Block-coordinate proximal algorithms for scale-free texture segmentation†

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† Supported by ANR-16-CE33-0020 MultiFracs, France.

Texture segmentation

Segmentation task

k-means

Piecewise constant image

Texture segmentation

Segmentation task

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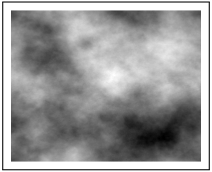
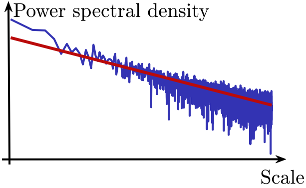
Piecewise constant image

Monofractal scale invariant texture



Slope : fractal parameter h [Abry1995]



High resolution necessary

Texture segmentation

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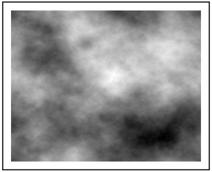

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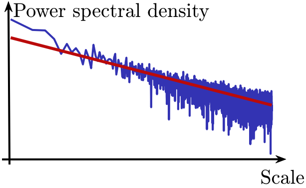


High resolution necessary

I) Detect constant h areas

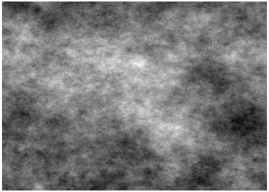
Estimation of local h

II) Effective implementation

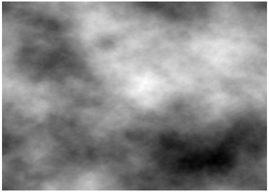
Block-coordinate algorithm

Monofractal textures

Synthetic texture with constant local regularity



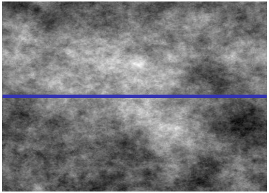
$h = 0.3$



$h = 0.9$

Monofractal textures

Synthetic texture with constant local regularity



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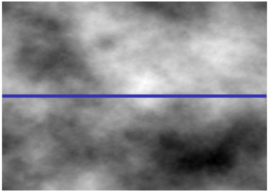

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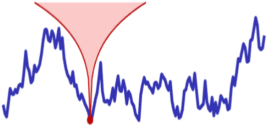
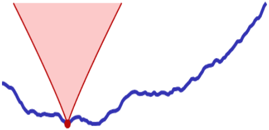




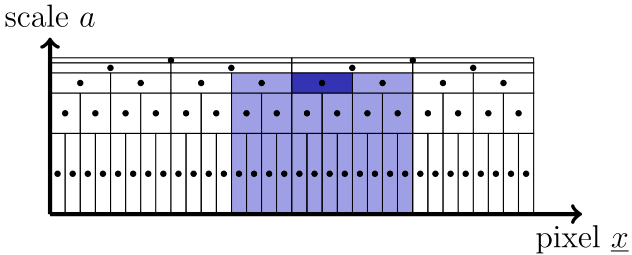
IDEA : fit local behavior with power law functions

$$|f(x) - f(y)| \leq C|x - y|^{h(x)}, \quad h(x) \equiv 0.3\,(\text{left}), \quad 0.9\,(\text{right})$$

Wavelet transform and leader coefficients

(i) **WT** of image X : $w_{a,\underline{n}}(X)$ at scale a and pixel $\underline{n}$,

(ii) **Local surpremum** of $|w_{a,\underline{n}}(X)|$: $\mathcal{L}_{a,\underline{n}}(X)$ (*leaders*)



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Linear regression [Wendt2009]

$$\boxed{\log\left(\mathcal{L}_{a,\underline{n}}\right) \simeq \log\left(\eta(\underline{n})\right) + \log(a)h(\underline{n})}$$

Multiscale analysis
Estimation of local regularity

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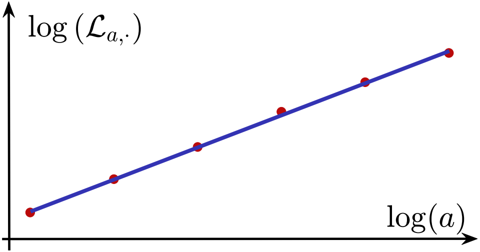
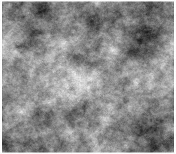
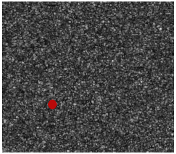


Texture X

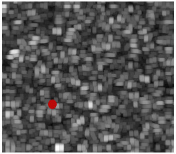


Scale $a = 2^1$



⋮

Scale $a = 2^6$



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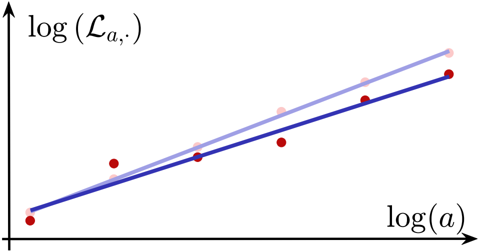

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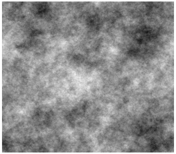

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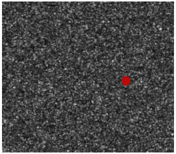

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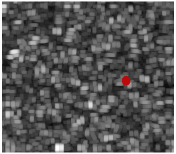

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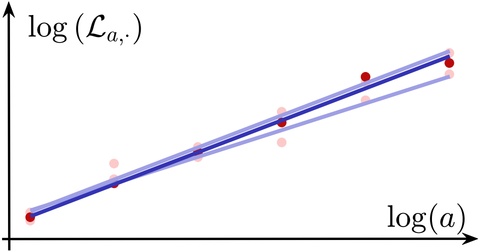

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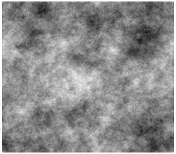

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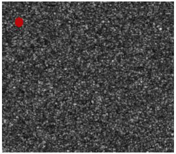

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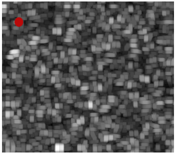
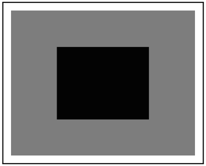
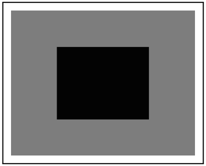




Piecewise constant h



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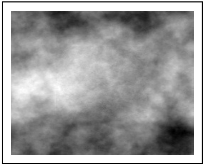
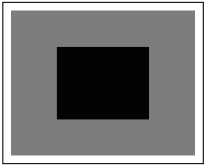


Texture sample X

Piecewise monofractal textures

Synthetic data



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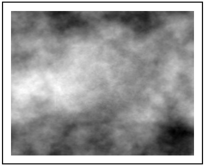

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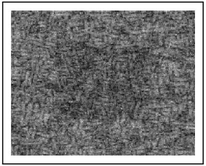

Linear fit $\widehat{h}$

Linear fit is not satisfactory !

Piecewise monofractal textures

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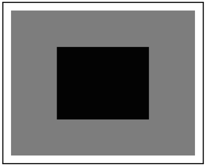

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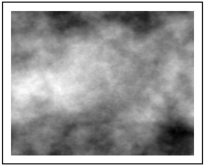

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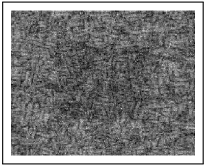

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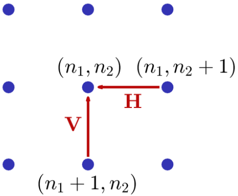
Objective function enforcing picewise constancy [Pustelnik2016]

$$\widehat{\widehat{h}} \in \underset{h}{\mathrm{Argmin}} \quad \underbrace{\mathbf{DF}(h, X)}_{\textbf{D}\text{ata }\textbf{F}\text{idelity}} + \underbrace{\lambda \mathbf{TV}(h)}_{\textbf{T}\text{otal }\textbf{V}\text{ariation}}$$

AIM : enforce piecewise behavior of estimate



Discrete difference operator

$$(\mathbf{D}h)_{n_1,n_2} = \frac{1}{2}\begin{pmatrix} h_{n_1,n_2+1} - h_{n_1,n_2} \\ h_{n_1+1,n_2} - h_{n_1,n_2} \end{pmatrix}$$

Total Variation denoising
Convex optimization

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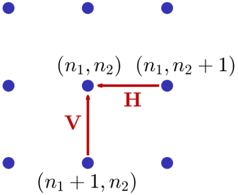

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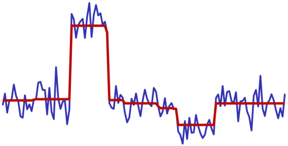
Total variation penalization

$$\|\mathbf{D}h\|_{2,1} = \sum_{n_1=1}^{N-1}\sum_{n_2=1}^{N-1}\sqrt{(\mathbf{H}h)^2_{n_1,n_2} + (\mathbf{V}h)^2_{n_1,n_2}}$$



TV denoising ($\widehat{\widehat{h}}$ piecewize constant estimate)

$$\widehat{\widehat{h}} \in \underset{h}{\operatorname{Argmin}} \frac{1}{2}\|h - \widehat{h}\|_2^2 + \lambda \|\mathbf{D}h\|_{2,1}$$

Reminder : $\widehat{h}$ linear fit estimate.

Total Variation denoising
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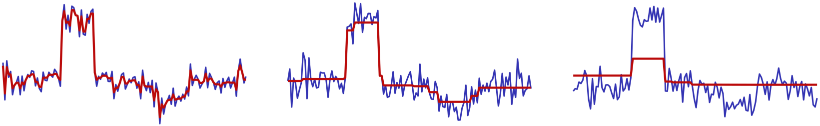
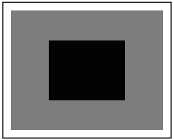
Regularization parameter (fine tuning of λ)

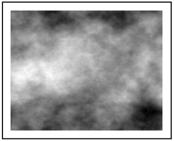


Too small | **Optimal** | Too large

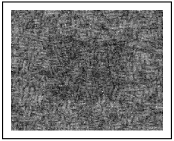
Results on synthetic data
Linear regression then TV denoising



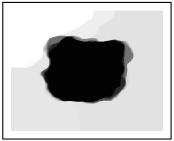
Mask



Texture sample

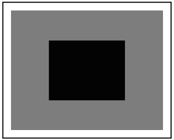


Linear fit $\widehat{h}$ ✗



TV denoised $\widehat{\widehat{h}}$ ✓

Results on synthetic data
Linear regression then TV denoising



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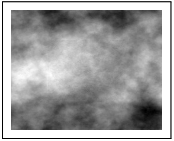

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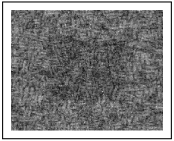

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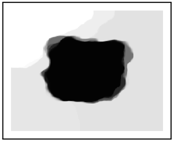

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Issues/Difficulties :

(i) fine tuning of the regularization parameter : $\lambda \in [\lambda_{\text{min}}, \dots, \lambda_{\text{max}}]$

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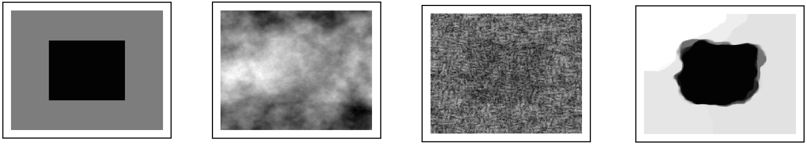
Linear regression then TV denoising



Mask | Texture sample | Linear fit $\widehat{h}$ ✗ | TV denoised $\widehat{\widehat{h}}$ ✓

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(ii) $\lambda_{\rm opt} \sim 10$ (compared to $\lambda_{\rm opt} \sim 10^{-2}$ for image denoising)
$\hookrightarrow$ needs large number of iterations

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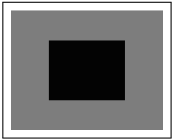
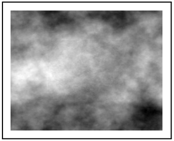
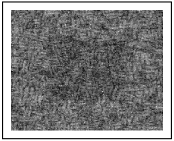
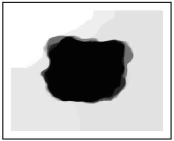
Linear regression then TV denoising









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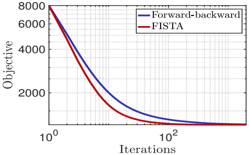
(iii) computational cost (time & memory) :

Image size	256×256	512×512	1024×1024
Computational time	3 min	16 min	86 min

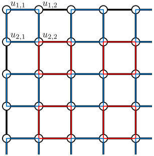
Proximal algorithms acceleration

State of the art

Over-relaxation : FISTA **[Dossal2014]**
Fast **I**terative **S**hrinkage **T**hresholding **A**lgorithm



Alternating methods : ADMM **[Chambolle2015]**
Proximal **A**lternating **D**escent



Block-coordinate approaches : (Random) block selection

- Stochastic gradient *(machine learning)* **[Le Roux2012]**
- Primal and/or dual splitting *(image processing)* **[Repetti2015, Feriel2017, Chambolle2017]**

TV denoising $\widehat{\widehat{h}} \in \operatorname*{Argmin}_{h} \frac{1}{2}\|h - \widehat{h}\|_2^2 + \lambda\|\mathbf{D}h\|_{2,1}$

for $k \in \mathbb{N}$ **do**

$$y^{[k+1]} = \mathrm{prox}_{\gamma\ \lambda\|.\|_{2,1}^*}\left(y^{[k]} + \gamma\ \mathbf{D}\ h^{[k]}\right)$$
$$h^{[k+1]} = h^{[k]} - \mathbf{D}^*\left(y^{[k+1]} - y^{[k]}\right)$$

end

Dual forward-backward algorithm [Feriel2017]

Proximal algorithms

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Dual forward-backward algorithm [Feriel2017]

TV denoising

$$\widehat{\widehat{h}} \in \operatorname{Argmin}_{h} \frac{1}{2}\|h - \widehat{h}\|_2^2 + \lambda\|\mathbf{D}h\|_{2,1}$$

for $k \in \mathbb{N}$ **do**

$$y^{[k+1]} = \mathrm{prox}_{\gamma\ \lambda\|.\|_{2,1}^*}\left(y^{[k]} + \gamma\ \mathbf{D}\ h^{[k]}\right)$$

$$h^{[k+1]} = h^{[k]} - \mathbf{D}^*\left(y^{[k+1]} - y^{[k]}\right)$$

end

Dual forward-backward algorithm [Feriel2017]

Quadratic convergence condition $\gamma < 1/\|\mathbf{D}\|^2$

Proximal algorithms

TV denoising

$$\widehat{\widehat{h}} \in \operatorname*{Argmin}_{h} \frac{1}{2}\|h-\widehat{h}\|_2^2 + \lambda \sum_{\ell=1}^{L} \|\mathbf{D}_\ell h\|_{2,1}$$

Block strategies

$$\mathbf{D} = [\mathbf{D}_1, \ldots, \mathbf{D}_\ell, \ldots, \mathbf{D}_L]^\top$$

$$\|\mathbf{D}h\|_{2,1} = \sum_{\ell=1}^{L} \|\mathbf{D}_\ell h\|_{2,1}$$

for $k \in \mathbb{N}$ **do**

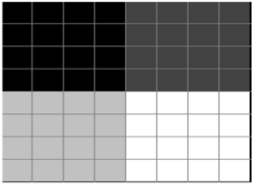
  for $\ell \in \{1, \ldots, L\}$ **do**

$$y_\ell^{[k+1]} = \mathrm{prox}_{\gamma_\ell \lambda \|.\|_{2,1}^*}\left(y_\ell^{[k]} + \gamma_\ell \mathbf{D}_\ell h^{[k]}\right)$$

$$h^{[k+1]} = h^{[k]} - \mathbf{D}_\ell^*\left({y_\ell}^{[k+1]} - y_\ell^{[k]}\right)$$

  end

end

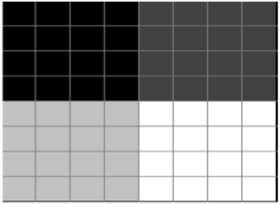


Block dual forward-backward algorithm [Feriel2017]

Quadratic convergence condition

$$\gamma_\ell < 1/\|\mathbf{D}_\ell\|^2$$

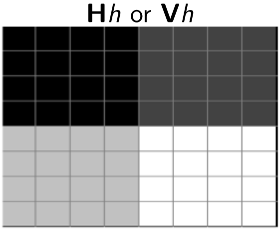
H*h* or **V***h*



Regions

Choice of the blocks

Regions and lattices



Regions

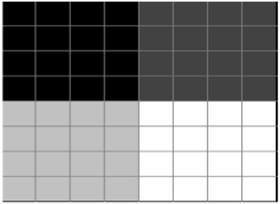
$$\forall \ell \in \{1, \ldots, 4\},$$

$$\|\mathbf{D}_\ell\| = \sqrt{2} \simeq 1.4142$$

$$= \|\mathbf{D}\|$$

no gain !

Hh or **V**h



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no gain !

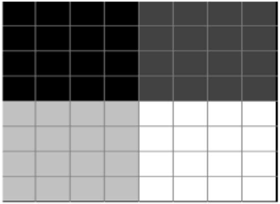
Descent steps

$$\gamma_\ell < 1$$

Choice of the blocks

Regions and lattices

$\mathbf{H}h$ or $\mathbf{V}h$



Regions

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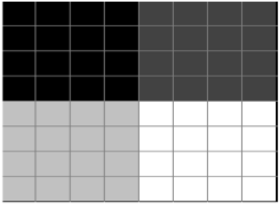

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Lattices

$$\forall \ell \in \{1, \dots, 4\},$$
$$\|\mathbf{D}_\ell\| = \sqrt{2} \simeq 1.4142$$
$$= \|\mathbf{D}\|$$

no gain !

$$\forall \ell \in \{1, \dots, 4\},$$
$$\|\mathbf{D}_\ell\| = \sqrt{3}/2 \simeq 0.8660$$
$$< \|\mathbf{D}\|$$

gain of factor $\simeq 1.6$

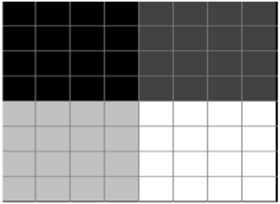
Descent steps

$$\gamma_\ell < 1$$

Choice of the blocks

Regions and lattices

$\mathbf{H}h$ or $\mathbf{V}h$



Regions

$\mathbf{H}h$ or $\mathbf{V}h$

Lattices

Regions:

$$\forall \ell \in \{1, \dots, 4\}, \quad \|\mathbf{D}_\ell\| = \sqrt{2} \simeq 1.4142 = \|\mathbf{D}\|$$

no gain !

Lattices:

$$\forall \ell \in \{1, \dots, 4\}, \quad \|\mathbf{D}_\ell\| = \sqrt{3}/2 \simeq 0.8660 < \|\mathbf{D}\|$$

gain of factor $\simeq 1.6$

Descent steps

Regions: $\gamma_\ell < 1$

Lattices: $\gamma_\ell < 8/3$

Duality gap

$$\delta(h, y) = \underbrace{\frac{1}{2}\|h - \widehat{h}\|_2^2 + \lambda\|\mathbf{D}h\|_{2,1}}_{\text{primal functional}} + \underbrace{\frac{1}{2}\|-\mathbf{D}^* y + \widehat{h}\|_2^2 - \frac{1}{2}\|\widehat{h}\|_2^2 + \iota_{2,\infty(\lambda)}(y)}_{\text{dual functional}}$$

Duality gap

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Convergence

$$\log \delta(h^{[k]}, y^{[k]}) \underset{k\to\infty}{\longrightarrow} -\infty$$

Duality gap

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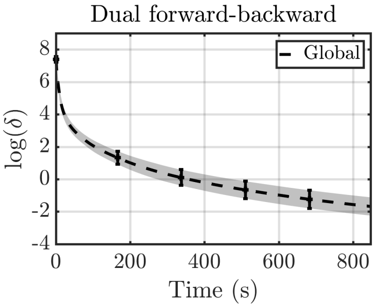

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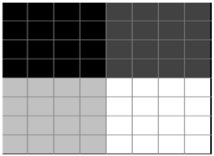
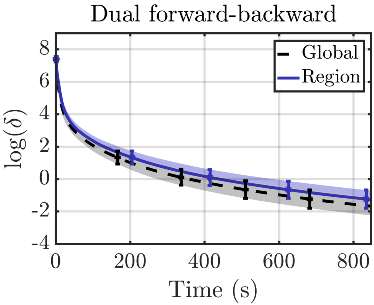


Region



Forward-backward algorithms convergence

Region Vs lattice strategies

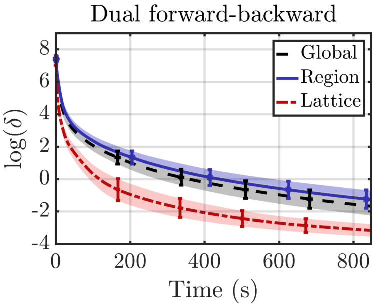
Duality gap

$$\delta(h, y) = \underbrace{\frac{1}{2}\|h - \widehat{h}\|_2^2 + \lambda\|\mathbf{D}h\|_{2,1}}_{\text{primal functional}} + \underbrace{\frac{1}{2}\|-\mathbf{D}^* y + \widehat{h}\|_2^2 - \frac{1}{2}\|\widehat{h}\|_2^2 + \iota_{2,\infty(\lambda)}(y)}_{\text{dual functional}}$$

Convergence

$$\log \delta(h^{[k]}, y^{[k]}) \underset{k \to \infty}{\longrightarrow} -\infty$$

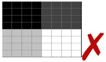
Lattice



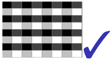
Extension to primal-dual algorithms

Block forward-backward

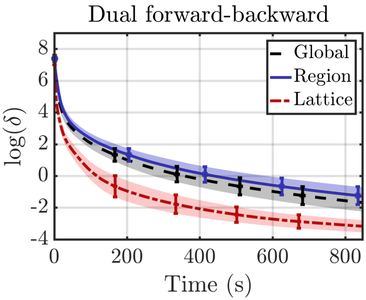
$$\gamma_\ell < \frac{1}{\|\mathbf{D}_\ell\|^2} \quad \text{[Feriel2017]}$$



Region



Lattice



Extension to primal-dual algorithms

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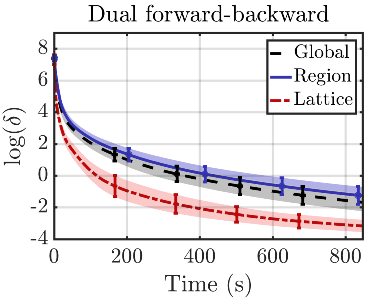


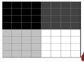
Block primal-dual

$$\sigma_\ell \tau < \frac{1}{L\|\mathbf{D}_\ell\|^2} \quad \text{[Chambolle2017]}$$

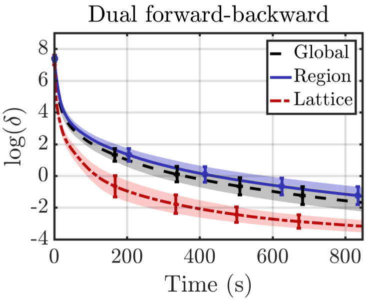
Extension to primal-dual algorithms

Block forward-backward

$$\gamma_\ell < \frac{1}{\|\mathbf{D}_\ell\|^2} \quad \text{[Feriel2017]}$$


✗
Region


✓
Lattice



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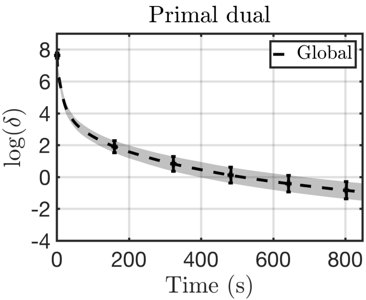

Extension to primal-dual algorithms

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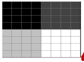
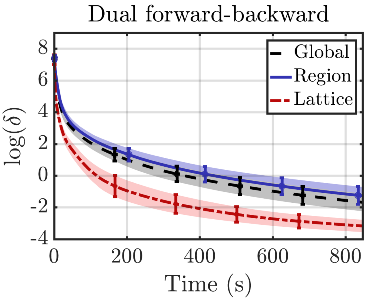

✗
Region


✓
Lattice



Block primal-dual

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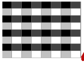
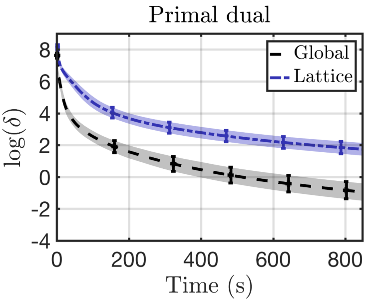
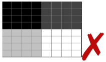

✗
Lattice



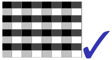
Extension to primal-dual algorithms

Block forward-backward

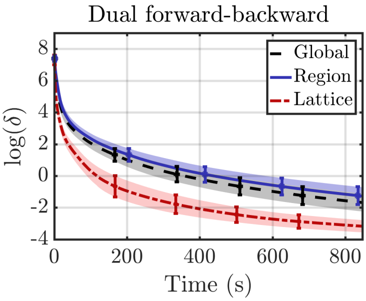
$$\gamma_\ell < \frac{1}{\|\mathbf{D}_\ell\|^2} \quad \text{[Feriel2017]}$$



Region

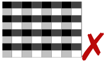


Lattice

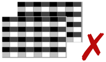


Block primal-dual

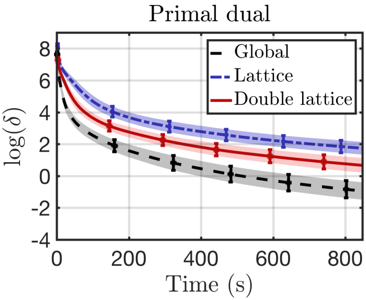
$$\sigma_\ell \tau < \frac{p_\ell}{\|\mathbf{D}_\ell\|^2} \quad \text{[Chambolle2017]}$$



Lattice



Double lattice

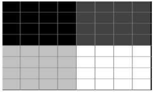


Take home message

Block strategies

Lower memory requirements ✓

[Repetti2015]



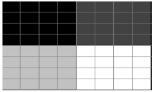
Larger computational time ✗

Take home message

Block strategies

Lower memory requirements ✓

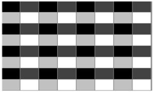
[Repetti2015]



Larger computational time ✗

Lattice splitting

Lower memory requirements ✓



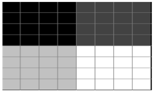
Reduced computational time ✓

Take home message

Block strategies

Lower memory requirements ✓

[Repetti2015]



Larger computational time ✗

Lattice splitting

Lower memory requirements ✓

Reduced computational time ✓

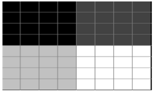
Possibilities

- explore large range of regularization parameter λ,
- process high resolution images,
- analyze huge amount of data.

Take home message

Block strategies

Lower memory requirements ✓ [Repetti2015]



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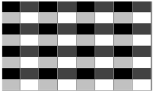

Reduced computational time ✓

Possibilities

- explore large range of regularization parameter λ,
- process high resolution images,
- analyze huge amount of data.

Applications

- medical imaging (cancer detection, ...) [Marin2017],
- meteorology (clouds characterization, ...) [Arrault1997],
- art (painting authentication, ...) [Abry2013].