsymbol{\mathcal{G}} = \sum_a \log(a) \log \boldsymbol{\mathcal{L}}_{a,.},$$

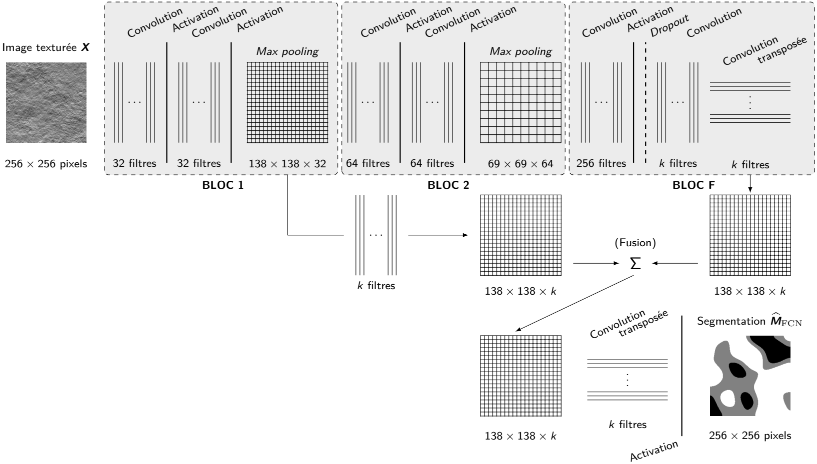
$$\forall m = \{0,1,2\}, S_m = \sum_a (\log a)^m, \quad \boldsymbol{\Phi}^*\boldsymbol{\Phi} = \begin{pmatrix} S_2 \boldsymbol{I} & S_1 \boldsymbol{I} \\ S_1 \boldsymbol{I} & S_0 \boldsymbol{I} \end{pmatrix}$$

$$\mathrm{MC}^*(\boldsymbol{h},\boldsymbol{v}) = \frac{1}{4}\langle(\boldsymbol{h},\boldsymbol{v}),(\boldsymbol{\Phi}^*\boldsymbol{\Phi})^{-1}(\boldsymbol{h},\boldsymbol{v})\rangle + \langle(\boldsymbol{\mathcal{G}},\boldsymbol{\mathcal{T}}),(\boldsymbol{\Phi}^*\boldsymbol{\Phi})^{-1}(\boldsymbol{h},\boldsymbol{v})\rangle + \mathcal{C}$$

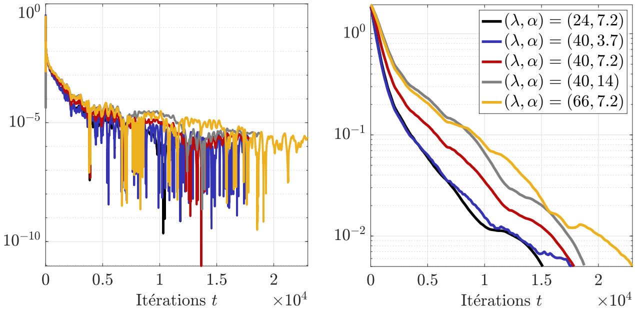
où $\mathcal{C}$ est une constante dépendant uniquement de $\boldsymbol{\mathcal{L}}$.

Architecture pour la segmentation de texture

Avec connexions *résiduelles*



Critère d'arrêt



Performances de segmentation
Configuration I

	2 classes	3 classes	4 classes
Entraîné sur la Config. I, testé sur la Config. I			
Segmentation à contours « libres »	$93{,}2 \pm 0{,}8\%$	$69{,}3 \pm 2{,}8\%$	$58{,}6 \pm 1{,}5\%$
Entraîné sur 2000 images			
Réseau à $8 \cdot 10^7$ poids / $P = 2000$	$97.3 \pm 0{,}6\%$	$97{,}8 \pm 0{,}3\%$	$97{,}1 \pm 0{,}4\%$
Réseau à $2 \cdot 10^6$ poids / $P = 2000$	$97{,}4 \pm 0{,}6\%$	$98.1 \pm 0{,}3\%$	$96{,}8 \pm 0{,}5\%$
Réseau à $4 \cdot 10^5$ poids / $P = 2000$	$96{,}9 \pm 0{,}7\%$	$98{,}0 \pm 0{,}3\%$	$96.5 \pm 0{,}5\%$
Entraîné sur 20 images			
Réseau à $8 \cdot 10^7$ poids / $P = 20$	$95{,}5 \pm 0{,}9\%$	$97{,}5 \pm 0{,}4\%$	$95{,}4 \pm 0{,}8\%$
Réseau à $2 \cdot 10^6$ poids / $P = 20$	$95{,}4 \pm 1{,}1\%$	$97{,}4 \pm 0{,}5\%$	$95{,}9 \pm 0{,}7\%$
Réseau à $4 \cdot 10^5$ poids / $P = 20$	$96{,}6 \pm 0{,}7\%$	$98{,}0 \pm 0{,}4\%$	$96{,}5 \pm 0{,}5\%$

Performances de segmentation
Configuration II

	2 classes	3 classes	4 classes
Entraîné sur la Config. II, testé sur la Config. II			
Segmentation à contours « libres »	$97{,}8 \pm 0{,}2\%$	$95{,}2 \pm 3{,}1\%$	$64{,}9 \pm 1{,}4\%$
Entraîné sur 2000 images			
Réseau à $8 \cdot 10^7$ poids / $P = 2000$	$99{,}1 \pm 0{,}2\%$	$98{,}3 \pm 0{,}3\%$	$95{,}7 \pm 0{,}5\%$
Réseau à $2 \cdot 10^6$ poids / $P = 2000$	$99{,}0 \pm 0{,}2\%$	$98{,}5 \pm 0{,}3\%$	$95{,}6 \pm 0{,}5\%$
Réseau à $4 \cdot 10^5$ poids / $P = 2000$	$99{,}1 \pm 0{,}2\%$	$98{,}4 \pm 0{,}3\%$	$95{,}2 \pm 0{,}6\%$
Entraîné sur 20 images			
Réseau à $8 \cdot 10^7$ poids / $P = 20$	$98{,}8 \pm 0{,}2\%$	$97{,}9 \pm 0{,}3\%$	$94{,}5 \pm 0{,}7\%$
Réseau à $2 \cdot 10^6$ poids / $P = 20$	$98{,}6 \pm 0{,}3\%$	$97{,}4 \pm 0{,}4\%$	$93{,}0 \pm 0{,}9\%$
Réseau à $4 \cdot 10^5$ poids / $P = 20$	$98{,}8 \pm 0{,}3\%$	$98{,}3 \pm 0{,}3\%$	$94{,}8 \pm 0{,}6\%$

Robustesse

Entraîné sur la Config. I, testé sur la Config. II

	2 classes	3 classes	4 classes
Entraîné sur la Config. I, testé sur la Config. II			
Segmentation à contours « libres »	$79{,}2 \pm 2{,}9\%$	$95{,}2 \pm 1{,}2\%$	$66{,}3 \pm 1{,}1\%$
Entraîné sur 2000 images			
Réseau à $8 \cdot 10^7$ poids / $P = 2000$	$91{,}2 \pm 2{,}1\%$	$65{,}7 \pm 7{,}2\%$	$55{,}6 \pm 3{,}4\%$
Réseau à $2 \cdot 10^6$ poids / $P = 2000$	$87{,}9 \pm 2{,}5\%$	$69{,}0 \pm 7{,}6\%$	$50{,}8 \pm 4{,}0\%$
Réseau à $4 \cdot 10^5$ poids / $P = 2000$	$81{,}8 \pm 3{,}8\%$	$65{,}2 \pm 7{,}2\%$	$46{,}4 \pm 3{,}7\%$
Entraîné sur 20 images			
Réseau à $8 \cdot 10^7$ poids / $P = 20$	$91{,}4 \pm 1{,}6\%$	$63{,}3 \pm 7{,}1\%$	$54{,}7 \pm 3{,}3\%$
Réseau à $2 \cdot 10^6$ poids / $P = 20$	$92{,}4 \pm 1{,}6\%$	$65{,}6 \pm 7{,}4\%$	$44{,}4 \pm 3{,}4\%$
Réseau à $4 \cdot 10^5$ poids / $P = 20$	$86{,}3 \pm 2{,}6\%$	$64{,}9 \pm 7{,}2\%$	$48{,}4 \pm 3{,}8\%$

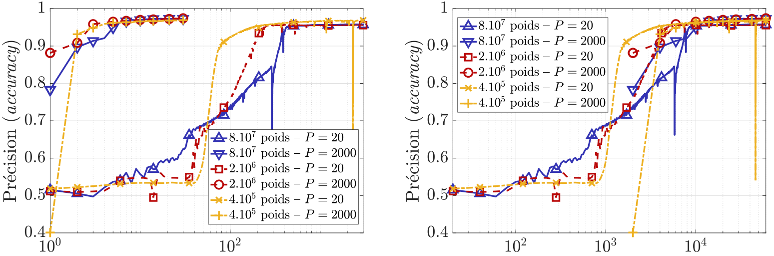
Robustesse

Entraîné sur la Config. II, testé sur la Config. I

	2 classes	3 classes	4 classes
Entraîné sur la Config. II, testé sur la Config. I			
Segmentation à contours « libres »	$90{,}9 \pm 2{,}8\%$	$66{,}7 \pm 2{,}5\%$	$52{,}0 \pm 1{,}5\%$
Entraîné sur 2000 images			
Réseau à $8 \cdot 10^7$ poids / $P = 2000$	$56{,}2 \pm 13{,}5\%$	$73{,}5 \pm 8{,}2\%$	$50{,}9 \pm 3{,}9\%$
Réseau à $2 \cdot 10^6$ poids / $P = 2000$	$55{,}1 \pm 14.0\%$	$74{,}9 \pm 8{,}2\%$	$51{,}3 \pm 4{,}3\%$
Réseau à $5\,10^5$ poids / $P = 2000$	$55{,}5 \pm 13{,}8\%$	$72{,}6 \pm 8{,}1\%$	$50{,}2 \pm 3.8\%$
Entraîné sur 20 images			
Réseau à $8 \cdot 10^7$ poids / $P = 20$	$57{,}1 \pm 13{,}3\%$	$71{,}1 \pm 8{,}2\%$	$52{,}6 \pm 3{,}8\%$
Réseau à $2 \cdot 10^6$ poids / $P = 20$	$55{,}3 \pm 14{,}0\%$	$71{,}7 \pm 8{,}4\%$	$49{,}6 \pm 4{,}2\%$
Réseau à $5\,10^5$ poids / $P = 20$	$62{,}3 \pm 11{,}5\%$	$71{,}0 \pm 8{,}2\%$	$54{,}1 \pm 3{,}7\%$

Convergence de la phase d'entraînement

Évolution du score de segmentation des trois réseaux au cours de l'entraînement sur la Config. I, avec deux classes $k = 2$

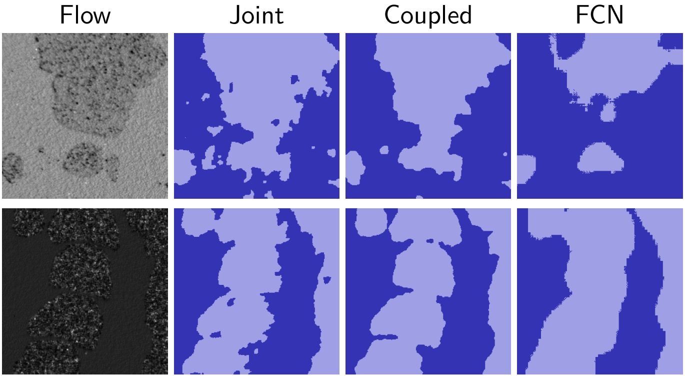


Comparaison effort de calcul $\mathcal{C}$

	$\mathcal{W}$	P	$\mathcal{I}$	$\mathcal{C}$
Segmentation à contours ≪ libres ≫	2	1	10^7	$2\,10^7$
Réseau à $8 \cdot 10^7$ poids / $P = \{20{,}2000\}$	$8 \cdot 10^7$	$\{20{,}2000\}$	$\{3000{,}30\}$	$4{,}8 \cdot 10^{12}$
Réseau à $2 \cdot 10^6$ poids / $P = \{20{,}2000\}$	$2 \cdot 10^6$	$\{20{,}2000\}$	$\{3000{,}30\}$	$1{,}2 \cdot 10^{11}$
Réseau à $4 \cdot 10^5$ poids / $P = \{20{,}2000\}$	$4 \cdot 10^5$	$\{20{,}2000\}$	$\{3000{,}30\}$	$2{,}4 \cdot 10^{10}$

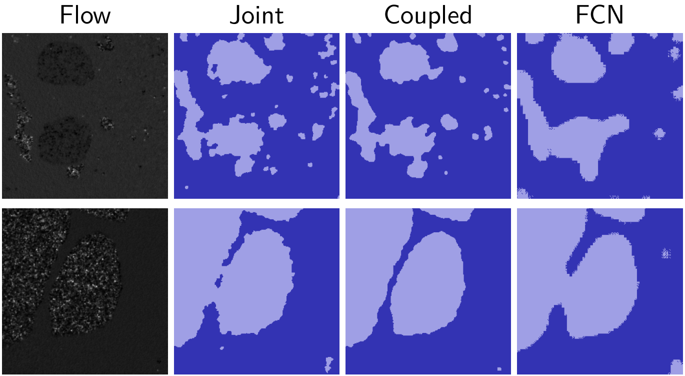
- $\mathcal{W}$: nombre de poids
- P : taille base d'entraînement
- $\mathcal{I}$: nombre d'*epochs*

Réseaux de neurones convolutionnels†



† V. Andrearczyk, https://arxiv.org/abs/1703.05230

Réseaux de neurones convolutionnels†



† V. Andrearczyk, `https://arxiv.org/abs/1703.05230`

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: vérité et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0},\rho^2\mathbf{I})$

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Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: vérité et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Estimateur paramétrique $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: vérité et $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Estimateur paramétrique $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$$

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: vérité et $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Estimateur paramétrique $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$$

Erreur quadratique $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 = \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ inconnue

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: vérité et $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Estimateur paramétrique $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$$

Erreur quadratique $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 = \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ inconnue

$$R(\lambda) = \mathbb{E}_\zeta \left\|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y} + \boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2$$

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: vérité et $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Estimateur paramétrique $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \operatorname{argmin}_{\boldsymbol{x}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$

Erreur quadratique $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 = \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ inconnue

$$\begin{aligned} R(\lambda) &= \mathbb{E}_\zeta \left\|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y} + \boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2 \\ &= \mathbb{E}_\zeta \left\|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\right\|^2 + 2\mathbb{E}_\zeta \left\langle \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}, \boldsymbol{y} - \bar{\boldsymbol{x}} \right\rangle + \mathbb{E}_\zeta \left\|\boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2 \end{aligned}$$

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: vérité et $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Estimateur paramétrique $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

$$\text{Ex.} \quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \operatorname{argmin}_{\boldsymbol{x}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$$

Erreur quadratique $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 = \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ inconnue

$$\begin{aligned} R(\lambda) &= \mathbb{E}_\zeta \left\|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y} + \boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2 \\ &= \mathbb{E}_\zeta \left\|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\right\|^2 + 2\mathbb{E}_\zeta \left\langle \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}, \boldsymbol{y} - \bar{\boldsymbol{x}} \right\rangle + \mathbb{E}_\zeta \left\|\boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2 \\ &= \mathbb{E}_\zeta \left\|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\right\|^2 + 2\mathbb{E}_\zeta \left\langle \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda), \zeta \right\rangle - 2\mathbb{E}_\zeta \left\langle \bar{\boldsymbol{x}} + \zeta, \zeta \right\rangle + \mathbb{E}_\zeta \left\|\zeta\right\|^2 \end{aligned}$$

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: vérité et $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Estimateur paramétrique $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$

Erreur quadratique $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 = \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ inconnue

$$
\begin{aligned}
R(\lambda) &= \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y} + \boldsymbol{y} - \bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\mathbb{E}_\zeta \langle \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}, \boldsymbol{y} - \bar{\boldsymbol{x}} \rangle + \mathbb{E}_\zeta \|\boldsymbol{y} - \bar{\boldsymbol{x}}\|^2 \\
&= \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\mathbb{E}_\zeta \langle \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda), \zeta \rangle - 2\mathbb{E}_\zeta \langle \zeta, \zeta \rangle + \mathbb{E}_\zeta \|\zeta\|^2
\end{aligned}
$$

Stein Unbiased Risk Estimate (Principe)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: vérité et $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

Estimateur paramétrique $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

$$\text{Ex.} \quad \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linéaire)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(non linéaire)} \end{cases}$$

Erreur quadratique $R(\lambda) \triangleq \mathbb{E}_\zeta \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 = \mathbb{E}_\zeta \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ inconnue

$$\begin{aligned} R(\lambda) &= \mathbb{E}_\zeta \left\|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y} + \boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2 \\ &= \mathbb{E}_\zeta \left\|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\right\|^2 + 2\mathbb{E}_\zeta \left\langle \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}, \boldsymbol{y} - \bar{\boldsymbol{x}} \right\rangle + \mathbb{E}_\zeta \left\|\boldsymbol{y} - \bar{\boldsymbol{x}}\right\|^2 \\ &= \mathbb{E}_\zeta \left\|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\right\|^2 + 2\mathbb{E}_\zeta \left\langle \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda), \zeta \right\rangle - 2\mathbb{E}_\zeta \left\langle \zeta, \zeta \right\rangle + \mathbb{E}_\zeta \left\|\zeta\right\|^2 \\ &= \mathbb{E}_\zeta \underline{\left\|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\right\|^2} + 2\mathbb{E}_\zeta \left\langle \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda), \zeta \right\rangle - \underline{\mathbb{E}_\zeta \left\|\zeta\right\|^2} \end{aligned}$$

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$$\mathbb{E}_\zeta \langle \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda), \zeta \rangle = \int \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \zeta; \lambda), \zeta \rangle \exp\left(-\frac{\|\zeta\|^2}{2\rho^2}\right) \mathrm{d}\zeta$$

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Observations $\quad \boldsymbol{y} = \bar{\boldsymbol{x}} + \zeta \in \mathbb{R}^P, \quad \bar{\boldsymbol{x}}$: vérité et $\zeta \sim \mathcal{N}(\mathbf{0},\rho^2\mathbf{I})$

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$$\mathbb{E}_\zeta \langle \widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda), \zeta \rangle = \int \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \zeta;\lambda), \zeta \rangle \exp\left(-\frac{\|\zeta\|^2}{2\rho^2}\right) \mathrm{d}\zeta \stackrel{\text{I.P.P.}}{=} \rho^2 \mathbb{E}_\zeta \operatorname{tr}\left(\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda)\right)$$

Estimateur séquentiel et différentiation récursive

$$\boldsymbol{\Lambda} \triangleq (\lambda,\alpha), \quad \mathbf{U}_{\boldsymbol{\Lambda}} : (\boldsymbol{h},\boldsymbol{v}) \mapsto \lambda[\alpha \mathbf{D}\boldsymbol{h}, \mathbf{D}\boldsymbol{v}]$$

Primal-dual accéléré

$$\widetilde{\boldsymbol{z}}^n = \boldsymbol{z}^n + \tau_n \mathbf{U}_{\boldsymbol{\Lambda}} \boldsymbol{w}^n$$
$$\boldsymbol{z}^{n+1} = \mathrm{prox}_{\tau_n(\|\cdot\|_{2,1})^*}\left(\widetilde{\boldsymbol{z}}^n\right)$$
$$\widetilde{\boldsymbol{x}}^n = \boldsymbol{x}^n - \sigma_n \mathbf{U}_{\boldsymbol{\Lambda}}^* \boldsymbol{z}^{n+1}$$
$$\boldsymbol{x}^{n+1} = \mathrm{prox}_{\sigma_n\|\mathbf{D}\mathcal{L}-\boldsymbol{\Phi}\cdot\|_2^2}\left(\widetilde{\boldsymbol{x}}^n\right)$$
$$\theta_n = (1+2\mu\sigma_n)^{-\frac{1}{2}},$$
$$\tau_{n+1} = \tau_n/\theta_n,\ \sigma_{n+1} = \theta_n\sigma_n$$
$$\boldsymbol{w}^{n+1} = \boldsymbol{x}^n + \theta^n\left(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n\right)$$

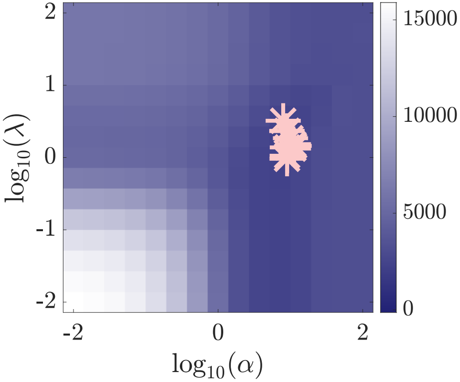
Primal-dual accéléré différentié

$$\partial_{\boldsymbol{\Lambda}}\widetilde{\boldsymbol{z}}^n = \partial_{\boldsymbol{\Lambda}}\boldsymbol{z}^n + \tau_n \mathbf{U}_{\boldsymbol{\Lambda}}\partial_{\boldsymbol{\Lambda}}\boldsymbol{w}^n + \tau_n\partial_{\boldsymbol{\Lambda}}\mathbf{U}_{\boldsymbol{\Lambda}}\boldsymbol{w}^n$$
$$\partial_{\boldsymbol{\Lambda}}\boldsymbol{z}^{n+1} = \partial_{\widetilde{\boldsymbol{z}}}\mathrm{prox}_{\tau_n(\|\cdot\|_{2,1})^*}\left(\widetilde{\boldsymbol{z}}^n\right)[\partial_{\boldsymbol{\Lambda}}\widetilde{\boldsymbol{z}}^n]$$
$$\partial_{\boldsymbol{\Lambda}}\widetilde{\boldsymbol{x}}^n = \partial_{\boldsymbol{\Lambda}}\boldsymbol{x}^n - \sigma_n \mathbf{U}_{\boldsymbol{\Lambda}}^*\partial_{\boldsymbol{\Lambda}}\boldsymbol{z}^{n+1} - \sigma_n\partial_{\boldsymbol{\Lambda}}\mathbf{U}_{\boldsymbol{\Lambda}}\boldsymbol{z}^{n+1}$$
$$\partial_{\boldsymbol{\Lambda}}\boldsymbol{x}^{n+1} = \partial_{\widetilde{\boldsymbol{x}}}\mathrm{prox}_{\sigma_n\|\mathbf{D}\mathcal{L}-\boldsymbol{\Phi}\cdot\|_2^2}\left(\widetilde{\boldsymbol{x}}^n\right)[\partial_{\boldsymbol{\Lambda}}\widetilde{\boldsymbol{x}}^n]$$
$$\partial_{\boldsymbol{\Lambda}}\boldsymbol{w}^{n+1} = \partial_{\boldsymbol{\Lambda}}\boldsymbol{x}^n + \theta^n\left(\partial_{\boldsymbol{\Lambda}}\boldsymbol{x}^{n+1} - \partial_{\boldsymbol{\Lambda}}\boldsymbol{x}^n\right)$$

Recherche automatique des paramètres de régularisation

Moyenne sur dix réalisations de texture

Matrice de covariance estimée $\boldsymbol{\mathcal{S}}$



$\widehat{\lambda}^{\mathrm{qN}} = 1{,}74 \pm 0{,}43 \qquad \widehat{\alpha}^{\mathrm{qN}} = 9{,}62 \pm 0{,}70$

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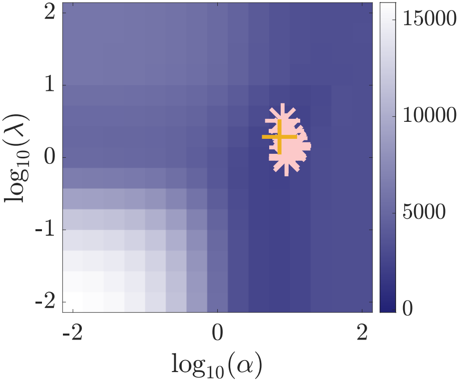

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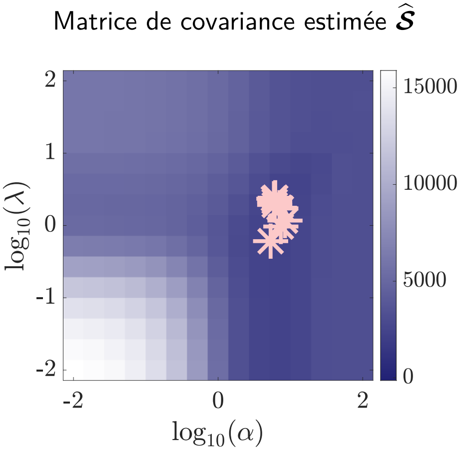

$\widehat{\lambda}^{\mathsf{qN}} = 1{,}68 \pm 0{,}37 \qquad \widehat{\alpha}^{\mathsf{qN}} = 6{,}70 \pm 0{,}58$

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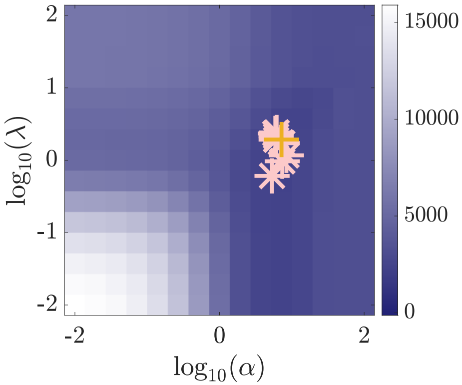

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Initialisation quasi-Newton

- Hyperparamètres $\mathbf{\Lambda}^{[0]} = \big(\lambda^{[0]}, \alpha^{[0]}\big)$, avec

$$\lambda^{[0]} = \frac{\mathrm{tr}(\boldsymbol{\mathcal{S}})}{2\,\mathrm{TV}(\widehat{\boldsymbol{v}}^{\mathrm{RL}}(\boldsymbol{\mathcal{L}}))}, \quad \text{et} \quad \alpha^{[0]} = \frac{\mathrm{TV}(\widehat{\boldsymbol{v}}^{\mathrm{RL}}(\boldsymbol{\mathcal{L}}))}{\mathrm{TV}(\widehat{\boldsymbol{h}}^{\mathrm{RL}}(\boldsymbol{\mathcal{L}}))}.$$

- Approximation de l'inverse de la hessienne

$$\boldsymbol{H}^{[0]} = \mathrm{diag}\left(\left|\frac{\kappa\lambda^{[0]}}{\partial_\lambda \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \mathbf{\Lambda}^{[0]} | \boldsymbol{\mathcal{S}})}\right|, \left|\frac{\kappa\alpha^{[0]}}{\partial_\alpha \widehat{R}_{\nu,\boldsymbol{\varepsilon}}(\boldsymbol{\mathcal{L}}; \mathbf{\Lambda}^{[0]} | \boldsymbol{\mathcal{S}})}\right|\right).$$