Combining Local Regularity Estimation and Total Variation Optimization for Scale-Free Texture Segmentation[†]

B. Pascal[1], N. Pustelnik[1], P. Abry[1]

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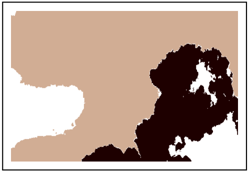
[†] Supported by Defi Imag'in SIROCCO and ANR-16-CE33-0020 MultiFracs, France.

Texture segmentation

Segmentation task

k-means



Piecewise constant image

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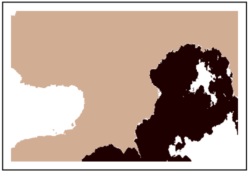
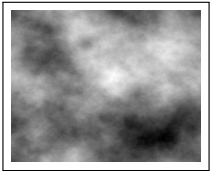
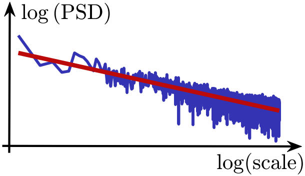


Piecewise constant image

Monofractal scale invariant texture



Slope: fractal parameter h **[Abry1995]**



High resolution necessary

Texture segmentation

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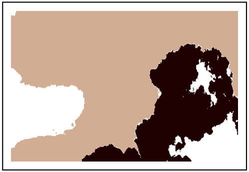

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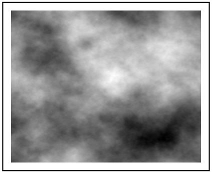

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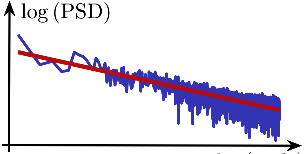
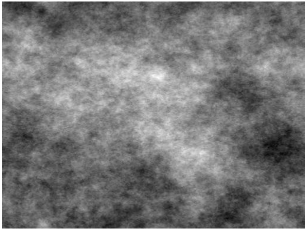


High resolution necessary

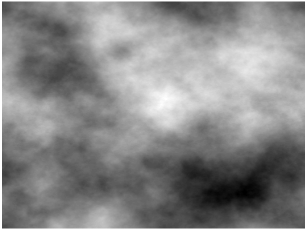
Contributions (i) Segmentation based on scale-free parameters
(ii) Effective implementation using strong-convexity

Monofractal textures

Synthetic texture with constant local regularity



$h = 0.3$



$h = 0.9$

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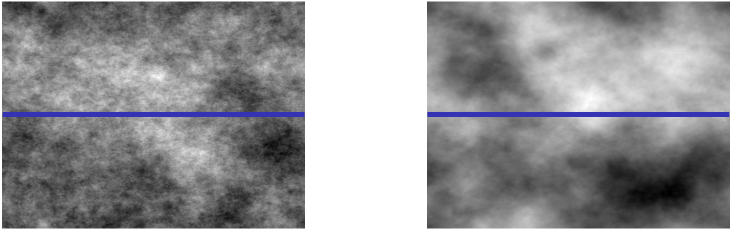


$h = 0.3$ $\qquad\qquad$ $h = 0.9$

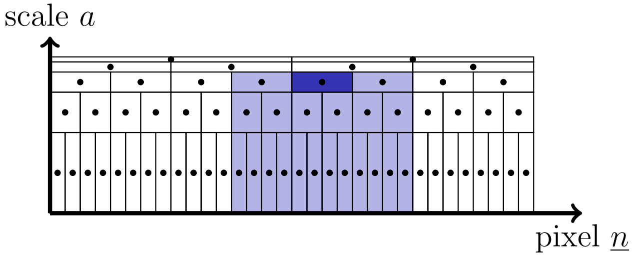
IDEA: fit local behavior with power law functions

$$|f(x) - f(y)| \leq C|x - y|^{h(x)}, \quad h(x) \equiv 0.3\,(\text{left}), \quad 0.9\,(\text{right})$$

Wavelet transform and leader coefficients

(i) **DWT** of image X: $w_{a,\underline{n}}(X)$ at scale a and pixel $\underline{n}$,

(ii) **Local surpremum** of $|w_{a,\underline{n}}(X)|$: $\mathcal{L}_{a,\underline{n}}(X)$ (*leaders*)



Multiscale analysis
Estimation of local regularity

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Linear regression $(\hat{v}_{\mathrm{reg}}(\underline{n}), \hat{h}_{\mathrm{reg}}(\underline{n}))$ [Wendt2009]

$$\boxed{\log\left(\mathcal{L}_{a,\underline{n}}\right) \simeq v(\underline{n}) + h(\underline{n}) \log(a)}$$

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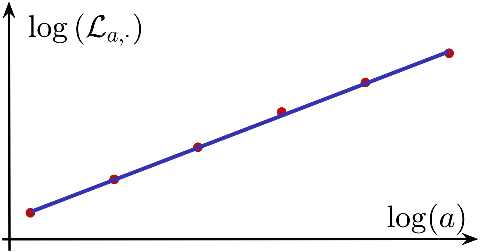
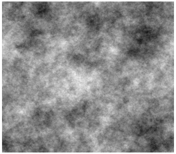
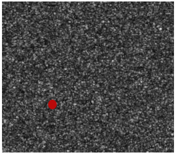


Texture X

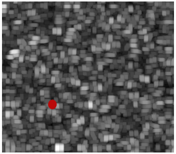


Scale $a = 2^1$



⋮

Scale $a = 2^6$



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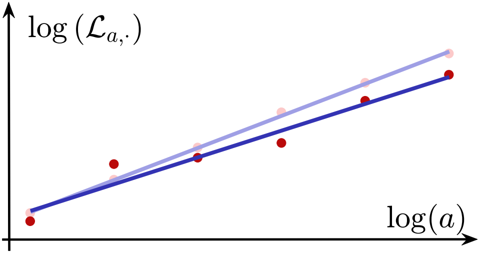

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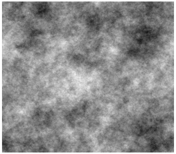

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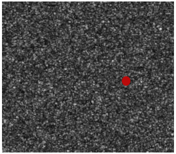

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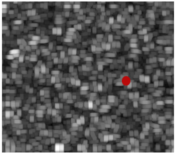

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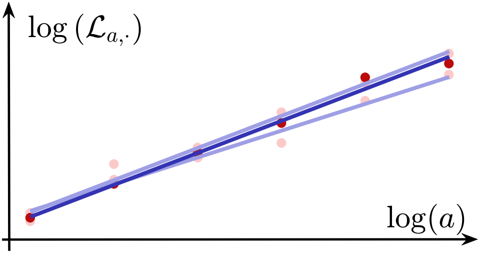

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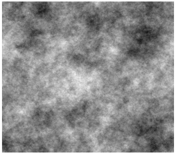

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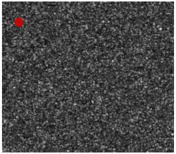

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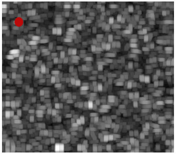
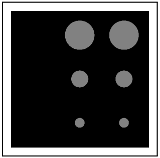
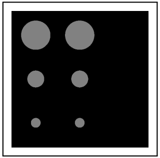


Piecewise monofractal textures

Synthetic data



Piecewise cst. v_0



Piecewise cst. h_0

Piecewise monofractal textures

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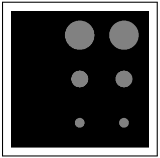

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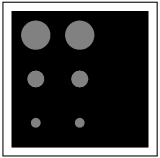
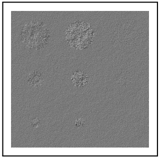


Piecewise cst. h_0



Texture sample X

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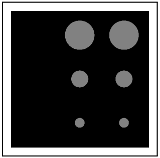

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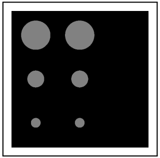

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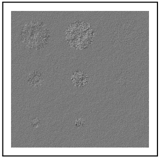

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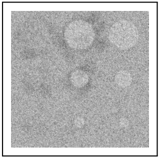
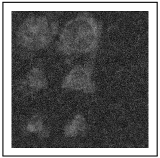


Lin. reg. $\widehat{v}_{\mathrm{reg}}$



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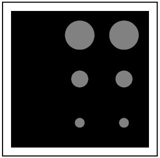

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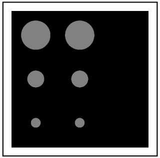

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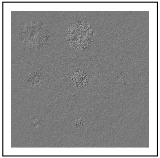

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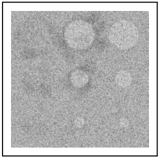

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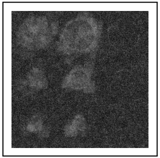
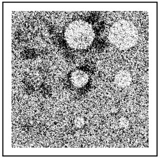
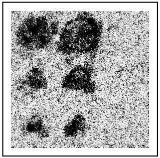


Lin. reg. $\widehat{h}_{\mathrm{reg}}$



k-means on $\widehat{v}_{\mathrm{reg}}$



k-means on $\widehat{h}_{\mathrm{reg}}$

✗ Linear regression estimator has a large variance (point-wise estimator)

Optimization scheme - Monofractal model and piecewise constancy

$$\left(\widehat{v}, \widehat{h}\right) \in \operatorname*{Argmin}_{v,h} \mathbf{DF}(v, h\,;\, \mathcal{L}(X)) + \lambda_v \mathbf{TV}(v) + \lambda_h \mathbf{TV}(h)$$

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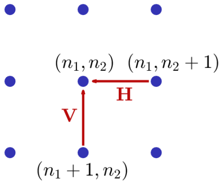
aim: enforce piecewise behavior of estimate

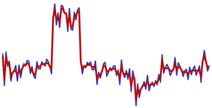


Discrete difference operator

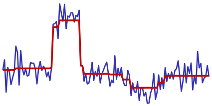
$$(\mathbf{D}x)_{n_1,n_2} = \frac{1}{2} \begin{pmatrix} x_{n_1,n_2+1} - x_{n_1,n_2} \\ x_{n_1+1,n_2} - x_{n_1,n_2} \end{pmatrix}$$

Total variation penalization

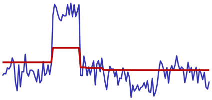
$$\mathbf{TV}(x) = \|\mathbf{D}x\|_{2,1} = \sum_{n_1=1}^{N-1} \sum_{n_2=1}^{N-1} \sqrt{(\mathbf{H}x)^2_{n_1,n_2} + (\mathbf{V}x)^2_{n_1,n_2}}$$



Too small



Optimal



Too large

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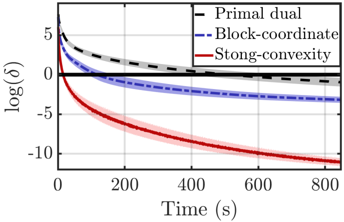
State-of-the-art - Segmentation on h only

$$\mathbf{DF}(h\,;\,\mathcal{L}) = \frac{1}{2}\|h - \widehat{h}_{\mathrm{reg}}\|_2^2$$

✓ only one parameter λ_h

✓ fast algorithms **[Pascal2018]**



✗ poor segmentation performance

$$\mathbf{DF}(h,\omega\,;\,\mathcal{L}) = \frac{1}{2}\|h - \sum_a \omega_a \mathcal{L}_{a,.}\|_2^2$$

✗ additional constraints on $\{\omega\}_a$

✗ time and memory consuming

✓ very good accuracy **[Pustelnik2016]**

Optimization scheme - Monofractal model and piecewise constancy

$$\left(\widehat{v}, \widehat{h}\right) \in \operatorname*{Argmin}_{v,h} \underline{\mathbf{DF}(v, h\,;\, \mathcal{L}(X))} + \lambda_v \mathbf{TV}(v) + \lambda_h \mathbf{TV}(h)$$

Proposed data fidelity term - Joint segmentation on v and h

$$\mathbf{DF}(v, h\,;\, \mathcal{L}) = \frac{1}{2} \sum_a \| v + \log(a) h - \log \mathcal{L}_{a,.} \|_2^2$$

Objectives

- match scale-free behavior
- couple the estimation of v and h
- do not impose that v and h have same edges

✓ two features estimated **jointly**

✓ strong-convexity of **DF**: accelerated algorithm **[Chambolle2011]**

✗ two regularization parameters to tune

Strong convexity

Definition

φ is $\boldsymbol{\alpha}$-strongly convex iff $\varphi - \frac{\boldsymbol{\alpha}}{2}\|\cdot\|^2$ is convex.

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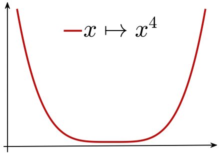
Examples



✓ strictly convex
✗ not strongly convex

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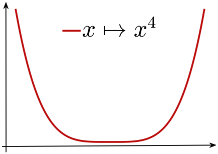

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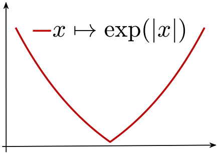

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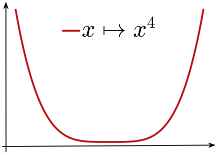

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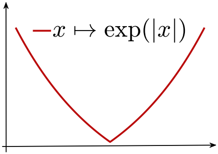

✓ strictly convex
✓ 1-strongly convex

Proposition

If φ is differentiable

(i) $\forall x, y,\ \langle \nabla\varphi(x) - \nabla\varphi(y), x - y\rangle \geq \alpha\|x - y\|^2$
$\Rightarrow \quad \varphi$ is α-strongly convex,

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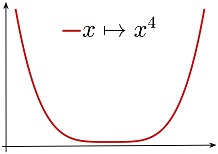

✓ strictly convex
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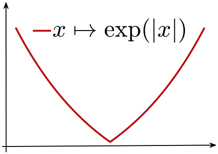

✓ strictly convex
✓ 1-strongly convex

Proposition

If φ is differentiable

(i) $\forall x, y,\ \langle \nabla\varphi(x) - \nabla\varphi(y), x - y\rangle \geq \alpha\|x - y\|^2$
$\Rightarrow \quad \varphi$ is α-strongly convex,

(ii) if $\nabla\varphi(x) = \mathbf{L}x - p, \quad \forall x,\ \langle \mathbf{L}x, x\rangle \geq \alpha\|x\|^2$
$\Rightarrow \quad \varphi$ is α-strongly convex, with α the smallest eigenvalue of $\mathbf{L}$.

Strong convexity

$$\Phi(v,h) = \underbrace{F_{\mathbf{A}}(v,h;\mathcal{L})}_{\text{?-convex}} + \underbrace{\lambda_v \|\mathbf{D}v\|_{2,1} + \lambda_h \|\mathbf{D}h\|_{2,1}}_{\text{convex}}$$

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$$F_{\mathbf{A}}(v,h\,;\,\mathcal{L}) = \frac{1}{2}\sum_{\mathrm{a_{min}}}^{a_{\max}} \|v + \log(a)h - \log\mathcal{L}_{a,.}\|_2^2 = \frac{1}{2}\|\mathbf{A}(v,h) - \log\mathcal{L}\|_2^2$$

where $\mathbf{A} : (v,h) \mapsto \{v + \log(a)h\}_a$ is **linear**.

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$$\nabla F_{\mathbf{A}}\left(v,h\,;\,\mathcal{L}\right) = \mathbf{A}^*\left(\mathbf{A}(v,h) - \log \mathcal{L}\right) = \underbrace{\mathbf{A}^*\mathbf{A}(v,h)}_{\mathbf{L}x} - \underbrace{\mathbf{A}^* \log \mathcal{L}}_{p}$$

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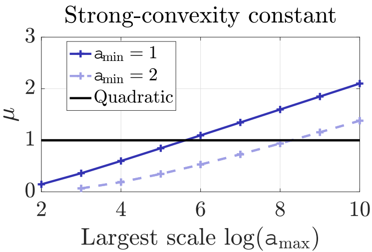
Proposition

$F_{\mathbf{A}}(v,h;\mathcal{L})$ is μ-strongly convex, with μ the smallest eigen value of $\mathbf{A}^*\mathbf{A}$.



Accelerated primal-dual algorithm

[Chambolle2011] Customized for our objective function

Primal variable $VH \underset{\text{def}}{\equiv} (v, h)$, dual variable $UL \underset{\text{def}}{\equiv} (u, \ell)$

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[Chambolle2011] Customized for our objective function

Primal variable $VH \underset{\text{def}}{\equiv} (v, h)$, dual variable $UL \underset{\text{def}}{\equiv} (u, \ell)$

for $k \in \mathbb{N}^*$ **do**

```
// Update of primal variable
```

$$VH^{[k+1]} = \operatorname{prox}_{\delta_k F_{\mathbf{A}}(.,\mathcal{L})}\left(VH^{[k]} - \delta_k \mathbf{D}^* \overline{UL}^{[k]}\right)$$

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```
// Update of dual variable
```

$$UL^{[k+1]} = \operatorname{prox}_{\nu_k \Lambda \|\cdot\|_{2,1}^*}\left(UL^{[k]} + \nu_k \mathbf{D}\, VH^{[k]}\right)$$

Accelerated primal-dual algorithm

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```
// Update of descent steps
```

$$\vartheta_k = (1 + 2\mu\delta_k)^{-1/2}, \ \underbrace{\delta_{k+1} = \vartheta_k \delta_k}_{\text{smaller}}, \ \ \underbrace{\nu_{k+1} = \nu_k / \vartheta_k}_{\text{larger}}$$

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```
// Update of auxiliary variable
```

$$\overline{UL}^{[k+1]} = UL^{[k+1]} + \vartheta_k \left(UL^{[k+1]} - UL^{[k]} \right)$$

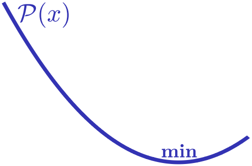
end

Duality gap

Measuring the convergence speed

Primal problem

$$\widehat{x} = \operatorname*{argmin}_{x} F(x) + G(\mathbf{L}x)$$



Duality gap

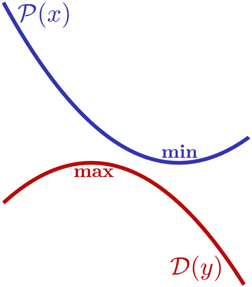
Measuring the convergence speed

Primal problem

$$\widehat{x} = \underset{x}{\operatorname{argmin}}\, F(x) + G(\mathbf{L}x)$$

Dual problem

$$\widehat{y} = \underset{y}{\operatorname{argmax}}\, -F^*(-\mathbf{L}^* y) - G^*(y)$$



Duality gap

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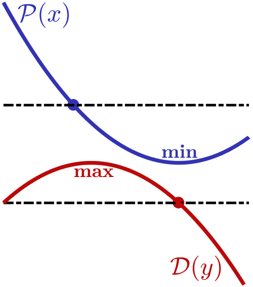
$$\widehat{y} = \operatorname*{argmax}_y \; -F^*(-\mathbf{L}^* y) - G^*(y)$$

Duality gap

$$\delta(x; y) \underset{\text{def.}}{=} F(x) + G(\mathbf{L}x) + F^*(-\mathbf{L}^* y) + G^*(y)$$



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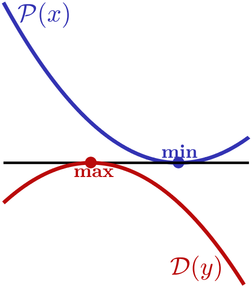
Characterization of the solution

$$\delta(\widehat{x}; \widehat{y}) \underset{\text{prop.}}{=} 0$$



Computing the duality gap

$$\delta(\quad ; \quad)$$

$$= \qquad + $$

Computing the duality gap

$$\delta(\underbrace{v, h}_{\text{primal}}; \quad)$$

$$= F_{\mathbf{A}}(v, h;\, \mathcal{L}) + G(\mathbf{D}v, \mathbf{D}v) +$$

Data fidelity

$$F_{\mathbf{A}}(v, h;\, \mathcal{L}) = \frac{1}{2} \sum_{a} \|v + \log(a)h - \mathcal{L}_{a,.}\|_2^2$$

Penalization

$$G(\mathbf{D}v, \mathbf{D}h) = \lambda_v \|\mathbf{D}v\|_{2,1} + \lambda_h \|\mathbf{D}h\|_{2,1}$$

Computing the duality gap

$$\delta(\underbrace{v, h}_{\text{primal}}; \underbrace{u, \ell}_{\text{dual}})$$

$$= F_{\mathbf{A}}(v, h\,;\,\mathcal{L}) + G(\mathbf{D}v, \mathbf{D}v) + F^*_{\mathbf{A}}(-\mathbf{D}^* u, -\mathbf{D}^*\ell) + G^*(u, \ell)$$

Data fidelity

$$F_{\mathbf{A}}(v, h\,;\,\mathcal{L}) = \frac{1}{2}\sum_a \|v + \log(a)h - \mathcal{L}_{a,.}\|_2^2$$

$$F^*_{\mathbf{A}}(-\mathbf{D}^* u, -\mathbf{D}^*\ell) = \,?$$

Penalization

$$G(\mathbf{D}v, \mathbf{D}h) = \lambda_v \|\mathbf{D}v\|_{2,1} + \lambda_h \|\mathbf{D}h\|_{2,1}$$

$$G^*(u, \ell) = \iota_{\mathcal{B}_{2,\infty}(\lambda_v)}(u) + \iota_{\mathcal{B}_{2,\infty}(\lambda_h)}(\ell)$$

$\mathcal{B}_{2,\infty}(\lambda)$: ball of radius λ w.r.t. $\|.\|_{2,\infty}$

Convex conjugate of data fidelity term

$$F^*_{\mathbf{A}}(v, h\,; \mathcal{L}) = \sup_{\widetilde{v}\in\mathbb{R}^{|\Omega|}, \widetilde{h}\in\mathbb{R}^{|\Omega|}} \langle \widetilde{v}, v\rangle + \langle \widetilde{h}, h\rangle - F_{\mathbf{A}}(\widetilde{v}, \widetilde{h}; \mathcal{L}).$$

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Euler condition

$$\begin{cases} v - \sum_a \left(\bar{v} + \log(a)\bar{h} - \log \mathcal{L}_{a,.}\right) = 0 \\ h - \sum_a \log(a)\left(\bar{v} + \log(a)\bar{h} - \log \mathcal{L}_{a,.}\right) = 0 \end{cases}$$

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$$\mathcal{S} = \sum_a \log \mathcal{L}_{a,.} \quad \text{and} \quad \mathcal{T} = \sum_a \log(a) \log \mathcal{L}_{a,.},$$

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$$\forall m = \{0, 1, 2\},\ S_m = \sum_a (\log a)^m, \quad \mathbf{A}^*\mathbf{A} = \begin{pmatrix} S_0\mathbb{I} & S_1\mathbb{I} \\ S_1\mathbb{I} & S_2\mathbb{I} \end{pmatrix}$$

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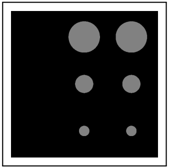
$$F^*_{\mathbf{A}}(v,h\,;\mathcal{L}) = \frac{1}{2}\langle (v,h), (\mathbf{A}^*\mathbf{A})^{-1}(v,h)\rangle + \langle (\mathcal{S},\mathcal{T}), (\mathbf{A}^*\mathbf{A})^{-1}(v,h)\rangle + \mathcal{C}$$

where $\mathcal{C}$ constant term only depending on $\mathcal{L}(X)$.

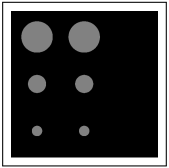
Performance assessment

Experiments on synthetic data

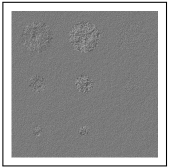
i) Generate a synthetic texture X from (v_0, h_0)



Piecewise constant v_0



Piecewise constant h_0



Texture sample X

Performance assessment

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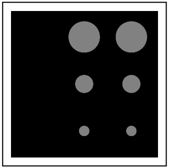

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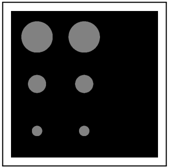
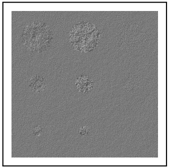


Piecewise constant h_0



Texture sample X

ii) Solve the minimization problem

$$\left(\widehat{v}, \widehat{h}\right) = \operatorname*{argmin}_{v,h} \mathbf{DF}(v, h, \mathcal{L}(X)) + \lambda_v \mathbf{TV}(v) + \lambda_h \mathbf{TV}(h)$$

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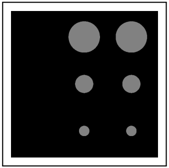

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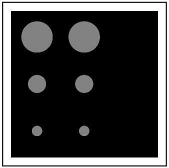

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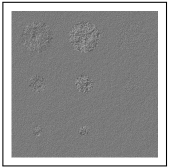

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Performance assessment

Experiments on synthetic data

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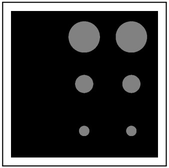

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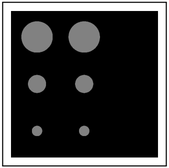

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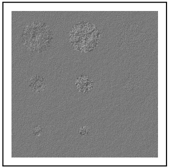

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iv) Re-estimation of v and h and computation of the error (SNR)

Performance assessment

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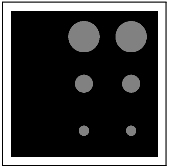

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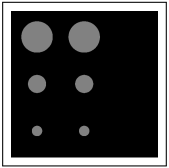

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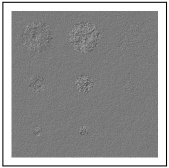

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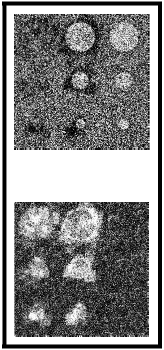
iii) k-means on $\widehat{v}$ and $\widehat{h}$ separately with $k = 2$

iv) Re-estimation of v and h and computation of the error (SNR)

v) Repeat ii) to iv) for different λ_v and λ_h ...

Results on synthetic data

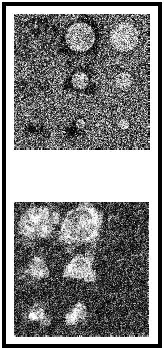
v first row, h second row



Linear regression

Results on synthetic data

v first row, h second row



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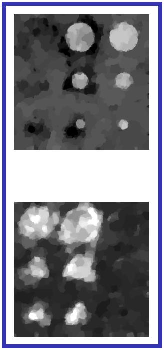
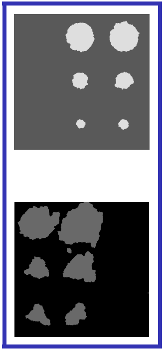


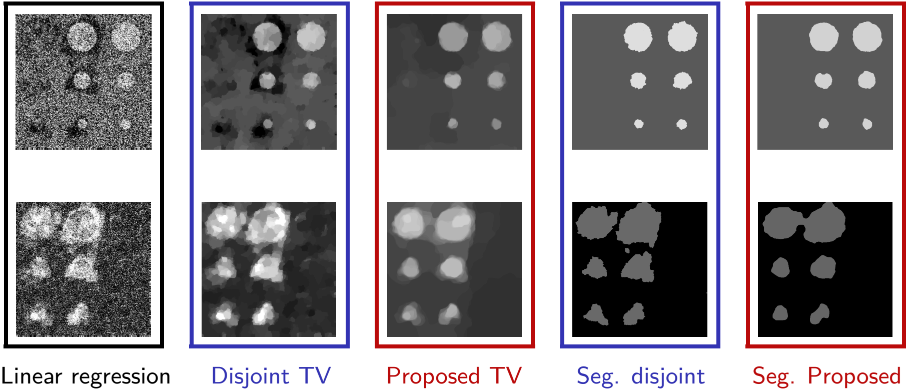
Disjoint TV



Seg. disjoint

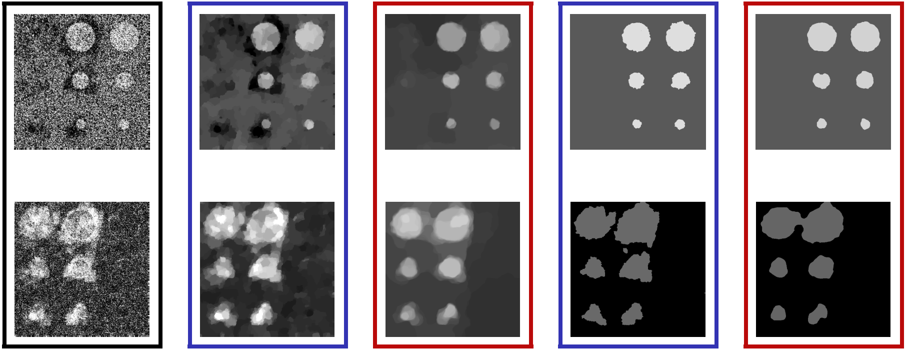
Results on synthetic data

v first row, *h* second row



Results on synthetic data

v first row, h second row



Linear regression | Disjoint TV | Proposed TV | Seg. disjoint | Seg. Proposed

$\mathrm{SNR}(v, v_0)$	2.7496	9.9722	**10.2854**	8.0758	8.0241
$\mathrm{SNR}(h, h_0)$	-5.3411	-4.2591	-4.1325	0.14181	**0.24025**

The experiment

PhD thesis of **Marion Serre** under supervision of **Valérie Vidal**

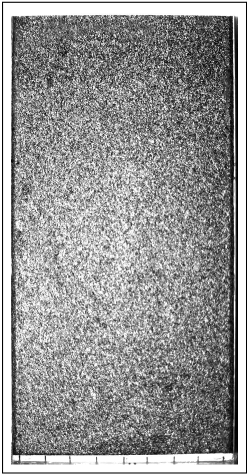
Multiphasic flow:

- porous media (solid)

- water (liquid)
- air (gas)

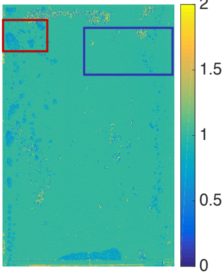
Hele-Shaw cell (quasi-2D)

- homogeneous liquid flow
- 9 gas injectors at the bottom
- height = 30cm, thickness = 2mm



2D flow gas & liquid

Image $x \in \mathbb{R}^{|\Omega|}$



Zoom of $x \in \mathbb{R}^{|\Omega|}$

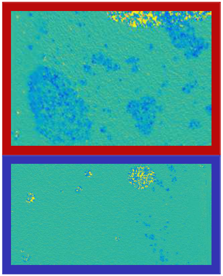
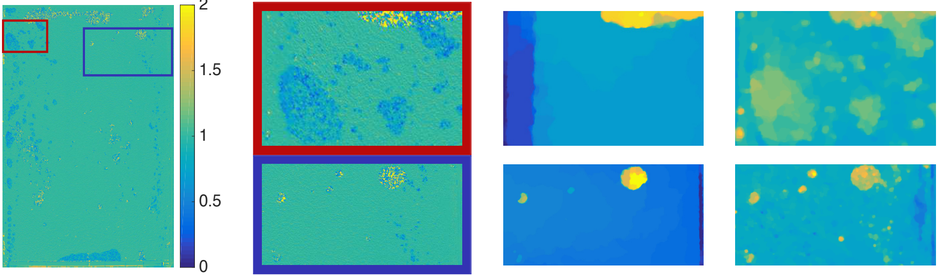


Image $x \in \mathbb{R}^{|\Omega|}$ Zoom of $x \in \mathbb{R}^{|\Omega|}$ Proposed TV: $\widehat{s}$ Proposed TV: $\widehat{h}$



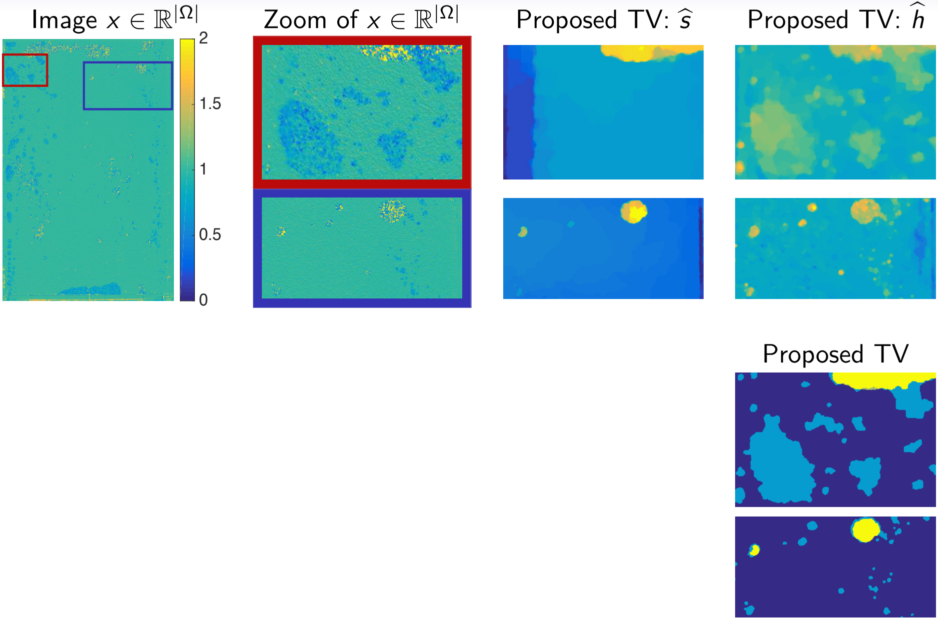


Image $x \in \mathbb{R}^{|\Omega|}$
2
1.5
1
0.5
0
Zoom of $x \in \mathbb{R}^{|\Omega|}$
Proposed TV: $\widehat{s}$
Proposed TV: $\widehat{h}$
Proposed TV

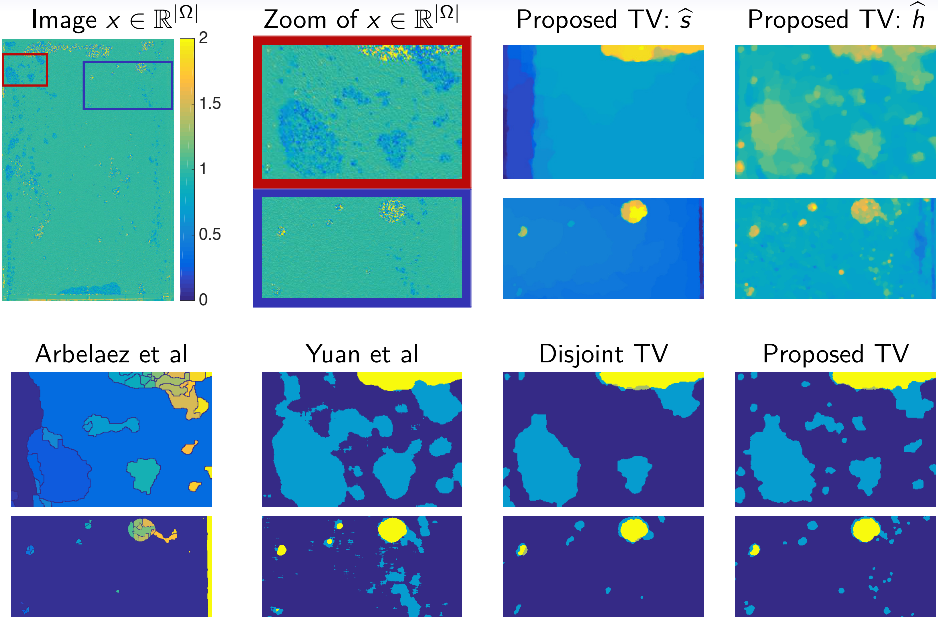
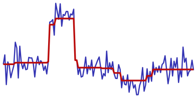


Image $x \in \mathbb{R}^{|\Omega|}$
2
1.5
1
0.5
0
Zoom of $x \in \mathbb{R}^{|\Omega|}$
Proposed TV: $\widehat{s}$
Proposed TV: $\widehat{h}$
Arbelaez et al
Yuan et al
Disjoint TV
Proposed TV

Prospects and future works

- automated tuning of regularization parameters λ_v, λ_h,



1D: $\lambda_{\text{opt}} = \frac{\sqrt{N}\sigma}{4}$ **[Dümbgen2009]**

2D: ?

N: number of points,
σ: std of noise

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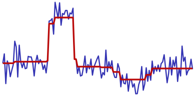

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2D: ?

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- use the same data fidelity term into other models

Mumford-Shah $$\min_{u,K} \int_{\Omega\setminus K} |u-g|^2 \, \mathrm{d}x + \int_{\Omega\setminus K} \|\nabla u\|^2 \, \mathrm{d}x + \mathcal{H}^1(K)$$

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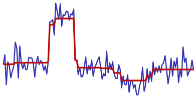

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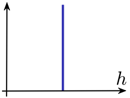
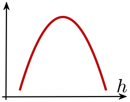
- studying multifractal textures



$\Longrightarrow$



Thank you for listening, I will be glad to answer your questions !